Methods of Vanishing Viscosity for Nonlinear Conservation Laws

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Special Session in Honor of Costas Dafermos

Continuum and Kinetic Methods in the Theory
Shocks, Fronts, Dislocations and Interfaces

Archimedes Center for Modeling, Analysis & Computation (ACMAC)
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Hyperbolic Conservation Laws:

\[ \partial_t U(t, x) + \partial_x F(U(t, x)) = 0, \quad U(t, x) \in \mathbb{R}^N \]

\[ F : \mathbb{R}^N \rightarrow \mathbb{R}^N \] Nonlinear: Eigenvalues of \( \nabla_u F(U) \) are real

\[ \partial_t U(t, x) + \partial_x F\{U(t, x), U^{(t)}(\cdot, x)\} = 0 \]

\[ U^{(t)}(\tau, x) := U(t - \tau, x) \]—the past history of \( U \) and \( x \) r.t. \( t \)

Approaches: Honor Physical or Design Artificial \( N \times N \) matrix function:

\[ D : \mathbb{R}^N \rightarrow M^{N \times N}, \quad D(U) \geq 0 \]

\[ \partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_x (D(U^\varepsilon) \partial_x U^\varepsilon) \]

admits a global solution \( U^\varepsilon(t, x) \) for each fixed \( \varepsilon > 0 \);

\[ U^\varepsilon(t, x) \rightarrow U(t, x) \] topology ?? \( U(t, x) \) an entropy solution??
Methods of Vanishing Viscosity

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\[ U^\varepsilon(t, x) \to U(t, x) \text{ topology ?? } \quad U(t, x) \text{ an entropy solution??} \]

Stokes (1848), Rayleigh (1910), Taylor (1910), Weyl (1949), \cdots
Hyperbolic Conservation Laws:
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Stokes (1848), Rayleigh (1910), Taylor (1910), Weyl (1949), ···

Theory: Entropy Conditions, Nonuniqueness, Existence, Solution Behavior, ···

Numerical Methods/Applications: Shock Capturing, Upwind, Kinetic, ···

Challenges: Singular limits, ··· \( \implies \) New Math Ideas/Methods ···

*Analogously for Multidimensional Setting
Scalar Conservation Laws: $BV$-Estimates

\[ \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon \]

**Estimates:** There exists $C$ independent of $\varepsilon$ such that

- **Maximum Principle:** $\|u^\varepsilon\|_{L^\infty} \leq C$
- **$BV$ Estimates:** $\|\partial_x u^\varepsilon\|_{L^1} + \|\partial_t u^\varepsilon\|_{L^1} \leq C$

Hopf, Oleinik, Lax, Volpert, Kruzhkov, ⋯

**Volpert’s Method:**

\[ \partial_t(|\partial_x u^\varepsilon|) + \partial_x(f'(u^\varepsilon)|\partial_x u^\varepsilon|) \leq \varepsilon \partial_{xx}(|\partial_x u^\varepsilon|), \]
\[ \partial_t(|\partial_t u^\varepsilon|) + \partial_x(f'(u^\varepsilon)|\partial_t u^\varepsilon|) \leq \varepsilon \partial_{xx}(|\partial_t u^\varepsilon|). \]

**BV-Compactness Theorem \implies** Strong Convergence of $u^\varepsilon(t, x)$

- Similar arguments to obtain for the $L^1$–Equicontinuity directly
- A corollary of the $L^1$-stability and the comparison principle via Kruzhkov’s Method
- $\partial_t u^\varepsilon + \nabla_x \cdot f(u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon$
Nonlocal Scalar Conservation Laws: \( BV \)-Estimates

\[
\partial_t u + \partial_x (f(u) + \int_0^t k_\delta(t - \tau)f(u(\tau))d\tau) = 0
\]

Memory kernel: \( k_\delta(t) \rightarrow (\alpha - 1)\delta(t) \) when \( \delta \rightarrow 0 \)

**Artificial Viscosity Method** (C-Christoforou 2007):

\[
\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) + \int_0^t k_\delta(t - \tau)f(u^\varepsilon(\tau))d\tau = \varepsilon\partial_{xx}(u^\varepsilon + \int_0^t k_\delta(t - \tau)u^\varepsilon(\tau)d\tau)
\]

**Idea:** Employ the resolvent kernel \( r_\delta(t) \) of \( k_\delta(t) \):

\[
r_\delta(t) + k_\delta \ast r_\delta = -k_\delta
\]

(MacCamy 1977, Dafermos 1988, Nohel-Rogers-Tzavaras 1988)

\[
\implies \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) + r_\delta(0)u^\varepsilon = r_\delta(t)u_0 - \int_0^t r_\delta'(t - \tau)u^\varepsilon(\tau)d\tau + \varepsilon\partial_{xx}u^\varepsilon
\]

**Estimates:** There exists \( C \) independent of \( \varepsilon, \delta \) such that

- **\( BV \cap L^\infty \) Estimates:** \( \| u^{\varepsilon,\delta} \|_{L^\infty} + \| \partial_x u^{\varepsilon,\delta} \|_{L^1} + \| \partial_t u^{\varepsilon,\delta} \|_{L^1} \leq C \)

**BV-Compactness Theorem** \( \implies \) **Strong Convergence of** \( u^{\varepsilon,\delta}(t, x) \)

- Existence and Stability of Entropy Solutions \( u^{\delta}(t, x) \) to hyperbolic conservation laws with memory as the limit of \( \varepsilon \rightarrow 0 \)
- When \( k_\delta(t) \rightarrow (\alpha - 1)\delta(t) \) as \( \delta \rightarrow 0 \), \( u^{\delta}(t, x) \rightarrow u(t, x) \) in \( L^1 \), a solution to the scalar conservation laws \( \partial_t u + \alpha\partial_x f(u) = 0 \)
\[
\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon
\]

**Estimates:** There exists $C$ independent of $\varepsilon$ such that
- **Maximum Principle:** $\|u^\varepsilon\|_{L^\infty} \leq C$ (or $\|u^\varepsilon\|_{L^p} \leq C$)
- **Dissipation Estimate:** $\|\sqrt{\varepsilon} u^\varepsilon_x\|_{L^2} \leq C$

**Energy Estimate:**
\[
\varepsilon (\partial_x u^\varepsilon)^2 = -\partial_t \left( \frac{(u^\varepsilon)^2}{2} \right) - \partial_x \left( \int_{u^\varepsilon}^u w f'(w) dw \right) + \varepsilon \partial_{xx} \left( \frac{(u^\varepsilon)^2}{2} \right)
\]

\[\Rightarrow \] For any $\eta \in C^2$ with entropy flux $q(u) = \int_u^u \eta'(w) f'(w) dw$, 
\[
\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \text{is compact in } H^{-1}_{loc}
\]

**Compensated Compactness** \[\Rightarrow\] **Strong Convergence of** $u^\varepsilon(t, x)$

- $\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x (D(u^\varepsilon) \partial_x u^\varepsilon)$
- Tartar, Schonbek, DiPerna, Chen-Lu, Tadmor-Rascle-Bagnerini, ⋯
- Scalar Conservation Laws with Memory: Dafermos 1988
Hyperbolic Systems: $BV$-Estimates via Glimm’s Scheme (1965)

- Wave Interaction Estimates
- $BV$-Estimate Techniques: Glimm Functional
- .......

$\Rightarrow$

- Global Existence of Solutions in $BV$ when the Total Variation of the Initial Data Is Small
- Structure of Solutions in $BV$
  Dafermos, DiPerna, ···
- Asymptotic Behavior: Decay of Periodic Solutions, ···
  Glimm-Lax, Dafermos, DiPerna, Liu, ···
- .........
\[ \partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_{xx} U^\varepsilon, \quad U(0, x) = U_0(x) \in \mathbb{R}^N \]

**Biachini-Bressan (2005):** For strictly hyperbolic systems on \( U \) in a n.b.h.d. of a compact set \( K \subset \mathbb{R}^N \), there exist constants \( \delta > 0 \) and \( C_j, j = 1, 2, 3 \), such that, if

\[
\text{Tot. Var.} \{ U_0 \} < \delta, \quad \lim_{x \to -\infty} U_0(x) \in K,
\]

then, \( \forall \varepsilon > 0 \), there exists a unique solution \( U^\varepsilon(t, \cdot) := S^\varepsilon_t U_0(\cdot) \) s.t.

**BV Bound:** \( \text{Tot. Var.} \{ S^\varepsilon_t U_0 \} \leq C_1 \text{ Tot. Var.} \{ U_0 \} \)

**\( L^1 \)-Stability:**

\[
\| S^\varepsilon_t U_0 - S^\varepsilon_t V_0 \|_{L^1} \leq C_2 \| U_0 - V_0 \|_{L^1} \\
\| S^\varepsilon_t U_0 - S^\varepsilon_s U_0 \|_{L^1} \leq C_3 (|t - s| + |\sqrt{\varepsilon t} - \sqrt{\varepsilon s}|)
\]

\[ \implies \text{Strong Convergence and } L^1\text{-Stability of the Limit Solution} \]

- Even for non-conservative strictly hyperbolic systems
- The approach requires the artificial viscosity form
- The total variation of the initial data is sufficiently small
Using the heat kernel to estimate the solution for \( t \in [0, \tau_\varepsilon] \):
\[
\| \partial_x U_\varepsilon(t, \cdot) \|_{L^1} \leq \kappa \delta, \quad \kappa \text{ indept. of } \varepsilon \text{ and } \delta
\]

Decompose \( \partial_x U_\varepsilon \) along a suitable basis of unit vectors \( \{r_1, \cdots, r_N\} \):

\[
\partial_x U_\varepsilon = \sum v_\varepsilon^i r_i \quad \text{(Sum of gradients of viscous travelling waves)}
\]

Obtain a system of \( N \) equations for these scalar components:

\[
\partial_t v_\varepsilon^i + \partial_x (\tilde{\lambda}_i v_\varepsilon^i) - \varepsilon \partial_{xx} v_\varepsilon^i = \phi_\varepsilon^i, \quad i = 1, \cdots, N.
\]

Then, as the scalar case, we obtain that, for all \( t \geq \tau_\varepsilon \),

\[
\| v_\varepsilon^i(t, \cdot) \|_{L^1} \leq \| v_\varepsilon^i(\tau_\varepsilon, \cdot) \|_{L^1} + \int_{\tau_\varepsilon}^\infty \int |\phi_\varepsilon^i(t, x)| \, dx \, dt.
\]

Construct the basis \( \{r_1, \cdots, r_N\} \) in a clever way so that, for \( t \geq \tau_\varepsilon \),

\[
\int_{\tau_\varepsilon}^\infty \int |\phi_\varepsilon^i(t, x)| \, dx \, dt \leq C, \quad C > 0 \text{ independent of } \varepsilon > 0.
\]

\[\implies\] Total Var. \( \{U_\varepsilon(t, \cdot)\} = \| U_\varepsilon^x(t, \cdot) \|_{L^1} \leq \sum \| v_\varepsilon^i(t, \cdot) \|_{L^1} \leq C \]
Further Problem: $BV$ Estimates and Convergence of vanishing viscosity approximation of the general form:

$$\partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_x (D(U^\varepsilon) \partial_x U^\varepsilon)$$

for general viscosity matrices $D(U)$.

Physical Viscosity: Navier-Stokes Viscosity Matrices?

Navier-Stokes Equations $\rightarrow$ Euler Equations??
Artificial Viscosity via Compensated Compactness

\[ \partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_{xx} U^\varepsilon \]

**Assume:** The system has a strictly convex entropy function \( \eta_*(U) \).

**Estimates:** There exists \( C \) independent of \( \varepsilon \) such that

- **Invariant Regions:** \( \|U^\varepsilon\|_{L^\infty} \leq C \) (or \( \|U^\varepsilon\|_{L^p} \leq C \))
- **Dissipation Estimate:** \( \|\sqrt{\varepsilon} \partial_x U^\varepsilon\|_{L^2} \leq C \)

**Energy Estimate:**

\[ \varepsilon (\partial_x U^\varepsilon)^\top \nabla^2 \eta_*(U^\varepsilon) \partial_x U^\varepsilon = -\partial_t \eta_*(U^\varepsilon) - \partial_x q_*(U^\varepsilon) + \varepsilon \partial_{xx} \eta_*(U^\varepsilon) \]

\[ \implies \text{For any } \eta \in C^2 \text{ with entropy flux } q \text{ (i.e., } \nabla q(U) = \nabla \eta(U) \nabla F(U) \text{),} \]

\[ \partial_t \eta(U^\varepsilon) + \partial_x q(U^\varepsilon) \text{ is compact in } H_{loc}^{-1} \]

**Compensated Compactness \implies Strong Convergence of } U^\varepsilon(t, x) \]

- \( \partial_t U^\varepsilon + \partial_x F(U^\varepsilon) = \varepsilon \partial_x (D(U^\varepsilon) \partial_x U^\varepsilon) \) for \( \nabla^2 \eta_*(U) D(U) \geq c_0 > 0 \).
- **Large Initial Data, \ldots**
Compactness: Young Measure and Commutation Identity

**Div-Curl Lemma** [Tartar (1979), Murat (1978)]
**Young Measure Rep.** [Tartar (1979), Ball (1989), Alberti-Müller (2001)]

\[
\langle \nu(\lambda), \eta_1(\lambda)q_2(\lambda) - q_1(\lambda)\eta_2(\lambda) \rangle \\
= \langle \nu(\lambda), \eta_1(\lambda) \rangle \langle \nu(\lambda), q_2(\lambda) \rangle - \langle \nu(\lambda), q_1(\lambda) \rangle \langle \nu(\lambda), \eta_2(\lambda) \rangle
\]

for any entropy-entropy flux pairs \((\eta_j, q_j), j = 1, 2\), where \(\nu = \nu_{t,x}(\lambda)\) is the associated Young measure (probability measure) for the sequence \(U^\varepsilon(t,x)\).

**Issue:** Is \(\nu\) a Dirac measure? \(\implies\) **Compactness of** \(U^\varepsilon(t,x)\) **in** \(L^1\)

**Remarks:**
- The viscosity matrix \(\nabla^2 \eta_*(U)D(U) > 0\) \(\implies\) Dissipation Estimate
- \(2 \times 2\) **Strict Hyperbolicity:** \(\implies\) Rich Entropy-Entropy Flux Pairs
  DiPerna, Dafermos, Serre, Morawetz, Perthame-Tzavaras, Chen-Li, · · · ·
Further Challenges

- Nonstrictly Hyperbolic Systems
- Initial Data of Large Oscillation without Bounded Variation
- Viscosity Matrix with $\nabla^2\eta_*(U)D(U) \geq 0$ but Not Positive Definite ??
- No $L^\infty$-Uniform Bound: Only Energy Bounds

Important Problems:

- The Compressible Navier-Stokes Equations
  $\Rightarrow$ the Compressible Euler Equations
- Global Spherically Symmetric Solutions to the Compressible Euler Equations
- Global Solutions to the Compressible Euler Equations with Random Forcing Terms
- Global Solutions to Nonlocal Conservation Laws with Memory

......
Navier-Stokes Equations: Inviscid Limit

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= \varepsilon u_{xx},
\end{align*}
\]

with the initial conditions:

\[
\rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x), \quad \lim_{x \to \pm \infty} (\rho_0(x), u_0(x)) = (\rho^\pm, u^\pm),
\]

\((\rho^\pm, u^\pm)\) are constant end-states with \(\rho^\pm > 0\)

The viscosity coefficient \(\varepsilon \in (0, \varepsilon_0]\) for some fixed \(\varepsilon_0\)

\(\rho\) – Density, \(u\) – Velocity of Fluid when \(\rho > 0\)

\(p = p(\rho) = \rho^2 e'(\rho)\) – Pressure with internal energy \(e(\rho)\)

For a polytropic perfect gas, \(p(\rho) = \kappa \rho^\gamma, \quad e(\rho) = \frac{\kappa}{\gamma-1} \rho^{\gamma-1}, \quad \gamma > 1\)

Existence of \(C^2\) solutions \((\rho^\varepsilon, u^\varepsilon)(t, x)\) for Large Data:

Ya Kanel 1968, Hoff 1998

Problem: **Inviscid Limit of** \((\rho^\varepsilon, u^\varepsilon)(t, x)\) when \(\varepsilon \to 0??\)

Limit System: Isentropic Euler Equations ??

\[
\begin{align*}
\partial_t \rho + \partial_x m &= 0, \\
\partial_t m + \partial_x \left( \frac{m^2}{\rho} + p(\rho) \right) &= 0
\end{align*}
\]

**Strict hyperbolicity:** Fails when \( \rho \to 0 \) (vacuum)

**Entropy Pair** \((\eta, q)\): \( \nabla q(U) = \nabla \eta(U) \nabla F(U) \)

**Convex Entropy:** \( \nabla^2 \eta(U) > 0 \) \quad **Weak Entropy:** \( \eta(\rho, \rho u)|_{\rho=0} = 0 \)

Weak entropy pairs are represented as

\[
\eta^\psi(\rho, \rho u) = \int_{\mathbb{R}} \chi(s) \psi(s) \, ds, \quad q^\psi(\rho, \rho u) = \int_{\mathbb{R}} (\theta s + (1 - \theta)u) \chi(s) \psi(s) \, ds
\]

by \( C^2 \) test functions \( \psi(s) \), where \( \chi(s) \) is the weak entropy kernel:

\[
\chi(s) := \left[ \rho^{2\theta} - (u - s)^2 \right]_+, \quad \theta = \frac{\gamma - 1}{2}, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}.
\]

**Physical Convex Entropy:** Mechanical energy-energy flux pair \((\eta_*, q_*)\):

\[
\eta_*(\rho, m) = \frac{1}{2} \frac{m^2}{\rho} + \rho e(\rho), \quad q_*(\rho, m) = \frac{1}{2} \frac{m^3}{\rho^2} + m(e(\rho) + \frac{p}{\rho})
\]
Artificial Viscosity: \[
\begin{align*}
\partial_t \rho + \partial_x m &= \varepsilon \partial_x^2 \rho, \\
\partial_t m + \partial_x \left( \frac{m^2}{\rho} + p(\rho) \right) &= \varepsilon \partial_x^2 m,
\end{align*}
\]

Numerical Viscosity: Lax-Friedrichs Scheme, Godunov Scheme, \cdots

Advantages: There exists \( C > 0 \), independent of \( \varepsilon > 0 \), such that

- **Invariant Regions** \implies \( L^\infty \) Estimate:
  \[
  0 \leq \rho^\varepsilon(t,x) \leq C, \quad 0 \leq |m^\varepsilon(t,x)| \leq C\rho^\varepsilon(t,x) \quad \text{a.e.}
  \]

- **Dissipation Estimate:**
  \[
  \sqrt{\varepsilon} \| \partial_x (\rho^\varepsilon, m^\varepsilon) \|_{L^2([0,T] \times \mathbb{R})} \leq C
  \]
  via the mechanical energy-entropy flux pairs \((\eta_*, q_*)\) that is strictly convex for \( 1 < \gamma \leq 2 \) (and convex for \( \gamma > 2 \): weighted dissipation estimate).

\implies For any \( C^2 \) weak entropy-entropy flux pair \((\eta, q)\),
\[
\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \quad \text{is compact in } H_{loc}^{-1}
\]
Physical Convex Entropy: Mechanical Energy-Energy Flux Pair \((\eta_*, q_*)\):

\[
\eta_*(\rho, m) = \frac{1}{2} m^2 \rho + e(\rho), \quad q_*(\rho, m) = \frac{1}{2} m^3 \rho^2 + me'(\rho)
\]

\[\implies \nabla^2 \eta_*(U) > 0 \quad \text{for } U = (\rho, m) \text{ when } 1 < \gamma \leq 2.\]

W.O.L.G. assume that \(\lim_{|x| \to \infty} U^\varepsilon = \bar{U} = U^\pm\). Set

\[
\Phi_*(U) = \eta_*(U) - \eta_*(\bar{U}) - \nabla \eta_*(\bar{U})(U - \bar{U}) \geq 0,
\]

\[
\Psi_*(U) = q_*(U) - q_*(\bar{U}) - \nabla \eta_*(\bar{U})(F(U) - F(\bar{U})).
\]

Multiply \(\nabla \Phi_*(U^\varepsilon)\), both sides of the system:

\[
\partial_t \Phi_*(U^\varepsilon) + \partial_x \Psi_*(U^\varepsilon) = \varepsilon \partial_x (\nabla \Phi_*(U^\varepsilon) \partial_x U^\varepsilon) - \varepsilon (\partial_x U^\varepsilon)^\top \nabla^2 \eta_*(U^\varepsilon) \partial_x U^\varepsilon.
\]

\[\implies \varepsilon \int_0^T \int_{-\infty}^{\infty} (\partial_x U^\varepsilon)^\top \nabla^2 \eta_*(U^\varepsilon) \partial_x U^\varepsilon \, dx \, dt
\leq \int_{-\infty}^{\infty} \Phi_*(U_0(x)) \, dx < \infty\]
Isentropic Euler Equations: Reduction of a Measure-Valued Solution $\nu_{t,x}$: \text{supp} \, \nu \text{ Is Bounded}

DiPerna: \[ \gamma = \frac{N+2}{N}, \; N \geq 5 \text{ odd} \]

Ding-Chen-Luo, Chen: \[ \gamma \in (1, \frac{5}{3}] \]

Lions-Perthame-Tadmor: \[ \gamma \geq 3 \]

Lions-Perthame-Souganidis: \[ \gamma \in (\frac{5}{3}, 3) \implies \gamma \in (1, 3) \]

Chen-LeFloch: \text{General Pressure Laws}

Conclusion: If \text{supp} \, \nu \text{ is bounded}, then

\[ \nu_{t,x} = \nu(\rho(t,x),m(t,x)), \]

that is, \( (\rho^\varepsilon(t,x), m^\varepsilon(t,x)) \rightarrow (\rho(t,x), m(t,x)) \) a.e. \( (t,x) \).

Key Point: Employ Effectively the Weak Entropy-Entropy Flux Pairs
Navier-Stokes Equations: Inviscid Limit

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Existence of \(C^2\) solutions \((\rho^\varepsilon, u^\varepsilon)(t, x)\) for Large Data:

Ya Kanel 1968, Hoff 1998

Problem: \textbf{Vanishing Viscosity Limit of} \((\rho^\varepsilon, u^\varepsilon)(t, x)\) \textit{when} \(\varepsilon \to 0??\)

Gilberg (1951), Hoff-Liu (1989), Gûes-Métivier-Williams-Zumbrun (2006), \cdots
Navier-Stokes Equations $\Rightarrow$ Euler Equations

**New Difficulties:**

- **No Invariant Regions:** Only Energy Norms
- **No Direct Derivative Estimates:**
  \[ \| \sqrt{\varepsilon} \partial_x (\rho^\varepsilon, m^\varepsilon) \|_{L^2([0,T] \times \mathbb{R})} \leq C \]
  although \[ \| \sqrt{\varepsilon} \partial_x u^\varepsilon \|_{L^2([0,T] \times \mathbb{R})} \leq C \]
- **No A-Priori Bounded Support of the Measure-Valued Solution** $\nu_{t,x}$

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Vanishing Viscosity/Conservation Laws  
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Strategies:

- Finite-Energy Bounds+Higher Integrability Bounds (replacing \(L^\infty\) bound)
- New Derivative Estimate for \(\varepsilon \partial_x \rho^\varepsilon\)
- \(H^{-1}\)-Compactness of Weak Entropy Dissipation Measures Only for Weak Entropy Pairs with Compactly Supported \(C^2\) Test Functions
- Prove that Any Connected Component of Support of the Measure-Valued Solution \(\nu_{t,x}\) Must Be Bounded

Related Earlier Work:
- Serre-Shearer 1994: 2 \(\times\) 2 system of elasticity with severe growth conditions
- LeFloch-Westdickenberg 2009: Energy Norms, \(\gamma \in (1,5/3]\), full dissipation

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- No Direct Derivative Estimates: \[ \| \sqrt{\varepsilon} \partial_x (\rho^\varepsilon, m^\varepsilon) \|_{L^2([0,T] \times \mathbb{R})} \leq C \]
  although \[ \| \sqrt{\varepsilon} \partial_x u^\varepsilon \|_{L^2([0,T] \times \mathbb{R})} \leq C \]
- No A-Priori Bounded Support of the Measure-Valued Solution \( \nu_{t,x} \)

Strategies:

- Finite-Energy Bounds+Higher Integrability Bounds (replacing \( L^\infty \) bound)
- New Derivative Estimate for \( \varepsilon \partial_x \rho^\varepsilon \)
- \( H^{-1} \)-Compactness of Weak Entropy Dissipation Measures Only for Weak Entropy Pairs with Compactly Supported \( C^2 \) Test Functions
- Prove that Any Connected Component of Support of the Measure-Valued Solution \( \nu_{t,x} \) Must Be Bounded

Related Earlier Work:

- Serre-Shearer 1994: \( 2 \times 2 \) system of elasticity with severe growth conditions
- LeFloch-Westdickenberg 2009: Energy Norms, \( \gamma \in (1, \frac{5}{3}] \), full dissipation
Theorem (Chen-Perepelitsa: CPAM 2010)

- Let the initial functions \((\rho_0, u_0)\) satisfy the finite-energy conditions.
- Let \((\rho^\varepsilon, m^\varepsilon), m^\varepsilon = \rho^\varepsilon u^\varepsilon\), be the solution of the Cauchy problem for the Navier-Stokes equations for each fixed \(\varepsilon > 0\).

\[\Rightarrow\] When \(\varepsilon \to 0\), there exists a subsequence of \((\rho^\varepsilon, m^\varepsilon)\) that converges strongly almost everywhere to a finite-energy entropy solution \((\rho, m)\) to the Cauchy problem for the isentropic Euler equations for any \(\gamma > 1\).
Navier-Stokes Equations: Key Estimates I

Let the initial functions \((\rho_0, u_0)\) satisfy

\[
\int_{-\infty}^{\infty} \Phi_\ast(U_\varepsilon^0(x)) \, dx \leq E_0 < \infty,
\]

\[
\int \left( \varepsilon^2 \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3} + 2\varepsilon \frac{|\rho_{0,x}(x)u_0(x)|}{\rho_0(x)} \right) \, dx \leq E_1 < \infty,
\]

where \(\Phi_\ast(U) = \eta_\ast(U) - \eta_\ast(\bar{U}) - \nabla \eta_\ast(\bar{U})(U - \bar{U}) \geq 0, \quad \eta_\ast(U) = \frac{1}{2} \frac{m^2}{\rho} + \rho e(\rho),\)

and \(E_0, E_1\) are independent of \(\varepsilon\).

Then there exists \(C > 0\) independent of \(\varepsilon, t > 0\) such that, for any \(t > 0\),

- **Energy Estimate:**

\[
\int_{-\infty}^{\infty} \Phi_\ast(U_\varepsilon(t, x)) \, dx + \int_0^t \int \varepsilon |u_\varepsilon^x|^2 \, dx d\tau \leq E_0;
\]
Let the initial functions \((\rho_0, u_0)\) satisfy

\[
\int_{-\infty}^{\infty} \Phi^*(U_0^\varepsilon(x)) \, dx \leq E_0 < \infty,
\]

\[
\int \left( \varepsilon^2 \frac{\rho_0^\varepsilon, x(x)^2}{\rho_0^\varepsilon(x)^3} + 2\varepsilon \frac{\rho_0^\varepsilon, x(x) u_0^\varepsilon(x)}{\rho_0^\varepsilon(x)} \right) \, dx \leq E_1 < \infty,
\]

where \(\Phi^*(U) = \eta^*(U) - \eta^*(\bar{U}) - \nabla \eta^*(\bar{U})(U - \bar{U}) \geq 0, \ \eta^*(U) = \frac{1}{2} \frac{m^2}{\rho} + \rho e(\rho),\)

and \(E_0, E_1\) are independent of \(\varepsilon\).

Then there exists \(C > 0\) independent of \(\varepsilon, t > 0\) such that, for any \(t > 0\),

- **Energy Estimate:**

\[
\int_{-\infty}^{\infty} \Phi^*(U^\varepsilon(t, x)) \, dx + \int_0^t \int \varepsilon |u^\varepsilon_x|^2 \, dxd\tau \leq E_0;
\]

- **New Derivative Estimate for Density:**

\[
\varepsilon^2 \int \frac{|\rho^\varepsilon_x(t,x)|^2}{\rho^\varepsilon(t,x)^3} \, dx + \varepsilon \int_0^t \int (\rho^\varepsilon)^{\gamma - 3} |\rho^\varepsilon_x|^2 \, dxd\tau \leq C(E_0 + E_1).
\]
Navier-Stokes Equations: Key Estimates II

**Case I:** $u^+ = u^- = 0$. Set $v = \frac{1}{\rho}$. Then the conservation of mass implies

$$v_t + uv_x = vu_x.$$ 

Differentiating it in $x$ and then multiplying by $2v_x$ to obtain

$$\left(\left|v_x\right|^2\right)_t + u\left(\left|v_x\right|^2\right)_x = 2v_x(vu_x)_x - 2u_x|v_x|^2.$$ 

Multiplying by $\rho$ and using the conservation of mass yield

$$\left(\rho|v_x|^2\right)_t + (\rho u|v_x|^2)_x = 2v_x u_{xx} = \frac{2}{\varepsilon}v_x p_x + \frac{2}{\varepsilon}v_x((\rho u)_t + (\rho u^2)_x)$$

$$= -\frac{(\gamma - 1)^2}{4} \rho^{\gamma-3} |\rho_x|^2 + \frac{2}{\varepsilon}(\rho u v_x)_t + R,$$

where $R = \frac{2}{\varepsilon}(\rho u(u v_x)_x - \rho u(vu_x)_x + v_x(\rho u^2)_x)$.

Note that

$$\int_{-\infty}^{\infty} R \, dx = \frac{2}{\varepsilon} \int_{-\infty}^{\infty} |u_x|^2 \, dx.$$
Then
\[ \varepsilon^2 \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3} \, dx + \frac{(\gamma - 1)^2}{2} \varepsilon \int_0^t \int \rho^{-3} |\rho_x|^2 \, dx \, d\tau \]
\[ = -2\varepsilon \int \left[ \frac{\rho_x u}{\rho} \right]^t_{\tau=0} \, dx + 2\varepsilon \int_0^t \int |u_x|^2 \, dx \, d\tau \]
\[ \leq \frac{\varepsilon^2}{2} \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3} \, dx + \varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3} \, dx + C. \]

That is,
\[ \varepsilon^2 \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3} \, dx + \varepsilon \int_0^t \int \rho^{-3} |\rho_x|^2 \, dx \, d\tau \]
\[ \leq C \left( \varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3} \, dx + 1 \right). \]

**Case II:** \( u^+ \neq u^- \): More technically involved.
Navier-Stokes Equations: Higher Integrability Estimates

Let the initial data \((\rho_0, u_0)\) satisfy

\[
\int_{-\infty}^{\infty} \Phi_\ast(U_0^\varepsilon(x)) \, dx \leq E_0 < \infty, \quad \int \rho_0^\varepsilon(x) |u_0^\varepsilon(x)| \, dx \leq M_0 < \infty
\]

\[
\int \left( \varepsilon^2 \frac{\rho_0^{\varepsilon, x}(x)^2}{\rho_0^\varepsilon(x)^3} + 2\varepsilon \frac{\rho_0^{\varepsilon, x}(x)u_0^\varepsilon(x)}{\rho_0^\varepsilon(x)} \right) \, dx \leq E_1 < \infty,
\]

where \(E_0, E_1, M_0\) are independent of \(\varepsilon\).

Then, for any compact set \(K \subset \mathbb{R}\) and \(t > 0\),

- There exists \(C_1 = C_1(K, E_0, \gamma, t) > 0\), indept. of \(\varepsilon > 0\), such that
  \[
  \int_0^t \int_K (\rho^\varepsilon(t, x))^{\gamma+1} \, dx \, d\tau \leq C_1;
  \]

- There exists \(C_2 = C_2(E_0, E_1, M_0, K, \gamma, t)\), indept. of \(\varepsilon > 0\), such that
  \[
  \int_0^t \int_K \left( \rho^\varepsilon |u^\varepsilon|^3 + (\rho^\varepsilon)^{\gamma+\theta} \right) \, dx \, d\tau \leq C_2.
  \]

*Perthame-Lions-Tadmor 1994, LeFloch-Westdickenberg 2009*
Weak entropy pairs are represented as

\[ \eta^\psi(\rho, \rho u) = \int_{\mathbb{R}} \chi(s) \psi(s) \, ds, \quad q^\psi(\rho, \rho u) = \int_{\mathbb{R}} (\theta s + (1 - \theta)u) \chi(s) \psi(s) \, ds \]

by \( C^2 \) test functions \( \psi(s) \), for \( \chi(s) := [\rho^{2\theta} - (u - s)^2]_+^\lambda \), \( \theta = \frac{\gamma - 1}{2} \), \( \lambda = \frac{3 - \gamma}{2(\gamma - 1)} \).

Let \( \nu_{t,x} \) be the Young measure determined by the solutions of the Navier-Stokes equations. Then \( \nu_{t,x} \) is a measure-valued solution of the Euler equations:

- For the test functions \( \psi \in \{\pm 1, \pm s, s^2\} \),

\[ \langle \nu_{t,x}, \eta^\psi \rangle_t + \langle \nu_{t,x}, q^\psi \rangle_x \leq 0, \quad \langle \nu_{t,x}, \eta^\psi \rangle(0, \cdot) = \eta^\psi(\rho_0, \rho_0 u_0), \]

in the sense of distributions in \( \mathbb{R}^2_+ \).

- In addition, denoting \( \overline{f(s)} := \langle \nu_{t,x}, f(s; \rho, u) \rangle \), \( \nu_{t,x} \) is confined by:

\[ \theta(s_2 - s_1)(\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)} \overline{\chi(s_2)}) \\
= (1 - \theta)(u\overline{\chi(s_2)} \overline{\chi(s_1)} - u\overline{\chi(s_1)} \overline{\chi(s_2)}) \quad \text{for a.e.} \ s_1, s_2 \in \mathbb{R} \]

**New Difficulty:** \( \text{supp} \ \nu_{t,x} \text{ Is Unbounded.} \)
When $\gamma = 3$, then $\theta = 1$ and the commutation relation becomes
\[
\chi(s_1)\chi(s_2) = \chi(s_1)\chi(s_2),
\]
which implies
\[
\chi(s)^2 = \chi(s)^2,
\]
by taking $s_1 = s_2$. That is,
\[
\langle \nu, (\chi(s) - \chi(s))^2 \rangle = 0 \quad \text{for any } s \in \mathbb{R}
\]
This implies that $\nu$ must be a Dirac mass on the set $\{\rho > 0\}$ or be supported completely in the vacuum $V = \{\rho = 0\}$, that is, the measure-valued solution $\nu_{t,x}$ is a Dirac mass in the phase coordinates $(\rho, m)$:
\[
\nu_{t,x}(\rho, m) = \delta_{(\rho(t,x), m(t,x))}(\rho, m).
\]
Measure-Valued Solution with Unbounded Support: $\gamma > 3$

Let $A := \cup \{(u - \rho^{\theta}, \rho^{\theta} + u) : (\rho, u) \in \text{supp } \nu\}$.

Let $J = (s_-, s_+)$ be any connected component of $A$.

Note that $\text{supp } \chi(s) = \{(\rho, u) : u - \rho^{\theta} \leq s \leq u + \rho^{\theta}\}$.

Claim: Any connected component $J$ of the support is bounded for $\gamma > 3$

Strategy: On the contrary, let $\inf\{s : s \in J\} = -\infty$.

Fix $M_0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0 + 1)$;
Choose sufficiently small $s_1 \leq -2|M_0|$ to reach the contradiction.

New Observation:
Measure-Valued Solution with Unbounded Support: $\gamma > 3$

Let $A := \bigcup \{(u - \rho^\theta, \rho^\theta + u) : (\rho, u) \in \text{supp } \nu\}$.

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Fix $M_0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0 + 1)$.
Choose sufficiently small $s_1 \leq -2|M_0|$ to reach the contradiction.

**New Observation:**

$$\int_{M_0}^{M_0+1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} ds_2 \leq C(\lambda)|s_1|^\lambda, \quad \lambda < 0.$$
Measure-Valued Solution with Unbounded Support: $\gamma > 3$

Let $A := \cup \{(u - \rho^\theta, \rho^\theta + u) : (\rho, u) \in \text{supp} \nu\}$.

Let $J = (s_-, s_+)$ be any connected component of $A$.

Note that $\text{supp} \chi(s) = \{(\rho, u) : u - \rho^\theta \leq s \leq u + \rho^\theta\}$.

Claim: Any connected component $J$ of the support is bounded for $\gamma > 3$

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New Observation: $\int_{M_0}^{M_0+1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} ds_2 \leq C(\lambda)|s_1|^\lambda, \quad \lambda < 0.$

Lions-Perthame-Tadmor’s argument: $\frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} \geq \chi(s_2)$ a.e. $s_1, s_2 \in J, s_1 < s_2$.

$\implies \int_{M_0}^{M_0+1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} ds_2 \geq \int_{M_0}^{M_0+1} \chi(s_2)ds_2 = C(M_0, \lambda) > 0$
Measure-Valued Solution with Unbounded Support: $\gamma > 3$

Let $A := \cup \{ (u - \rho^\theta, \rho^\theta + u) : (\rho, u) \in \text{supp} \nu \}$.

Let $J = (s_-, s_+)$ be any connected component of $A$.

Note that $\text{supp} \chi(s) = \{ (\rho, u) : u - \rho^\theta \leq s \leq u + \rho^\theta \}$.

**Claim:** Any connected component $J$ of the support is bounded for $\gamma > 3$

**Strategy:** On the contrary, let $\inf \{ s : s \in J \} = -\infty$.

Fix $M_0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0 + 1)$; choose sufficiently small $s_1 \leq -2|M_0|$ to reach the contradiction.

**New Observation:** $\int_{M_0}^{M_0+1} \frac{\chi(s_1) \chi(s_2)}{\chi(s_1)} ds_2 \leq C(\lambda)|s_1|^{\lambda}, \quad \lambda < 0$.

**Lions-Perthame-Tadmor’s argument:** $\frac{\chi(s_1) \chi(s_2)}{\chi(s_1)} \geq \chi(s_2)$ a.e. $s_1, s_2 \in J, s_1 < s_2$.

$$\implies \int_{M_0}^{M_0+1} \frac{\chi(s_1) \chi(s_2)}{\chi(s_1)} ds_2 \geq \int_{M_0}^{M_0+1} \chi(s_2) ds_2 = C(M_0, \lambda) > 0$$

$$\implies 0 < C(M_0, \lambda) = \int_{M_0}^{M_0+1} \frac{\chi(s_1) \chi(s_2)}{\chi(s_1)} ds_2 \leq C(\lambda)|s_1|^{\lambda} \to 0 \quad \text{when } s_1 \to -\infty.$$  

*The case when $J$ is unbounded from above can be treated similarly.

$$\implies$$ Lions-Perthame-Tadmor’s case for $\gamma > 3$.  

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On the contrary, suppose that \( J \) is unbounded from below. Let \( M_0 = \sup\{s : s \in J\} \in (-\infty, \infty] \).

Let \( s_1, s_2, s_3 \in (-\infty, M_0) \) with \( s_1 < s_2 < s_3 \). The commutation relation \( \Rightarrow \)

\[
(s_2 - s_1) \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} + (s_3 - s_2) \frac{\chi(s_3)\chi(s_2)}{\chi(s_3)} = (s_3 - s_1)\frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)}.
\]

Differentiating this equation in \( s_2 \) and dividing by \( (s_3 - s_1) \), we obtain

\[
\frac{\chi'(s_2)}{\chi(s_1)} \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} = \frac{s_2 - s_1}{s_3 - s_1} \frac{\chi(s_1)\chi'(s_2)}{\chi(s_1)} + \frac{s_3 - s_2}{s_3 - s_1} \frac{\chi(s_3)\chi'(s_2)}{\chi(s_3)}
+ \frac{1}{s_3 - s_1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} - \frac{1}{s_3 - s_1} \frac{\chi(s_3)\chi(s_2)}{\chi(s_3)}.
\]

**Strategy:**
On the contrary, suppose that $J$ is unbounded from below.

Let $M_0 = \sup\{s : s \in J\} \in (-\infty, \infty]$.

Let $s_1, s_2, s_3 \in (-\infty, M_0)$ with $s_1 < s_2 < s_3$. The commutation relation

\[
(s_2 - s_1) \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} + (s_3 - s_2) \frac{\chi(s_3)\chi(s_2)}{\chi(s_3)} = (s_3 - s_1) \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)} \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)}.
\]

Differentiating this equation in $s_2$ and dividing by $(s_3 - s_1)$, we obtain

\[
\frac{\chi'(s_2)}{\chi(s_1)\chi(s_3)} \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} = \frac{s_2 - s_1}{s_3 - s_1} \frac{\chi(s_1)\chi'(s_2)}{\chi(s_1)} + \frac{s_3 - s_2}{s_3 - s_1} \frac{\chi(s_3)\chi'(s_2)}{\chi(s_3)}
+ \frac{1}{s_3 - s_1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} - \frac{1}{s_3 - s_1} \frac{\chi(s_3)\chi(s_2)}{\chi(s_3)}.
\]

**Strategy:** Take $s_1 \to -\infty$ and show that the left-hand side has a smaller order than the right-hand side $\implies$ Contradiction.
As before: For any $s_1, s_3 \in J$, \( \frac{\chi(s_1)\chi(s_3)}{\chi(s_1) \chi(s_3)} \geq 1; \)

\( \chi(s) \geq 0 \) is not identically zero and \( \chi(s) \to 0 \) as \( s \to \inf J, \sup J, \)
\( \implies \) there exists \( s_2 \) such that \( \chi'(s_2) > 0, \chi(s_2) > 0. \)

Let \( s_3 > s_2 \) be points such that \( \chi(s_3) > 0 \) and let \( s_1 \to -\infty. \) From the 1st identity, \( \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} = \chi(s_2) \frac{\chi(s_1)\chi(s_3)}{\chi(s_1) \chi(s_3)} + o(1), \) as \( s_1 \to -\infty. \)

\( \lceil \chi'(s) \rceil + \leq \frac{2\lambda}{s - s_1} \chi(s). \)

From the 2nd equation, by throwing away the negative terms, we obtain
\[
\frac{\chi'(s_2)}{\chi(s_1) \chi(s_3)} \leq \frac{2\lambda + 1}{s_3 - s_1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} + o(1).
\]
As before: For any $s_1, s_3 \in J$, \( \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} \geq 1; \)

\( \chi(s) \geq 0 \) is not identically zero and \( \chi(s) \to 0 \) as \( s \to \inf J, \sup J, \) \( \implies \) there exists \( s_2 \) such that \( \chi'(s_2) > 0, \chi(s_2) > 0. \)

Let \( s_3 > s_2 \) be points such that \( \chi(s_3) > 0 \) and let \( s_1 \to -\infty. \) From the 1st identity, \( \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} = \chi(s_2) \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} + o(1), \) as \( s_1 \to -\infty. \)

\[ [\chi'(s)]_+ \leq \frac{2\lambda}{s-s_1} \chi(s). \]

From the 2nd equation, by throwing away the negative terms, we obtain

\[ \frac{\chi'(s_2)\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} \leq \frac{2\lambda+1}{s_3-s_1} \frac{\chi(s_1)\chi(s_2)}{\chi(s_1)} + o(1). \]

\[ \implies \left( \frac{\chi'(s_2)}{\chi(s_2)} - \frac{2\lambda+1}{s_3-s_1} \chi(s_2) \right) \frac{\chi(s_1)\chi(s_3)}{\chi(s_1)\chi(s_3)} \leq o(1). \]

**Contradiction as \( s_1 \to -\infty.**

*Another different proof is given by LeFloch-Westdickenberg 2009 for \( 1 < \gamma \leq \frac{5}{3}. \)*
**Theorem (Chen-Perepelitsa: CPAM 2010)**

- Let the initial functions \((\rho_0, u_0)\) satisfy the finite-energy conditions.

- Let \((\rho^\varepsilon, m^\varepsilon), m^\varepsilon = \rho^\varepsilon u^\varepsilon,\) be the solution of the Cauchy problem for the Navier-Stokes equations for each fixed \(\varepsilon > 0\).

\[\Rightarrow\]
When \(\varepsilon \to 0\), there exists a subsequence of \((\rho^\varepsilon, m^\varepsilon)\) that converges strongly almost everywhere to a finite-energy entropy solution \((\rho, m)\) to the Cauchy problem for the isentropic Euler equations for any \(\gamma > 1\).
Multi-D Isentropic Euler Eqns with Spherical Symmetry

\[
\begin{aligned}
\rho_t + \nabla_x (\rho \mathbf{v}) &= 0, \\
(\rho \mathbf{v})_t + \nabla_x (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_x p &= 0
\end{aligned}
\]

\( \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \nabla_x \) – Gradient w.r.t. \( \mathbf{x} \in \mathbb{R}^d \)

\( \rho \) – Density, \( \mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d \) – Velocity, \( p \) – Pressure

Pressure-density constitutive relation (by scaling):

\[ p = p(\rho) = \frac{\rho^\gamma}{\gamma}, \quad \gamma > 1. \]

Spherically Symmetric Solutions:

\[ \rho(t, \mathbf{x}) = \rho(t, r), \quad \mathbf{v}(t, \mathbf{x}) = u(t, r) \frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}|. \]

Then the functions \((\rho, m) = (\rho, \rho u)\) are governed by

\[
\begin{aligned}
\rho_t + m_r + \frac{d-1}{r} m &= 0, \\
m_t + (\frac{m^2}{\rho} + p(\rho))_r + \frac{d-1}{r} \frac{m^2}{\rho} &= 0
\end{aligned}
\]
Defocusing: Expanding Spherically Symmetric Solutions


\[ 0 \leq \rho(t, x)^{\frac{\gamma-1}{2}} \leq u(t, x) \leq C < \infty. \]
Focusing: Imploding Spherically Symmetric Solutions

Guderley 1942, Courant-Fridrichs 1945: Singularity of Self-Similar Solutions
Rauch 1986: No $BV$ or $L^\infty$ Bounds

Longstanding Problem: Does the Concentration Phenomenon Occur?
$\iff$ Does the Density Develop a Measure at the Origin?
Viscosity Methods for the Euler Equations with Spherical Symmetry

\[
\begin{aligned}
\rho_t + m_r + \frac{d-1}{r} m &= \varepsilon \left( \rho_{rr} + \frac{d-1}{r} \rho_r \right), \\
m_t + \left( \frac{m^2}{\rho} + p \right)_r + \frac{d-1}{r} \frac{m^2}{\rho} &= \varepsilon \left( m_{rr} + \frac{d-1}{r} m \right)_r,
\end{aligned}
\]

with the initial conditions: \( \rho(0, r) = \rho_0(r), \ m(0, r) = m_0(r) \),
with the bdry condition: \( (\rho_r, m)|_{r=a(\varepsilon)} = (0, 0), \ a(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

**Chen-Perepelitsa 2011 (Uniform Estimates):** For any compact set \( K \subset \mathbb{R}_+ \) and \( T > 0 \), there exists \( C > 0 \) independent of \( \varepsilon > 0 \) such that

\[
\int_{a(\varepsilon)}^{\infty} \left( \frac{1}{2} \rho u^2 + \rho e(\rho) \right) r^{d-1} \, dr \\
+ \varepsilon \int_0^T \int_{a(\varepsilon)}^{\infty} \left( \rho^{\gamma-2} |\rho_r|^2 + \rho |u_r|^2 + \frac{\rho u^2}{2r^2} \right) r^{d-1} \, dr \, dt \leq C;
\]

\[
\int_0^t \int_K \left( \rho^{\varepsilon} |u^{\varepsilon}|^3 + \rho^{\varepsilon(3\gamma-1)/2} + (\rho^{\varepsilon})^{\gamma+1} \right) r^{d-1} \, dr \, dt \leq C.
\]
Spherically Symmetric Solutions for the Euler Equations via Vanishing Viscosity Limit

\[
\begin{aligned}
\rho_t + m_r + \frac{d-1}{r}m &= \varepsilon \left( \rho_{rr} + \frac{d-1}{r} \rho_r \right), \\
\rho_t + \left( \frac{m^2}{\rho} + p \right)_r + \frac{d-1}{r} \frac{m^2}{\rho} &= \varepsilon \left( m_{rr} + \frac{d-1}{r} m \right)_r,
\end{aligned}
\]

with the initial conditions: \( \rho(0, r) = \rho_0(r) \), \( m(0, r) = m_0(r) \),
with the boundary condition: \( (\rho_r, m)|_{r=a(\varepsilon)} = (0, 0) \)

Chen-Perepelitsa 2011:

Let the initial functions \((\rho_0, m_0)\) satisfy the finite-energy conditions. Then

- For each fixed \(\varepsilon > 0\), there exists a global viscous solution \((\rho^\varepsilon, m^\varepsilon)\);
- When \(\varepsilon \rightarrow 0\), there exists a subsequence of \((\rho^\varepsilon, m^\varepsilon)\) that converges strongly almost everywhere to a finite-energy spherically symmetric entropy solution \((\rho, m)\) for any \(\gamma > 1\).

*Answer to the Longstanding Problem: No Concentration Occurs at the Origin!!

*Vanishing Physical Viscosity Limit?
Conservation Laws of Viscoelastic Materials with Memory

\[
\begin{align*}
\partial_t u(t, x) - \partial_x v(t, x) &= 0, \\
\partial_t v(t, x) - \partial_x \sigma(t, x) &= 0
\end{align*}
\]

- Elastic Medium: \( \sigma(t, x) = f(u(t, x)) \)
- Nohel-Rogers-Tzavaras 1988:
  \[
  \sigma(t, x) = f(u(t, x)) + \int_0^t k(t - \tau)f(u(\tau, x))d\tau
  \]
- C-Dafermos 1997:
  \[
  \sigma(t, x) = f(u(t, x) + \int_0^t k(t - \tau)g(u(\tau, x))d\tau)
  \]

When \( k(t) = \frac{1}{\delta}\exp(-\frac{t}{\delta}) \), the system is equivalent to

\[
\begin{align*}
\partial_t u - \partial_x v &= 0, \\
\partial_t v - \partial_x f(w) &= 0, \\
\partial_t w - \partial_x v + \frac{1}{\delta}(w - (g(u) + u)) &= 0
\end{align*}
\]

\[
w(t, x) = u(t, x) + \int_0^t k(t - \tau)g(u(\tau, x))d\tau
\]
Conservation Laws of Viscoelastic Materials with Memory  
Method of Vanishing Viscosity: C-Dafermos 1997

\[
\begin{align*}
\partial_t u - \partial_x v &= \varepsilon \partial_{xx} (u + \int_0^t k(t - \tau)g(u(\tau, \cdot))d\tau), \\
\partial_t v - \partial_x f(u + \int_0^t k(t - \tau)g(u(\tau, \cdot))d\tau) &= \varepsilon \partial_{xx} v
\end{align*}
\]

Set
\[
w(t, x) = u(t, x) + \int_0^t k(t - \tau)g(u(\tau, x))d\tau \\
J[u] = k(0)g(u) + \int_0^t k'(t - \tau)g(u(\tau, x))d\tau
\]

\[
\Rightarrow \left\{ \begin{array}{l}
\partial_t w - \partial_x v = \varepsilon \partial_{xx} w + J[u], \\
\partial_t v - \partial_x f(w) = \varepsilon \partial_{xx} v
\end{array} \right.
\]

Estimates: \( \exists M_T > 0 \) such that
\[
\| (w^\varepsilon, v^\varepsilon) \|_{L^\infty \cap L^2(\mathbb{R}^2_+)} + \| \sqrt{\varepsilon}(w_x^\varepsilon, v_x^\varepsilon) \|_{L^2(\mathbb{R}^2_+)} \leq M_T \quad \text{for } 0 \leq t \leq T
\]

Compensated Compactness
\[
\Rightarrow \quad \ast \text{Strong Convergence of } (u^\varepsilon(t, x), v^\varepsilon(t, x)) \text{ to } (u(t, x), v(t, x)) \\
\ast (u(t, x), v(t, x)) \text{ is an entropy solution}
\]
Stochastic Compressible Euler Equations with Random Forcing:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= \sigma(\rho, u, t, x) \partial_t W(t),
\end{align*}
\]

*\(W(t)\) is a Gaussian white noise martingale random measure

Chen & Qian Ding 2011: Method of Artificial Viscosity:

\[\implies\text{Global Existence of Stochastic Entropy Solutions}\]
Stochastic Compressible Euler Equations with Random Forcing:

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Chen & Qian Ding 2011: Method of Artificial Viscosity:

\[\implies\text{Global Existence of Stochastic Entropy Solutions}\]


\[
\begin{aligned}
v_x - u_y &= \varepsilon \nabla \cdot (\sigma_1(\rho) \nabla \theta), \\
(\rho u)_x + (\rho v)_y &= \varepsilon \nabla \cdot (\sigma_2(\rho) \nabla \rho)
\end{aligned}
\]

\[q^2 = u^2 + v^2, \quad (u, v) = (q \cos \theta, q \sin \theta)\]

\[\rho = \rho(q) = (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma - 1}}, \quad \gamma > 1\]

\[\implies\text{Invariant Regions & Compensated Compactness Framework}\]
Concluding Remarks

- **Methods of Vanishing Viscosity** are fundamental approaches.
  
  *Theory of Hyperbolic Conservation Laws:*
  
  Entropy Conditions, Nonuniqueness, Existence, Solution Behavior
  
  *Nonuniqueness:* In general, the limit solutions of vanishing viscosity limit may depend on the viscosity matrices $D$ in the systems under consideration: MHD, multiphase flow models, ⋯
  
  *Numerical Methods:* Shock capturing methods, unwind, kinetic, ⋯
  
  Design correct numerical viscosity, ⋯

- **Challenges:** Singular limits, ⋯ $\Rightarrow$ New Math Ideas/Methods ⋯

- In this talk, we have presented several examples to show you how various methods of vanishing viscosity can be developed to obtain global entropy solutions. These examples show that every solution of a vanishing viscosity limit problem will lead to new ideas, techniques, approaches, frameworks, etc. which are useful for solving other important problems...

- Many fundamental problems are widely open and require new further methods of vanishing viscosity and related new ideas, techniques and approaches....
Happy 70th Birthday

Costas!