Mean-Curvature Reconstruction with Linear Finite Elements

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Numerical Mathematics
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Heraklion, EFEF 2013

Setting of the problem

- reconstruction of a discrete curvature vector from p.w. linear data
- control of the reconstruction at least w.r.t. to the L^2-norm
- applicable to triangulated surfaces
- applicable to level-sets of p.w. linear functions

- applications can be problems with free capillary boundaries or interfaces, or geometric PDEs

Weak formulation

- let $\Gamma \subset \mathbb{R}^{d+1}$ a closed surface, smooth enough
- $X_\Gamma : \Gamma \to \mathbb{R}^{d+1} : \xi \mapsto \xi$ its coordinate field
- $\nu_\gamma : \Gamma \to S^d$ its outer unit normal field
- H_Γ the mean curvature of Γ, up to sign convention
- classical fact: $Y_\Gamma := H_\Gamma \, \nu_\Gamma = -\Delta_\Gamma X_\Gamma$
- weak formulation

$$\int_\Gamma Y_\Gamma \cdot \psi = \int_\Gamma \nabla_\Gamma X_\Gamma \cdot \nabla_\Gamma \psi = \int_\Gamma \nabla_\Gamma \cdot \psi$$

for suitable test functions ψ, ∇_Γ is the tangential gradient

One discretization attempt: L^2-projection

- let Γ_h be a p.w. linear approximation of Γ
  - triangulation, or discrete level set
- consider standard linear Lagrange F.E.-space $\mathcal{S}(\Gamma_h)$
- try to define a discrete curvature vector $Y_h \in \mathcal{S}(\Gamma_h)$ by

$$\int_\Gamma Y_h \cdot \psi_h = \int_\Gamma \nabla_{\Gamma_h} X_h \cdot \nabla_{\Gamma_h} \psi_h \quad \forall \psi_h \in \mathcal{S}(\Gamma_h)$$

Known issues

- Y_h converges as functional to Y_Γ in H^{-1}
- Y_h does not converge w.r.t. to the L^2-norm
- higher order discretizations of Γ yield convergence w.r.t. L^2

Regularized L^2-projection

- add a regularization term
- define a discrete curvature vector $Y_h^\varepsilon \in \mathcal{S}(\Gamma_h)$ by

$$\int_\Gamma Y_h^\varepsilon \cdot \psi_h + \varepsilon \int_\Gamma \nabla_{\Gamma_h} Y_h^\varepsilon \cdot \nabla_{\Gamma_h} \psi_h = \int_\Gamma \nabla_{\Gamma_h} X_h \cdot \nabla_{\Gamma_h} \psi_h \quad \forall \psi_h$$

- ε can be chosen such that Y_h^ϵ converges towards Y_Γ in the L^2 norm
  - provided that the geometry approximation Γ_h is good enough
- choose $\varepsilon = h^{2/3}$: control over L^2 error of order $h^{2/3}$
- even control over the gradient of Y_h^ε of order $h^{1/3}$

Planar model problem

- let $\Omega = (0,1)^d \subset \mathbb{R}^d$ with periodic boundaries
- $\{\mathcal{T}_h\}_{h>0}$ family of quasi-uniform admissible triangulations
- $u \in H^2(\Omega)$, $v := -\Delta u \in H^2(\Omega)$
- $u_h \in \mathcal{S}(\mathcal{T}_h)$ with $||\nabla(u - u_h)||_{L^2(\Omega)} \leq C_u\, h$
- consider the weak formulation for v

$$\int_\Omega v\,\psi = \int_\Omega \nabla u \cdot \nabla \psi \quad \forall \psi \in H^1(\Omega)$$

- consider the definition for v_h

$$\int_\Omega v_h\,\psi_h + \varepsilon \int_\Omega \nabla v_h \cdot \nabla \psi_h = \int_\Omega \nabla u_h \cdot \nabla \psi_h \quad \forall \psi_h \in \mathcal{S}(\mathcal{T}_h)$$

Error estimate via Galerkin orthogonality

- standard ansatz

$$
\begin{aligned}
\|v - v_h\|^2 + \varepsilon \|\nabla(v - v_h)\|^2 \\
&= (v - v_h,\, I_h v - v_h) + \varepsilon\,(\nabla(v - v_h),\, \nabla(I_h v - v_h)) \\
&\quad + (v - v_h,\, v - I_h v) + \varepsilon\,(\nabla(v - v_h),\, \nabla(v - I_h v)) \\
&=: E_1 + E_2.
\end{aligned}
$$

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- interpolation estimates

$$
|E_2| \le \frac{1}{4}\,\|v - v_h\|^2 + c\,h^4 + \frac{1}{4}\,\varepsilon\,\|\nabla(v - v_h)\|^2 + \varepsilon\,c\,h^2.
$$

Error estimate via Galerkin orthogonality

- standard ansatz

$$\begin{aligned}\|v - v_h\|^2 + \varepsilon \|\nabla(v - v_h)\|^2 &= (v - v_h,\, I_h v - v_h) + \varepsilon (\nabla(v - v_h),\, \nabla(I_h v - v_h)) \\ &\quad + \text{interpol.} \\ &=: E_1 + \text{interpol.}\end{aligned}$$

- plug in weak formulations

$$E_1 = (\nabla(u - u_h),\, \nabla(I_h v - v_h)) + \varepsilon(\nabla v,\, \nabla(I_h v - v_h))$$

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&= (\nabla(u - u_h),\, \nabla(v - v_h)) - \varepsilon(\Delta v,\, v - v_h) + \text{interpol.}
\end{aligned}
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- plug in weak formulations

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$$|E_1| = \frac{1}{\delta}\,c\,h^2 + \frac{\delta}{4}\|\nabla(v - v_h)\|^2 + \varepsilon^2\|\Delta v\|^2 + \frac{1}{4}\|v - v_h\|^2 + O(h^2).$$

Error estimate via Galerkin orthogonality

- finally

$$\frac{1}{2}\,\|v-v_h\|^2+\frac{3\,\varepsilon-\delta}{4}\,\|\nabla(v-v_h)\|^2 \le C_u\,\frac{h^2}{\delta}+\varepsilon^2\,\|\Delta v\|+O(h^2).$$

- choosing $\varepsilon = \eta\, h^{2/3}$ yields

$$\|v-v_h\|^2 + \eta\, h^{2/3}\,\|\nabla(v-v_h)\|^2 \le c(\eta)\, h^{4/3} + O(h^2).$$

- η does not alter the asymptotic order, but has a significant impact on the magitude of the error for “large” h
- mean curvature estimates follow by standard techniques, c.f. Dziuk '88 (triangulated surfaces) and Reusken et. al. '09

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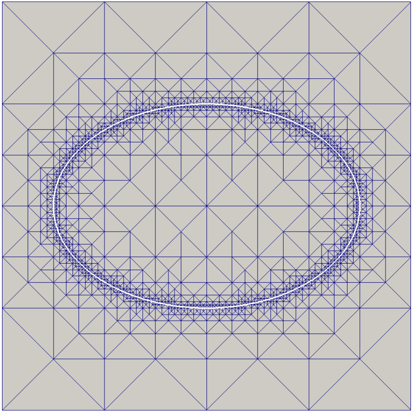
- elliptic arc with half-axes 1 and 1.5
- computation on discrete level-set

h_{max}	$\|Y - Y_h\|$	EOC
2.50e-01	2.29e-01	
1.25e-01	1.52e-01	0.59
6.25e-02	1.04e-01	0.56
3.13e-02	6.89e-02	0.59
1.56e-02	4.53e-02	0.61
7.81e-03	2.94e-02	0.62
3.91e-03	1.89e-02	0.64
1.95e-03	1.21e-02	0.65
9.77e-04	7.68e-03	0.65
4.88e-04	4.87e-03	0.66
2.44e-04	3.08e-03	0.66
1.22e-04	1.94e-03	0.66



- zero-level set of fourth-order polynomial $(x_1 - x_2^2)^2 + x_2^2 + x_3^2 - 1$
- computation on surface mesh

h_{max}	$\|Y - Y_h\|$	EOC
3.72e-01	4.55e+00	
1.94e-01	2.82e+00	0.73
1.02e-01	1.34e+00	1.16
5.19e-02	5.84e-01	1.22
2.61e-02	2.61e-01	1.17
1.31e-02	1.26e-01	1.05
6.53e-03	6.76e-02	0.89
3.26e-03	3.96e-02	0.77