Finite elements for immersed boundary method

Lucia Gastaldi

Joint work with: Daniele Boffi, Nicola Cavallini, Francesca Gardini, Pavia

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IBM – Immersed boundary method

Introduced by Peskin for the simulation of the blood flow in the heart.

<Peskin '72-'77>
<McQueen-Peskin '83->
<Peskin '02>

Successfully applied to many biological problems, where a fluid interacts with a flexible structure.

The main feature is that the structure is considered as a part of the fluid by introducing suitable additional forces and masses. The Navier–Stokes equations are solved in the whole domain (fluid + solid) by *finite differences* and the interaction with the structure is obtained by means of singular force and mass terms defined by a Dirac delta function localized in the solid domain.

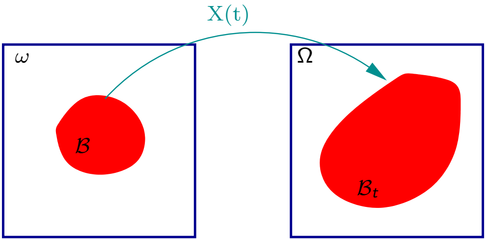
Finite elements for IBM

At the beginning, we used *finite elements* mainly because we thought this would simplify the mathematical analysis. Indeed, it turned out that this is a good choice also from the practical point of view.

<Boffi–G. '03>
<Boffi–G.–Heltai '04–'07>
<Boffi–G.–Heltai–Peskin '08>
<Boffi–Cavallini–G. '12>

- No need to approximating the Dirac delta functions, since the variational formulation takes care of it in a natural way
- Better interface approximation (less diffusion, sharp pressure jump)
- The fluid equations can be approximated with standard mixed schemes ($Q_2 - P_1$, Hood–Taylor, $P_1\mathrm{iso}P_2 - P_1^c$, ...)

Notation



Ω fluid + solid	$\mathcal{B}_t$ deformable structure domain
$\Omega \subset \mathbb{R}^d$, $d = 2, 3$	$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$
$\mathbf{x}$ Euler. var. in Ω	s Lagrangian var. in $\mathcal{B}$
	$\mathcal{B}$ reference domain
$\mathbf{u}(\mathbf{x}, t)$ fluid velocity	$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$p(\mathbf{x}, t)$ fluid pressure	$\mathbb{F} = \dfrac{\partial \mathbf{X}}{\partial s}$ deformation grad. $(\det \mathbb{F} > 0)$

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(s, t) \text{ where } \mathbf{x} = \mathbf{X}(s, t)$$

Model assumptions

From conservation of momenta, in absence of external forces, it holds

$$\rho\dot{\mathbf{u}} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\,\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} \quad \text{in } \Omega$$

In our case the Cauchy stress tensor has the following form

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s & \text{in } \mathcal{B}_t \end{cases}$$

- Incompressible fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f = -p\mathbb{I} + \mu(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^T)$
- Visco-elastic material: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s$ with $\boldsymbol{\sigma}_s$ elastic part of the stress

The structural material can have a density ρ_s different from the fluid density ρ_f

$$\rho = \begin{cases} \rho_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \rho_s & \text{in } \mathcal{B}_t \end{cases}$$

Variational formulation

- Navier–Stokes $\quad \mathbf{u}(t) \in H_0^1(\Omega)^d \;\; p \in L_0^2(\Omega)$

$$\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t)) = \langle \mathbf{d}(t), v\rangle + \langle \mathbf{F}^{\mathrm{FSI}}(t),\mathbf{v}\rangle \qquad \forall \mathbf{v}\in H_0^1(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

- Excess Lagrangian mass density

$$\langle \mathbf{d}(t), v\rangle = -\delta\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2}\mathbf{v}(\mathbf{X}(s,t))ds \qquad \delta\rho = \rho_s - \rho_f$$

- Load

$$\langle \mathbf{F}^{\mathrm{FSI}}(t),\mathbf{v}\rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\,ds$$

- Body motion

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

- Initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \;\forall \mathbf{x}\in\Omega, \quad \mathbf{X}(s,0) = \mathbf{X}_0(s) \;\forall s \in \mathcal{B}.$$

Stability

<Boffi–Cavallini–G. '10>

Recalling that

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

it holds

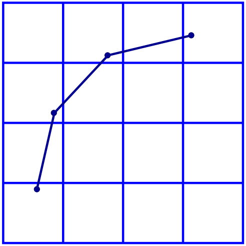
$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\nabla \mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}\delta\rho\frac{d}{dt}\left\|\frac{\partial \mathbf{X}}{\partial t}\right\|_B^2 = 0$$

$E(\mathbf{X}(t)) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))\,ds$ total elastic potential energy

W potential energy density related with $\mathbb{P}$ by $\mathbb{P}(\mathbb{F}) = \dfrac{\partial W}{\partial \mathbb{F}}$

Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair



$$V_h \subseteq H^1_0(\Omega)^d, \quad Q_h \subseteq L^2_0(\Omega)$$

- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- Piecewise linear finite element space for $\mathbf{X}$
  $S_h = \{\mathbf{Y} \in C^0(\mathcal{B};\Omega) : \mathbf{Y} \in P1\}$

Notation

- $T_k \;\; k = 1, \dots, M_e$ elements of $\mathcal{S}_h$
- $\mathbf{s}_j$, $j = 1, \dots, M$ vertices of $\mathcal{S}_h$
- $\mathcal{E}_h$ set of the edges e of $\mathcal{S}_h$
- $[\![\mathbb{P}]\!] = \mathbb{P}^+\mathbf{N}^+ + \mathbb{P}^-\mathbf{N}^-$ jump of $\mathbb{P}$ across e for internal edges

Fully discrete problem

Backward Euler – **BE**

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ e $\mathbf{X}_h^{n+1} \in S_h$ such that

$$\langle \mathbf{F}_h^{n+1}, \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^{n+1} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s)) dA \qquad \forall \mathbf{v} \in V_h$$

$$\textbf{NS} \begin{cases} \rho_f \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ \quad -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\ \quad -\delta\rho \displaystyle\int_{\mathcal{B}} \dfrac{\mathbf{X}_h^{n+1} - 2\mathbf{X}_h^n + \mathbf{X}_h^{n-1}}{\Delta t^2} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s)) ds \\ \qquad + < \mathbf{F}_h^{n+1}, \mathbf{v} >_h \qquad \forall \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h; \end{cases}$$

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n+1}) \quad \forall i = 1, \dots, M.$$

Fully discrete problem

Modified backward Euler

Step 1. $\langle \mathbf{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\, dA \qquad \forall \mathbf{v}\in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$\mathbf{NS}\begin{cases} \rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}, \mathbf{v}\right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ \quad -(\operatorname{div}\mathbf{v}, p_h^{n+1}) = \\ \quad -\delta\rho\displaystyle\int_{\mathcal{B}} \frac{\mathbf{u}_h^{n+1}(\mathbf{X}_h^n(s)) - \mathbf{u}_h^n(\mathbf{X}_h^{n-1}(s))}{\Delta t}\cdot \mathbf{v}(\mathbf{X}_h^n(s))ds \\ \qquad + \langle \mathbf{F}_h^n, \mathbf{v}\rangle_h \qquad \forall \mathbf{v}\in V_h \\ (\operatorname{div}\mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q\in Q_h; \end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1}-\mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1,\dots,M.$

Discrete Energy Estimate
<Boffi–Cavallini–G. '11>

Convexity of the potential energy density

W has bounded second der. $\kappa_{min}|\mathbb{D}|^2 \leq \mathbb{D} : \mathbb{H} : \mathbb{D} \leq \kappa_{max}|\mathbb{D}|^2$

Artificial Viscosity Theorem

Let $\mathbf{u}_h^n$, p_h^n and $\mathbf{X}_h^n$ be a solution to the FE-IBM, then

$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + (\mu + \mu_a)\|\nabla \mathbf{u}_h^{n+1}\|_0^2 + \frac{1}{\Delta t}\left(E\left[\mathbf{X}_h^{n+1}\right] - E\left[\mathbf{X}_h^n\right]\right) + \frac{1}{2\Delta t}(\rho_s - \rho_f)\left(\|\mathbf{u}_h^{n+1}(\mathbf{X}_h^n)\|_{0,\mathcal{B}}^2 - \|\mathbf{u}_h^n(\mathbf{X}_h^{n-1}\|_{0,\mathcal{B}}^2\right) \leq 0$$

CFL Conditions: $\mu + \mu_a \geq 0$, $\rho_s \geq \rho_f$ (might be relaxed)

CFL condition

BE (implicit) is unconditionally stable, while MBE (semi-implicit) requires the term μ_a to be not too large

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^n$$

$$L^n := \max_{T_k \in S_h} \left\{ \max_{\mathbf{s}_j, \mathbf{s}_i \in V(T_k)} |\mathbf{X}_{hj}^n - \mathbf{X}_{hi}^n| \right\}$$

space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \leq C h_x h_s$
2	2	$L^n \Delta t \leq C h_x$
3	2	$L^n \Delta t \leq C h_x^2$
3	3	$L^n \Delta t \leq C h_x^2 / h_s$

Fully variational approach

Body motion

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

We introduce a variational formulation:

$$\left\langle \mu, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}(t)}{\partial t} \right\rangle_{\mathcal{B}} = 0 \quad \forall \mu \in \Lambda$$

where Λ is the dual space of $H^1(\mathcal{B})^d$.

Fully variational approach

Body motion

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

We introduce a variational formulation:

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where Λ is the dual space of $H^1(\mathcal{B})^d$.

This equation plays the role of a constraint on the difference between the velocities in the fluid and in the solid, hence we introduce the Lagrange multiplier associated to it.

The IBM as a DLM/FD method

<Boffi–G., in progress>

For $t \in [0,T]$, find $\mathbf{u}(t) \in H^1_0(\Omega)^d$, $p(t) \in L^2_0(\Omega)$, $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$, and $\lambda(t) \in \Lambda$ such that

$$\rho \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t)) + \langle \lambda, \mathbf{v}(\mathbf{X}(\cdot,t))\rangle_{\mathcal{B}} = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

$$\delta\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2}\mathbf{Y}\,ds + \int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(s,t)) : \nabla_s \mathbf{Y}\,ds - \langle \lambda, \mathbf{Y}\rangle_{\mathcal{B}} = 0 \qquad \forall \mathbf{Y} \in H^1(\mathcal{B})^d$$

$$\left\langle \mu, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}(t)}{\partial t} \right\rangle_{\mathcal{B}} = 0 \qquad \forall \mu \in \Lambda$$

Time advancing scheme

Modified backward Euler

Given $\mathbf{u}_0 \in H^1_0(\Omega)^d$ and $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, for $n = 1, \dots, N$ find $\mathbf{u}^n, p^n \in H^1_0(\Omega)^d \times L^2_0(\Omega)$, $\mathbf{X}^n \in W^{1,\infty}(\mathcal{B})^d$, and $\lambda^n \in \Lambda$, such that

$$\rho_f\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t},\mathbf{v}\right) + a(\mathbf{u}^{n+1},\mathbf{v}) - (\operatorname{div}\mathbf{v}, p^{n+1}) + \langle\lambda^{n+1}, v(\mathbf{X}^n(\cdot))\rangle_{\mathcal{B}} = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}^{n+1}, q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

$$\delta\rho\left(\frac{\mathbf{X}^{n+1}-2\mathbf{X}^n+\mathbf{X}^{n-1}}{\Delta t^2},\mathbf{Y}\right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}^{n+1}), \nabla_s\mathbf{Y})_{\mathcal{B}} - \langle\lambda^{n+1},\mathbf{Y}\rangle_{\mathcal{B}} = 0 \qquad \forall \mathbf{Y} \in H^1(\mathcal{B})$$

$$\left\langle \mu, \mathbf{u}^{n+1}(\mathbf{X}^n(\cdot)) - \frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right\rangle_{\mathcal{B}} = 0 \qquad \forall \mu \in \Lambda.$$

Energy estimate for the time discrete problem

Proposition Unconditional stability

Assume that W is convex and $\delta\rho = \rho_s - \rho_f > 0$.
The following estimate holds true for all $n = 1, \dots, N$

$$\frac{\rho_f}{2\Delta t}\left(\|u^{n+1}\|_0^2 - \|u^n\|_0^2\right) + \mu\|\nabla \mathbf{u}^{n+1}\|_0^2$$
$$+ \frac{\delta\rho}{2\Delta t}\left(\left\|\frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right\|_{0,\mathcal{B}}^2 - \left\|\frac{\mathbf{X}^{n}-\mathbf{X}^{n-1}}{\Delta t}\right\|_{0,\mathcal{B}}^2\right)$$
$$+ \frac{1}{\Delta t}(E(\mathbf{X}^{n+1}) - E(\mathbf{X}^n)) \le 0$$

IBM versus DLM-IBM

Time advancing scheme for IBM formulation

- Compute F^{FSI} using $\mathbf{X}^n$
- Solve at time t^{n+1}

$$\begin{pmatrix} A_{\mathrm{NS}}(\mathbf{u}^n) & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} F^{\mathrm{FSI}} \\ 0 \end{pmatrix}$$

- Update pointwise the structure by

$$\frac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t}(s) = \mathbf{u}^{n+1}(\mathbf{X}^n(s))$$

Time advancing scheme for DLM-IBM formulation

$$\left(\begin{array}{cc|c|c} A_{\mathrm{NS}}(\mathbf{u}^n) & B^\top & 0 & L_f^\top(\mathbf{X}^n) \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & A_s & -L_s^\top \\ \hline L_f(\mathbf{X}^n) & 0 & -L_s & 0 \end{array}\right) \left(\begin{array}{c} \mathbf{u} \\ p \\ \hline \mathbf{X} \\ \hline \lambda \end{array}\right) = \left(\begin{array}{c} \mathbf{f}(\mathbf{u}^n) \\ 0 \\ \hline \mathbf{g}(\mathbf{X}^n) \\ \hline \mathbf{d}(\mathbf{X}^n) \end{array}\right)$$

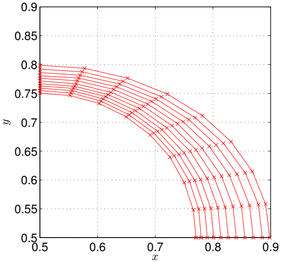
Ellipse immersed in a static fluid

$\mathbb{P} = \kappa \mathbb{F}$

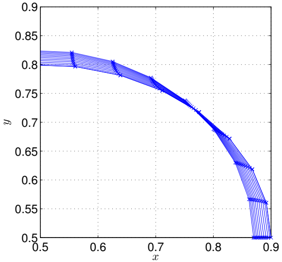
Fluid initially at rest: $\mathbf{u}_{0h} = 0$

$$\mathbf{X}_0(s) = \begin{pmatrix} 0.2\cos(2\pi s) + 0.45 \\ 0.1\sin(2\pi s) + 0.45 \end{pmatrix} \quad s \in [0,1],$$

$h_x = 1/32$, $h_s = 1/32$, $\Delta t = 10^{-2}$, $\mu = 1$, $\kappa = 5$



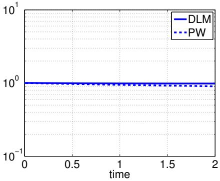
Standard IBM with PW update of the immersed boundary



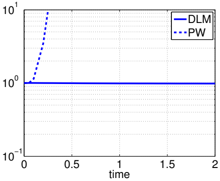
IBM with DLM

Stability analysis

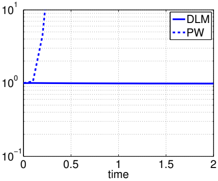
$\rho = 1,\ \mu = 1,\ \kappa = 5,\ h_x = 1/64,\ h_s = 1/128$



$\Delta t = 0.01$

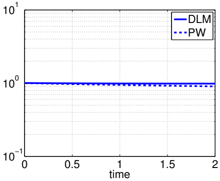


$\Delta t = 0.05$

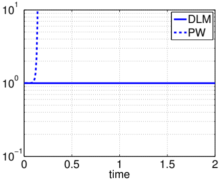


$\Delta t = 0.1$

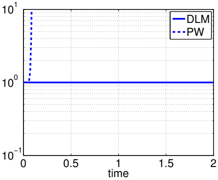
$\rho = 1,\ \mu = 1,\ \kappa = 5,\ h_x = 1/64,\ \Delta t = 0.01$



$h_s = 1/128$



$h_s = 1/256$



$h_s = 1/512$

Saddle point problem

Let $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ be invertible with Lipschitz inverse. Given $\mathbf{f} \in L^2(\Omega)^d$, $\mathbf{g} \in L^2(\mathcal{B})^d$ and $\mathbf{d} \in L^2(\mathcal{B})^d$, find $\mathbf{u} \in H^1_0(\Omega)$, $p \in L^2_0(\Omega)$, $\mathbf{X} \in H^1(\mathcal{B})$ and $\lambda \in (H^1(\mathcal{B}))^*$ such that

$$
\begin{aligned}
& a_f(\mathbf{u},\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) + \langle \lambda, \mathbf{v}(\overline{\mathbf{X}}(\cdot))\rangle_{\mathcal{B}} = (\mathbf{f},\mathbf{v}) && \forall \mathbf{v} \in H^1_0(\Omega)\\
& (\operatorname{div}\mathbf{u}, q) = 0 && \forall q \in L^2_0(\Omega)\\
& a_s(\mathbf{X},\mathbf{Y}) - \langle \lambda, \mathbf{Y}\rangle_{\mathcal{B}} = (\mathbf{g},\mathbf{Y})_{\mathcal{B}} && \forall \mathbf{Y} \in H^1(\mathcal{B})\\
& \langle \mu, \mathbf{u}(\overline{\mathbf{X}}(\cdot)) - \mathbf{X}\rangle_{\mathcal{B}} = \langle \mu, \mathbf{d}\rangle_{\mathcal{B}} && \forall \mu \in (H^1(\mathcal{B}))^*.
\end{aligned}
$$

where

$$
\begin{aligned}
& a_f(\mathbf{u},\mathbf{v}) = \alpha(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) && \forall \mathbf{u},\mathbf{v} \in H^1_0(\Omega)\\
& a_s(\mathbf{X},\mathbf{Y}) = \beta(\mathbf{X},\mathbf{Y}) + \gamma(\mathbb{P}(\nabla_s \mathbf{X}), \nabla_s \mathbf{Y})_{\mathcal{B}} && \forall \mathbf{X},\mathbf{Y} \in H^1(\mathcal{B})
\end{aligned}
$$

Well-posedness of the saddle point problem

We set

$$\mathbb{V} = H^1_0(\Omega)^d \times L^2_0(\Omega) \times H^1(\mathcal{B})$$

$$\mathbb{K} = \left\{(\mathbf{v}, q, \mathbf{Y}) \in \mathbb{V} : \ \langle \mu, v(\overline{\mathbf{X}}(\cdot)) - \mathbf{Y}\rangle_{\mathcal{B}} = 0 \ \forall \mu \in (H^1(\mathcal{B}))^*\right\}$$

Ellipticity on the kernel

There exists $\alpha_0 > 0$ such that

$$\inf_{(\mathbf{u},p,\mathbf{X})\in\mathbb{K}} \sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{K}} \frac{a_f(\mathbf{u},\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (\operatorname{div}\mathbf{u}, q) + a_s(\mathbf{X},\mathbf{Y})}{\|(\mathbf{u},p,\mathbf{X})\|_{\mathbb{V}} \|(\mathbf{v},q,\mathbf{Y})\|_{\mathbb{V}}} \ge \alpha_0.$$

Inf-sup condition

There exists $\beta_0 > 0$ such that

$$\sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{V}} \frac{\langle \mu, v(\overline{\mathbf{X}}(\cdot)) - \mathbf{Y}\rangle_{\mathcal{B}}}{\|(\mathbf{v},q,\mathbf{Y})\|_{\mathbb{V}}} \ge \beta \|\mu\|_{(H^1(\mathcal{B}))^*}$$

Finite element discretization

We consider

- $(V_h, Q_h) \subseteq H^1_0(\Omega)^d \times L^2_0(\Omega)$ stable pair for the Stokes equations;
- $S_h \subseteq H^1(\mathcal{B})$ continuous Lagrange elements;
- $\Lambda_h \subseteq (H^1(\mathcal{B}))^*$ continuous Lagrange elements (e.g. $\Lambda_h = S_h$)

Discrete saddle point problem

Find $\mathbf{u}_h \in V_h$, $p_h \in Q_h$, $\mathbf{X}_h \in S_h$ and $\lambda_h \in \Lambda_h$ such that

$$
\begin{aligned}
& a_f(\mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) + \langle \lambda_h, \mathbf{v}(\overline{\mathbf{X}}(\cdot)) \rangle_{\mathcal{B}} = (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in V_h \\
& (\operatorname{div} \mathbf{u}_h, q) = 0 && \forall q \in Q_h \\
& a_s(\mathbf{X}_h, \mathbf{Y}) - \langle \lambda_h, \mathbf{Y} \rangle_{\mathcal{B}} = (\mathbf{g}, \mathbf{Y})_{\mathcal{B}} && \forall \mathbf{Y} \in S_h \\
& \langle \mu, \mathbf{u}_h(\overline{\mathbf{X}}(\cdot)) - \mathbf{X}_h \rangle_{\mathcal{B}} = \langle \mu, \mathbf{d} \rangle_{\mathcal{B}} && \forall \mu \in \Lambda_h.
\end{aligned}
$$

Analysis of the discrete problem

$$\mathbb{V}_h = V_h \times Q_h \times S_h$$

$$\mathbb{K}_h = \left\{(\mathbf{v}, q, \mathbf{Y}) \in \mathbb{V}_h : \; \langle \mu, v(\overline{\mathbf{X}}(\cdot)) - \mathbf{Y}\rangle_{\mathcal{B}} = 0 \; \forall \mu \in \Lambda_h \right\}$$

Ellipticity on the kernel

There exists $\alpha_1 > 0$ independent of h such that

$$\inf_{(\mathbf{u},p,\mathbf{X})\in\mathbb{K}_h} \sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{K}_h} \frac{a_f(\mathbf{u},\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (\operatorname{div}\mathbf{u}, q) + a_s(\mathbf{X},\mathbf{Y})}{\|(\mathbf{u},p,\mathbf{X})\|_{\mathbb{V}} \|(\mathbf{v},q,\mathbf{Y})\|_{\mathbb{V}}} \geq \alpha_1.$$

Inf-sup condition

Assume that the mesh sequence is quasi-uniform, then there exists $\beta_1 > 0$ independent of h such that

$$\sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{V}} \frac{\langle \mu, v(\overline{\mathbf{X}}(\cdot)) - \mathbf{Y}\rangle_{\mathcal{B}}}{\|(\mathbf{v},q,\mathbf{Y})\|_{\mathbb{V}}} \geq \beta\|\mu\|_{(H^1(\mathcal{B}))^*}$$