

An Instant Optimal Adaptive Finite Element Method

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joint work with L. Diening and R. Stevenson

European Finite Element Fair 2013, Heraklion, Crete

Outline

Framework

AFEM and Main Result

Proof of the Main Result

Framework

Poisson Problem:

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. For $f \in L^2(\Omega)$ find $u \in H^1_0(\Omega)$, such that

$$-\Delta u = f \qquad \text{in } H^{-1}(\Omega) \quad \text{and} \quad u = 0 \text{ on } \partial \Omega.$$

$$\mathcal{J}(u) := \int_{\Omega} \frac{1}{2} |\nabla u|^2 - fu \, \mathrm{d}x = \min \left\{ \mathcal{J}(v) \colon v \in H_0^1(\Omega) \right\}.$$

Finite Elements and Ritz Approximation

Continuous piece wise affine finite elements

$$\mathbb{V}(\mathcal{T}) := \{ V \in H_0^1(\Omega) \mid V_{|T} \in \mathbb{P}_1(T), \ T \in \mathcal{T} \}.$$

over conforming triangulation \mathcal{T} of Ω with nodes \mathcal{N} and edges \mathcal{S} .

$$U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}): \quad \int_{\mathcal{O}} \nabla U_{\mathcal{T}} \cdot \nabla V \, \mathrm{d}x = \int_{\mathcal{O}} fV \, \mathrm{d}x \quad \text{for all } V \in \mathbb{V}(\mathcal{T}).$$

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Error Estimation

Edge Based Residual Indicators

$$\text{For } S \in \mathcal{S} \text{ define } \qquad \mathcal{E}^2_{\mathcal{T}}(S) := \int_S h_S \left| [\![\nabla U \tau]\!] \right|^2 \ \mathrm{d}s + \sum_{\partial T \supset S} \int_T h_T^2 f^2 \, \mathrm{d}x$$

Error Bounds

$$|||u - U_{\mathcal{T}}|||_{\Omega}^{2} \lesssim \mathcal{E}_{\mathcal{T}}^{2}(\mathcal{S}) := \sum_{S \in \mathcal{S}} \mathcal{E}_{\mathcal{T}}^{2}(S) \lesssim |||u - U_{\mathcal{T}}|||_{\Omega}^{2} + \operatorname{osc}^{2}(\mathcal{T})$$

Discrete Error Bounds

Let $\mathcal{T}_* > \mathcal{T}$ be a refinement of \mathcal{T} . Then

$$|||U_{\mathcal{T}} - U_{\mathcal{T}_*}|||_{\Omega}^2 \lesssim \mathcal{E}_{\mathcal{T}}^2(\mathcal{S} \setminus \mathcal{S}_*) \lesssim |||U_{\mathcal{T}} - U_{\mathcal{T}_*}|||_{\Omega}^2 + \sum_{T \in \mathcal{T} \setminus \mathcal{T}_*} \int_T h_T^2 f^2 \, \mathrm{d}x$$

Christian Kreuzer An Instant Optimal AFEM

Refinement Framework

Admissible Triangulations

- ▶ Initial conforming Refinement \mathcal{T}_0 of Ω with refinement edges labeled as in [Binev, Dahmen, DeVore '04] or [Stevenson, 08].
- $ightharpoonup \mathbb{T} := \{ \mathcal{T} : \text{ conforming refinement of } \mathcal{T}_0 \text{ using NVB} \}$
- ▶ Refinement Routine: Let $\mathcal{T} \in \mathbb{T}$ with edges \mathcal{S} then, for $S \in \mathcal{S}$, let

$$\mathcal{T} \leq \mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, S) \in \mathbb{T},$$

be the **smallest** refinement of \mathcal{T} s.t. $S \notin \mathcal{S}_*$.

Complexity of REFINE [BDD '04], [Stevenson, 08]

Let $\mathcal{T}_0 = \mathcal{T}_0 < \mathcal{T}_1 \leq \mathcal{T}_2 \leq \ldots \subset \mathbb{T}$ such that for some $\mathcal{M}_k \subseteq \mathcal{S}_k$ we have

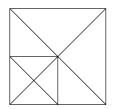
$$\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k) \quad \Rightarrow \quad \#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{r \in I_k} \#\mathcal{M}_n$$

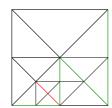
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For
$$S \in \mathcal{S} = \mathcal{S}(\mathcal{T}), \mathcal{T} \in \mathbb{T}$$
, define $\operatorname{refined}(\mathcal{T}, S) := \mathcal{S} \setminus \operatorname{REFINE}(\mathcal{T}, S)$.

Accumulated Error Indicators

$$\bar{\mathcal{E}}_{\mathcal{T}}^2(S) := \mathcal{E}_{\mathcal{T}}^2(\mathsf{refined}(\mathcal{T},\,S)) \ \Rightarrow \ \| U_{\mathcal{T}} - U_{\mathcal{T}_*} \|_{\Omega}^2 + \sum_{T \in \mathcal{T} \backslash \mathcal{T}_*} \int_T h_T^2 f^2 \, \mathrm{d}x \simeq \bar{\mathcal{E}}_{\mathcal{T}}^2(S)$$







AFEM and Main Result

The Adaptive Finite Element Method (AFEM)

Choose $\mu \in (0,1]$ and set k=0.

SOLVE: compute $U_k \in \mathbb{V}(\mathcal{T}_k)$;

ESTIMATE: compute $\bar{\mathcal{E}}_{\max}^2(k) := \max \{\mathcal{E}_k^2(\mathsf{refined}(\mathcal{T}_k, S)) : S \in \mathcal{S}_k\};$

MARK: $\mathcal{M}_k := \emptyset$, $\mathcal{C}_k := \mathcal{S}(\mathcal{T}_k)$, $\widetilde{\mathcal{M}}_k := \emptyset$;

while $C_k \neq \emptyset$ do

select $S \in \mathcal{C}_k$;

if $\mathcal{E}_k^2(\mathsf{refined}(\mathcal{T}_k,\,S)\setminus\widetilde{\mathcal{M}}_k)\geq\mu\,\bar{\mathcal{E}}_{\max}^2(k);$

then $\mathcal{M}_k := \mathcal{M}_k \cup \{S\};$

 $\widetilde{\mathcal{M}}_k := \widetilde{\mathcal{M}}_k \cup \mathsf{refined}(\mathcal{T}_k, S);$

end if;

 $C_k := C_k \setminus \mathsf{refined}(\mathcal{T}_k, S);$

end while;

REFINE: compute $\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$ and increment k.

The Main Result: Instant Optimality

Theorem [Diening, K., Stevenson 2013]

There exist constants $C=C(\mathcal{T}_0), \ \tilde{C}=\tilde{C}(\mathcal{T}_0,\mu)$, such that for $k,m\in\mathbb{N}$ with

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \ge \tilde{C} m$$
,

we have

$$C\left(\left\|u - U_{\mathcal{T}}\right\|_{\Omega}^{2} + \operatorname{osc}^{2}(\mathcal{T})\right) \ge \left\|u - U_{\mathcal{T}_{k}}\right\|_{\Omega}^{2} + \operatorname{osc}^{2}(\mathcal{T}_{k})$$

for all $\mathcal{T} \in \mathbb{T}$ with $\#\mathcal{T} - \#\mathcal{T}_0 < m$.

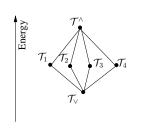
Outline

Framework

AFFM and Main Result

Proof of the Main Result

Proof: Lower Diamond Estimate



Minimal Diamond

A set $(\mathcal{T}^{\wedge}, \mathcal{T}_{\vee}; \mathcal{T}^{1}, \dots, \mathcal{T}^{m})$ is called *minimal diamond*, if

- $lackbox{}{\mathcal{T}}^\wedge$ is the finest coarsening and
- $ightharpoonup \mathcal{T}_{\lor}$ is the coarsest refinement of $\mathcal{T}^1, \ldots, \mathcal{T}^m$.

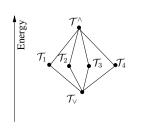
Lower Diamond Estimate

Let $(\mathcal{T}^{\wedge}, \mathcal{T}_{\vee}; \mathcal{T}^{1}, \dots, \mathcal{T}^{m})$ be a minimal diamond, such that the sets $\Omega(\mathcal{T}^{j} \setminus \mathcal{T}_{\vee}) := \operatorname{interior} \bigcup \{T \in \mathcal{T}^{j} \setminus \mathcal{T}_{\vee}\}$ are pairwise disjoint. Then

$$\mathcal{J}(U_{\wedge}) - \mathcal{J}(U_{\vee}) = \frac{1}{2} |U_{\wedge} - U_{\vee}|_{H^{1}(\Omega)}^{2}$$
$$\simeq \frac{1}{2} \sum_{i=1}^{m} |U_{i} - U_{\vee}|_{H^{1}(\Omega_{j})}^{2} = \sum_{i=1}^{m} \mathcal{J}(U_{j}) - \mathcal{J}(U_{\vee})$$

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Proof: Total Energy

$$\text{For } \mathcal{T} \in \mathbb{T} \text{, we set:} \qquad \mathcal{G}(\mathcal{T}) := \mathcal{J}(U_{\mathcal{T}}) + \sum_{T \in \mathcal{T}} h_T^2 \|f\|_{L^2(T)}^2.$$

$$\mathcal{G}(\mathcal{T}) - \mathcal{J}(u) \simeq |U_{\mathcal{T}} - u|_{H^1(\Omega)}^2 + \operatorname{osc}^2(\mathcal{T})$$

$$\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_*) \simeq \mathcal{E}_{\mathcal{T}}^2(\mathcal{S} \setminus \mathcal{S}_*).$$

$$\mathcal{T}_m^{\mathrm{opt}} \in \mathbb{T}: \quad \mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) = \min \left\{ \mathcal{G}(\mathcal{T}) \colon \mathcal{T} \in \mathbb{T} \text{ mit } \#\mathcal{T} - \#\mathcal{T}_0 \le m \right\}$$

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Properties

- $\mathcal{G}(\mathcal{T}) \mathcal{J}(u) \simeq |U_{\mathcal{T}} u|_{H^1(\Omega)}^2 + \operatorname{osc}^2(\mathcal{T}).$
- ▶ For \mathcal{T}_* being a refinement of \mathcal{T} , we have

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▶ The energy $\mathcal{G}: \mathbb{T} \to \mathbb{R}$ satisfies the lower diamond estimate.

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For $m \in \mathbb{N}$, we define

$$\mathcal{T}_m^{\mathrm{opt}} \in \mathbb{T} : \quad \mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) = \min \left\{ \mathcal{G}(\mathcal{T}) \colon \mathcal{T} \in \mathbb{T} \text{ mit } \#\mathcal{T} - \#\mathcal{T}_0 \le m \right\}$$

Proof: How to choose \mathcal{C} and \mathcal{U} ?

Population

For a triangulation \mathcal{T} , we define the corresponding *population* as $\mathcal{P} := \mathcal{N}(\mathcal{T})$.

$$gen(child) = gen(parent) + 1.$$

$$\sup_{\mathcal{D}} \sup_{P \in \mathcal{D}} \sup_{k \in \mathbb{N}} \# \big(\operatorname{ancestors}(P) \cap \operatorname{gen}^{-1}(k) \big) =: c_{\mathrm{GD}} < \infty$$



An Instant Optimal AFEM

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Populations have a tree structure. Thanks to the initial

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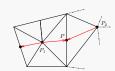
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An Instant Optimal AFEM

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Proof of the Main Result

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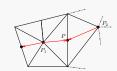
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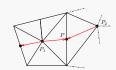
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Limited Genetic Diversity

$$\sup_{\mathcal{P}} \sup_{P \in \mathcal{P}} \sup_{k \in \mathbb{N}} \# \left(\operatorname{ancestors}(P) \cap \operatorname{gen}^{-1}(k) \right) =: c_{\mathrm{GD}} < \infty.$$



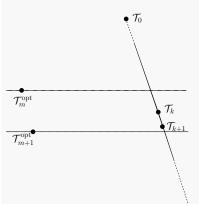
$$\mathcal{G}(\mathcal{T}_{k}) - \mathcal{G}(\mathcal{T}_{k+1}) \simeq \bar{\mathcal{E}}_{\max}^{2}(k)
\geq \frac{1}{\#\mathcal{U}} \sum_{\mathcal{T} \in \mathcal{U}} \mathcal{E}_{k}^{2} (\mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_{k})
\geq \frac{1}{\#\mathcal{U}} (\mathcal{G}(\mathcal{T}_{k}) - \mathcal{G}(\mathcal{T}_{\vee}))
\geq \frac{1}{\#\mathcal{U}} (\mathcal{G}(\mathcal{T}^{\wedge}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}))
\geq \frac{1}{\#\mathcal{U}} \sum_{\mathcal{T} \in \mathcal{C}} (\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}))
\geq \frac{\#\mathcal{C}}{\#\mathcal{U}} (\mathcal{G}(\mathcal{T}^{\text{opt}}) - \mathcal{G}(\mathcal{T}^{\text{opt}}))$$

Proof: Main Idea of Optimality

Assume for simplicity: In each iteration, AFEM marks only one largest element.

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})$.

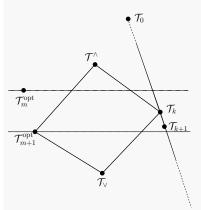


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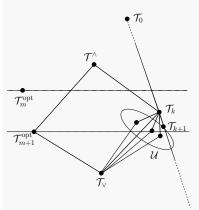
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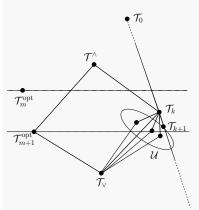
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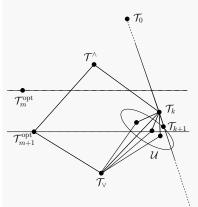


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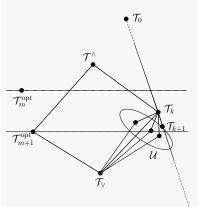
For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.



$$egin{aligned} \mathcal{G}(\mathcal{T}_k) &- \mathcal{G}(\mathcal{T}_{k+1}) \simeq ar{\mathcal{E}}_{ ext{max}}^2(k) \ &\geq rac{1}{\#\mathcal{U}} \, \mathcal{E}_k^2ig(igcup_{ au \in \mathcal{U}} \mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_kig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{\lor})ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}^{\wedge}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}})ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, \sum_{\mathcal{T} \in \mathcal{C}} ig(\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}})ig) \ &\gtrsim rac{\#\mathcal{C}}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}})ig) \end{aligned}$$

Induction:

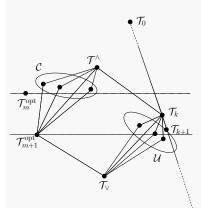
For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.



$$egin{aligned} \mathcal{G}(\mathcal{T}_k) &- \mathcal{G}(\mathcal{T}_{k+1}) \simeq ar{\mathcal{E}}_{ ext{max}}^2(k) \ &\geq rac{1}{\#\mathcal{U}} \, \mathcal{E}_k^2 ig(igcup_{\mathcal{T} \in \mathcal{U}} \, \mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_k ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{\lor}) ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}^{\wedge}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, \sum_{\mathcal{T} \in \mathcal{C}} ig(\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) ig) \ &\gtrsim rac{\#\mathcal{C}}{\mathbb{Z}_{m+1}} \, ig(\mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) ig) \end{aligned}$$

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})$.



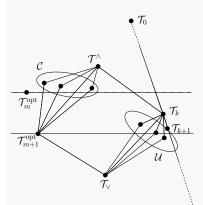
$$egin{aligned} \mathcal{G}(\mathcal{T}_k) &- \mathcal{G}(\mathcal{T}_{k+1}) \simeq ar{\mathcal{E}}_{ ext{max}}^2(k) \ &\geq rac{1}{\#\mathcal{U}} \, \mathcal{E}_k^2 ig(igcup_{\mathcal{T} \in \mathcal{U}} \mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_k ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{\lor}) ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, ig(\mathcal{G}(\mathcal{T}^{\land}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) ig) \ &\gtrsim rac{1}{\#\mathcal{U}} \, \sum_{\mathcal{T} \in \mathcal{C}} ig(\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) ig) \ &\geq rac{\#\mathcal{C}}{\mathbb{C}} \, ig(\mathcal{G}(\mathcal{T}_{opt}) - \mathcal{G}(\mathcal{T}_{opt}^{ ext{opt}}) ig) \end{aligned}$$

Proof of the Main Result



Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_m^{\mathrm{opt}})$.



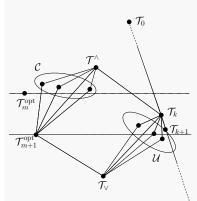
$$\begin{split} \mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq \bar{\mathcal{E}}_{\max}^2(k) \\ &\geq \frac{1}{\#\mathcal{U}} \, \mathcal{E}_k^2 \big(\bigcup_{\mathcal{T} \in \mathcal{U}} \mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_k \big) \\ &\gtrsim \frac{1}{\#\mathcal{U}} \, \big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{\vee}) \big) \\ &\gtrsim \frac{1}{\#\mathcal{U}} \, \big(\mathcal{G}(\mathcal{T}^{\wedge}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \big) \\ &\gtrsim \frac{1}{\#\mathcal{U}} \, \sum_{\mathcal{T} \in \mathcal{C}} \big(\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \big) \\ &\gtrsim \frac{\#\mathcal{C}}{\#\mathcal{U}} \, \big(\mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \big) \end{split}$$

Proof: Main Idea of Optimality

Assume for simplicity: In each iteration, AFEM marks only one largest element.

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.



$$egin{aligned} \mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq ar{\mathcal{E}}_{ ext{max}}^2(k) \ &\gtrsim rac{\#\mathcal{C}}{\#\mathcal{U}} \left(\mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}})
ight) \ &\geq \mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) \end{aligned}$$

Proper choice of ${\cal U}$ and ${\cal C}\colon \ \ \dfrac{\#{\cal C}}{\#{\cal U}} \geq c_{
m GD}.$

Induction yields $C \in \mathbb{N}$, such that

$$\mathcal{G}(\mathcal{T}_{Cm}) - \mathcal{J}(u) \le \mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{J}(u)$$

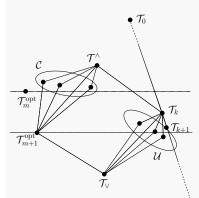
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Proof: Main Idea of Optimality

Assume for simplicity: In each iteration, AFEM marks only one largest element.

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \ge \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.



$$egin{aligned} \mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq ar{\mathcal{E}}_{ ext{max}}^2(k) \ &\gtrsim rac{\#\mathcal{C}}{\#\mathcal{U}} \left(\mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}})
ight) \ &\gtrsim \mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) \end{aligned}$$

Proper choice of \mathcal{U} and \mathcal{C} : $\frac{\#\mathcal{C}}{\#\mathcal{U}} \geq c_{\mathrm{GD}}$.

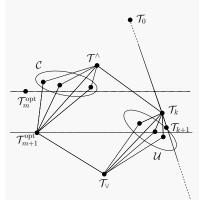
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$$\mathcal{G}(\mathcal{T}_{Cm}) - \mathcal{J}(u) \le \mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{J}(u)$$

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Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.



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ight) \ &\gtrsim \mathcal{G}(\mathcal{T}_m^{ ext{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{ ext{opt}}) \end{aligned}$$

Proper choice of \mathcal{U} and \mathcal{C} : $\frac{\#\mathcal{C}}{\#\mathcal{U}} \geq c_{\mathrm{GD}}$.

Induction yields $C \in \mathbb{N}$, such that

$$\mathcal{G}(\mathcal{T}_{Cm}) - \mathcal{J}(u) \le \mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{J}(u)$$

for all $m \in \mathbb{N}_0$.

Last Slide

Thank you for your attention!