Stationary and Traveling Waves in Nonlinear Lattices

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1 Introduction
   - NLS with Saturable nonlinearity
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   - Stability of Solitons saturable DNLS near the Anti-Continuum Limit
   - Traveling Waves in FPU Lattices with Saturable Nonlinearities

2 Periodic and Decaying Solutions in saturable DNLS
   - Main Result: Existence Theorem
   - Variational Approach
   - J and J_k functionals and Nehari Manifolds
   - Minimization problem

3 TW in FPU Lattices with Saturable Nonlinearities
   - Review of some interesting results
   - Statement of Problem and Main Results
   - Assumptions
   - Main Theorems
   - Variational Setting
   - Some Remarks

4 Conclusions
   - Concluding Remarks
   - References
Solitons in photo-refractive media. The equation describing these media is a modification of the original NLS, which consists in substituting the Kerr nonlinearity term by another one of saturable type:

\[ iu_t + u_{xx} + \beta \frac{|u|^2 u}{1 + |u|^2} = 0 \]  \hspace{1cm} (2.1)

where \( u \) is a normalized slowly varying envelope of the electric field of the light wave.

Some Properties

Saturable NLS (SNLS) equation is nonintegrable and the soliton collision processes are inelastic, leading to annihilation, fusion or creation of solitons. Another important feature of the SNLS is that the behaviour of the solutions is quite robust, being independent of the details of the mathematical model.
The discrete version of the NLS equation can be used to describe nonlinear waveguide arrays within the tight binding approximation Christodoulides et al 2003. The optical pulse propagation in 1D equidistant nonlinear waveguide arrays with saturable nonlinearity can be modeled, within the nearest-neighbor approximation and with neglected influence of diffusion of charge carriers,

**Saturable DNLS**

\[ i\dot{\psi}_n + \psi_{n+1} + \psi_{n-1} - 2\psi_n + \frac{\nu |\psi_n|^2}{1 + \mu |\psi_n|^2} \psi_n = 0 \]  
(2.2)

where \( \mu > 0 \) and \( \nu \neq 0 \).

**Standing wave solutions**

\( \psi_n = \exp(-i\omega t)u_n, \ u_n \in \mathbb{R} \). Consider two types of solutions:

- k-periodic, i.e. \( u_{n+k} = u_n \);
- vanishing at infinity, i.e. \( \lim_{n \to \pm\infty} u_n = 0, \ u_n \in l^2 \).
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We present a theory for the localized excitations in the vicinity of the anti-continuum limit (where the sites are decoupled from each other) in the case of saturable DNLS models where the nonlinearity is an arbitrary function of the field modulus. This situation encompasses as case examples the photorefractive nonlinearity.

**Discrete Solitons of saturable DNLS**

\[ u_n(t) = \phi_n e^{i(\mu - 2\varepsilon)t + i\theta_0}, \quad \mu \in \mathbb{R}, \quad \phi_n \in \mathbb{C}, \quad n \in \mathbb{Z} \quad (2.3) \]

where \( \theta_0 \in \mathbb{R} \) is a parameter and \( (\mu, \phi_n) \) solve the difference equation on \( n \in \mathbb{Z} \): \( (\mu - g(|\phi_n|^2))\phi_n = \varepsilon(\phi_{n+1} + \phi_{n-1}) \).

**Properties of Solitons near anti-continuum limit**

\[ \lim_{\varepsilon \to 0^+} \phi_n = \phi_n^{(0)} = \begin{cases} e^{i\theta_n} & n \in S \\ 0 & n \in \mathbb{Z}\setminus S \end{cases} \quad (2.4) \]

\[ \lim_{|n| \to \infty} e^{\kappa|n|}\phi_n = \phi_\infty, \quad \phi_n \in \mathbb{R}, \quad n \in \mathbb{Z} \quad (2.5) \]

where \( S \) is a finite set of nodes of the lattice \( n \in \mathbb{Z} \) and \( \theta_n = \{0, \pi\}, n \in S \).
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A Fermi-Pasta-Ulam (FPU) lattice is an infinite system of identical particles on a line with nearest neighbor interaction $q_n$ is the coordinate of the nth particle and $U(r)$ is the potential.

$$\ddot{q}_n = U'(q_{n+1} - q_n) - U'(q_n - q_{n-1}), \quad n \in \mathbb{Z}$$

**FPU solitary waves**

- Saturable Nonlinearities: $U$ is asymptotically quadratic at infinity.
- We prove the existence of super-sonic traveling waves when the wave speed belongs to a natural range.
- We prove the existence of periodic waves with arbitrarily large periods.
- We obtain solitary waves passing to the limit as the period tends to infinity and making use of concentration compactness argument.
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Existence Theorem (Pankov & Rothos, 2008)

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\[ -Δu_n - ωu_n = f(u_n), \quad f(u) = \frac{\nu u^3}{1 + µu^2}. \]

Suppose that either

\[ ω < 0 \quad \text{and} \quad ω + ν/µ > 0 \quad \text{or} \quad ω > 4 \quad \text{and} \quad ω + ν/µ < 4. \]

Then for every k ≥ 2 there exist two nontrivial k-periodic solutions (u_{n+k} = u_n) ±u^{(k)} as well as two nontrivial solution ±u ∈ l^2, of equation

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Let us point out that the spectrum of $-\Delta$ in $l^2$ coincides with the interval $[0, 4]$. The assumption of main Theorem means that $\omega$ does not belong to the spectrum, while $\omega + \nu/\mu$ is on the opposite side of the spectrum endpoint closest to $\omega$.

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We prove Main Theorem by means of a variational approach.
Variational Approach

- Introduce functionals $J_k$ and $J$ on the spaces of $k$-periodic sequences and $l^2$, respectively, whose critical points are solutions of equation $-\Delta u_n - \omega u_n = f(u_n)$.

- To produce nontrivial critical points, we use Nehari manifold $N_k$, $N$. These are $C^1$ submanifolds of the spaces $X_k$, $X$.

- On Nehari manifolds $J_k$ and $J$ are bounded below by positive constants.

- We minimize $J_k$ and $J$ over $N_k$, $N$.

- A minimum point of the functional over the Nehari manifold, this is automatically a solution of equation $-\Delta u_n - \omega u_n = f(u_n)$.

- In the periodic case $N_k$ is finite dimensional. The second part of our Theorem concerning $l^2$ solutions is more involved because the functional $J$ does not satisfy the Palais–Smale condition. Our idea is to pass to the limit as $k \to \infty$. 
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Basic Assumptions

We consider the following equation

$$-\Delta u - \omega u = \sigma f(u), \quad \sigma = \pm 1, \quad \omega < 0.$$ 

We suppose that the nonlinearity $f(t)$ satisfies the following assumptions in which

$$F(t) = \int_0^t f(s)ds$$

is the primitive function of $f(t)$.

- (h1) $f(t) = o(t)$ as $t \to 0$,
- (h2) $\lim_{t \to \pm \infty} \frac{f(t)}{t} = l < \infty$,
- (h3) $f \in C^1(\mathbb{R})$ and $f(t)t < f'(t)t^2$ for $t \neq 0$,
- (h4) $\lim \left[ \frac{1}{2} f(t)t - F(t) \right] = \infty$ as $t \to \pm \infty$.

$X_k$ the space of all $k$–periodic sequences, $X = l^2$ with $(\cdot, \cdot)_k$ and $(\cdot, \cdot)$ the natural inner products.
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J and $J_k$

- We define the bounded and self-adjoint operator $L = -\Delta - \omega$ acting either on $X_k$ or on $X$.

- we introduce the action functionals
  \[
  J_k(u) = \frac{1}{2} (Lu, u)_k - \sum_{Q_k} F(u_n), \quad Q_k = \{ k \in \mathbb{Z} : -\left[ \frac{k}{2} \right] \leq n \leq k - \left[ \frac{k}{2} \right] - 1 \} \\
  J(u) = \frac{1}{2} (Lu, u) - \sum_{\mathbb{Z}} F(u_n)
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- $J_k, J$ are $C^1$–functionals and the derivatives are given by
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Critical points of \( J_k \) and \( J \) are \( k \)-periodic and \( l^2 \)-solutions of

\[-\Delta u - \omega u = \sigma f(u), \quad \sigma = \pm 1, \quad \omega < 0\]

- \( \sigma = 1 \), a solution of the difference equation is a ground state solution if it minimizes the action among all solution of the same type.
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Nehari manifolds associated with functionals $J, J_k$

$$\begin{align*}
N_k & = \{ v \in X_k : \langle J'_k(u), v \rangle = 0, v \neq 0 \} \subset X_k \\
N & = \{ v \in X : \langle J'(u), v \rangle = 0, v \neq 0 \} \subset X
\end{align*}$$

Let $I_k(u) = \langle J'_k(u), v \rangle$, $I(u) = \langle J'(u), v \rangle$. These are $C^1$ functionals and

$$\begin{align*}
\langle I'_k(u), v \rangle &= 2(Lu, v)_k - \sum_{Q_k} [f(u_n) + f'(u_n)u_n]v_n, \\
\langle I'(u), v \rangle &= 2(Lu, v) - \sum_{Z} [f(u_n) + f'(u_n)u_n]v_n.
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**Lemma**

The sets $N_k$ and $N$ are nonempty closed $C^1$ submanifolds in $X_k$ and $X$, respectively. The derivatives $I'_k$ and $I'$ are nonzero on corresponding Nehari manifolds. Moreover, there exists $\beta_0 > 0$ such that $\|u\|_k \geq \beta_0, u \in N_k$, and $\|u\| \geq \beta_0, u \in N$. 
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Minimization problem set-up

The tangent spaces $T_u N_k$ and $T_u N$ at $u \in N_k$ or $u \in N$, respectively, are

$$T_u N_k = \{ v \in X_k : \langle I'_k(u), v \rangle = 0 \}, \quad T_u N = \{ v \in X : \langle I'(u), v \rangle = 0 \},$$

and the line $\mathbb{R} u = \{ tu : t \in \mathbb{R} \}$ is a transverse line.

Lemma

For $u \in N_k$ the function $J_k(tu)$, $t > 0$, has a unique critical point at $t = 1$ which is, actually, a global maximum. The same statement holds for $N$ and $J$.

Minimization problems

Minimum points of $J_k$ and $J$ on corresponding Nehari manifolds are solutions of equation $-\Delta u - \omega u = \sigma f(u)$.

$$m_k = \inf \{ J_k(v) : v \in N_k \} \quad \text{and} \quad m = \inf \{ J(v) : v \in N \}.$$
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Theorem (Periodic Solutions, 2008)

Assume that the nonlinearity satisfies (h1) – (h4) and either \( \sigma = 1, \omega < 0 \) and \( 1 + \omega > 0 \), or \( \sigma = -1, \omega > 4 \) and \( -1 + \omega < 4 \). Then for every \( k > 1 \) equation

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- possesses a nontrivial \( k \)-periodic ground state solution \( u^{(k)} \in X_k \),
- when \( f \) is odd, there are two nontrivial ground states \( \pm u^{(k)} \) and one of them is positive provided \( \sigma = 1 \).
- Let \( u^{(k)} \in X_k \) be the solution. Then there exists a ground state solution \( u \in l^2 \) and \( b_k \in \mathbb{Z} \) such that

\[ ||u^{(k)}(\cdot + b_k) - u(\cdot)||_k \to 0, \quad k \to \infty.\]
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Theorem (Exponential Decay Solutions, 2008)

Under the assumptions (h1)–(h4)

1. there exists a nontrivial ground state solution \( u \in l^2 \) and \( u \) decays exponentially fast

\[ |u_n| \leq Ce^{-\alpha|n|} \]

for some \( \alpha > 0 \) and \( C > 0 \),

2. if \( f \) is odd, there are two nontrivial ground states \( \pm u^{(k)} \) and one of them is positive provided \( \sigma = 1 \).

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Under the assumptions (h1)–(h4), let \( u^k \in X_k \) be the solution obtained in that Theorem. Then there exists a ground state solution \( u \in l^2 \) and \( b_k \in \mathbb{Z} \) such that

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The function $r(s) = (Au)(s)$ is called the relative displacement profile.
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Two types of traveling waves are of interest.

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To formulate the assumptions on the potential $U(r)$, we separate its harmonic and anharmonic parts

$$U(r) = \frac{c_0^2}{2} r^2 + V(r).$$

The constant $c_0 \geq 0$ is called the speed of sound. Our main assumptions on the potential are

(i) the function $V$ is a continuously differentiable function on $\mathbb{R}$, $V(0) = V'(0) = 0$ and $V'(r) = o(r)$ as $r \to 0$;

(ii) the limit $\lim_{r \to \pm \infty} V'(r)/r = a$ exists and is finite, and the function

$$g(r) = V'(r) - ar$$

is bounded;

(iii) for every $r_0 > 0$ there exists $\delta_0 = \delta_0(r_0) > 0$ such that

$$\frac{1}{2} r V'(r) - V(r) \geq \delta_0$$

for $|r| \geq r_0$, and $V(r) \geq 0$ for all $r \in \mathbb{R}$.

For shortness, we also set

$$f(r) = V'(r) = ar + g(r).$$
Existence Theorem of periodic traveling waves (Pankov & Rothos 2010)

Suppose that assumptions (i)–(iii) are satisfied and \( k \geq 1 \). If \( c_0^2 < c^2 < c_0^2 + a \) and either

\[
G(r) := \int_0^r g(\rho) \, d\rho \to -\infty \quad \text{as} \quad r \to \pm \infty
\]

or

\[
c^2 \left( \frac{\pi n}{k} \right)^2 - 4(c_0^2 + a) \sin^2 \frac{\pi n}{2k} \neq 0 \quad \forall n \in \mathbb{N},
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then there exist nonconstant non-decreasing, as well as non-increasing, \( 2k \)-periodic traveling waves.
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Existence Theorem of solitary waves (Pankov & Rothos 2010)

Suppose that assumptions (i)–(iii) are satisfied and \( k \geq 1 \). If \( c_0^2 < c^2 < c_0^2 + a \), then there exist both increasing and decreasing solitary traveling waves. Moreover, these waves are exponentially decaying in the sense that

\[
|u'(s)| \leq C \exp(-\alpha |s|),
\]

with certain positive constants \( C \) and \( \alpha \).
We introduce the following Hilbert spaces

\[ X_k = \{ u \in H^1_{\text{loc}}(\mathbb{R}) : u(s + 2k) = u(s), u(0) = 0 \} \quad (u, v)_k = \int_{-k}^{k} u'(s)v'(s)ds, \]

\[ X = \{ u \in H^1_{\text{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R})u(0) = 0 \}, \quad (u, v)_k = \int_{-\infty}^{\infty} u'(s)v'(s)ds. \]

The corresponding norms are denoted by \( \| \cdot \|_k \) and \( \| \cdot \| \), respectively. On the spaces \( X_k \) and \( X \) we consider the functionals defined by

\[ J_k(u) = \int_{-k}^{k} \left[ \frac{c^2}{2}u'(s)^2 - \frac{c_0^2}{2}(Au(s))^2 - V(Au(s)) \right] ds, \]

\[ J(u) = \int_{-\infty}^{\infty} \left[ \frac{c^2}{2}u'(s)^2 - \frac{c_0^2}{2}(Au(s))^2 - V(Au(s)) \right] ds, \]

respectively. The space \( X \) and functional \( J \) can be considered as limit cases \( X = X_\infty \) and \( J = J_\infty \) of \( X_k \) and \( J_k \), respectively.
The derivatives of the functionals $J_k$ and $J$ are given by

$$\langle J'_k(u), v \rangle_k = \int_{-k}^{k} [c^2u'(s)v'(s) - c_0^2Au(s)Av(s) - f(Au(s))Av(s)] \, ds$$

$$\langle J'(u), v \rangle = \int_{-\infty}^{\infty} [c^2u'(s)v'(s) - c_0^2Au(s)Av(s) - f(Au(s))Av(s)] \, ds$$

for all $v \in X_k$ and $v \in X$, respectively. Here $\langle \cdot, \cdot \rangle_k$ (respectively, $\langle \cdot, \cdot \rangle$) stands for canonical duality between $X_k$ (respectively, $X$) and its dual space. The norms in the dual spaces $X^*_k$ and $X^*$ are denoted by $\| \cdot \|_{k,*}$ and $\| \cdot \|_*$, respectively. Since $U'$ is continuous and weak solutions are continuous, equation (??) implies that weak solutions are actually of class $C^2$ and, hence, classical solutions.
The proof of Existence Theorem is based on the following version of the Mountain Pass Theorem:

Let $E$ be a Banach space and $\varphi$ a $C^1$-functional on $E$ that satisfies the Palais-Smale condition

$\text{(PS)}$ every sequence $u_n \in E$ such that $\varphi(u_n)$ is bounded and $\varphi(u_n) \to 0$ contains a convergent subsequence.

Suppose that there exist $r > 0$ and $e \in E$ such that $\|e\| > r$ and

$$ b = \inf_{\|u\|=r} \varphi(u) > \varphi(0) > \varphi(e). \quad (4.2) $$

Let $P : E \to E$ be a continuous map such that $P(0) = 0$, $P(e) = e$ and $\varphi(P(u)) \leq \varphi(u)$ for all $u \in E$. Then the Mountain Pass Value defined by

$$ d = \inf_{C} \max_{t \in [0,1]} \varphi(\gamma(t)), $$

$$ C = \{ \gamma \in C([0,1]; E) : \gamma(0) = 0, \gamma(1) = e \} $$

is a critical value of the functional $\varphi$, and there exists a critical point $u \in \overline{P(E)}$ (the closure of $P(E)$) of $\varphi$ such that $d = \varphi(u)$. 
what happens if the wave speed \( c \) does not belong to the allowed interval 
\( (c_0, \sqrt{c_0 + a}) \). If \( c^2 \geq c_0^2 + a? \)

**Proposition**

Suppose that assumptions (i) and (ii) are satisfied, \( a > 0 \), and \( g(r)r < 0 \) for all \( r \neq 0 \). Then nontrivial traveling waves, both periodic and solitary, with wave speed \( c \geq \sqrt{c_0 + a} \) do not exist.

- The case of sub-sonic and sonic waves, when \( 0 \leq c \leq c_0 \), is not clear. Assuming (i)-(iii) and certain non-resonance condition, one can prove the existence of (non-monotone) periodic traveling waves.
- as in the superlinear case, we can not derive any uniform upper bound for such waves and, hence, pass to the limit to obtain a solitary wave.
Conjecture 1

Assume (i). If $0 < c \leq c_0$, then there exists no nontrivial solitary wave whose relative displacement profile $r(s) = Au(s)$ has exponential, or even certain negative power, rate of decay at infinity.

We consider $L$ as a self-adjoint operator in the space $L^2(\mathbb{R})$:

$$Lu(s) = -u''(s) + b(u(s + 1) + u(s - 1) - 2u(s)) = -u''(s) - b(A^*Au)(s),$$

where $b$ is a real constant and $A^*$ is the adjoint operator to $A$ in the sense of $L^2(\mathbb{R})$.

The spectrum $\sigma(L)$ of $L$ is absolutely continuous and forms a semi-axis. If $W(s)$ is a real function vanishing at infinity, then the essential spectrum of the operator $L - A^*WA$ coincides with $\sigma(L)$ and also $L - A^*WA$ may have eigenvalues.

Conjecture 2

If $W$ decays at infinity sufficiently fast, then the operator $L - A^*WA$ has no eigenvalues embedded into the essential spectrum.
Existence of periodic and Decaying Solutions in DNLS with saturable nonlinearity.

- The solutions decay exponentially at infinity.
- Variational Methods, Nehari Manifolds, Minimize of functionals.

Traveling Waves in Fermi-Pasta-Ulam Lattices with Saturable Nonlinearities

- Saturable Nonlinearities: U is asymptotically quadratic at infinity.
- Existence of super-sonic traveling waves when the wave speed belongs to a natural range.
- Existence of periodic waves with arbitrarily large periods.
- We obtain solitary waves passing to the limit as the period tends to infinity and making use of concentration compactness argument.
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References


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