RUHR-UNIVERSITÄT BOCHUM

An Instant Optimal Adaptive Finite Element Method

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joint work with
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Outline

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Framework

AFEM and Main Result

Proof of the Main Result

Problem

Poisson Problem:

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. For $f \in L^2(\Omega)$ find $u \in H^1_0(\Omega)$, such that

$$-\Delta u = f \qquad \text{in } H^{-1}(\Omega) \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

$$\mathcal{J}(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 - fu\,\mathrm{d}x = \min\big\{\mathcal{J}(v) \colon v \in H^1_0(\Omega)\big\}.$$

Finite Elements and Ritz Approximation

Continuous piece wise affine finite elements

$$\mathbb{V}(\mathcal{T}) := \{V \in H^1_0(\Omega) \mid V_{|T} \in \mathbb{P}_1(T),\ T \in \mathcal{T}\}.$$

over conforming triangulation $\mathcal{T}$ of Ω with nodes $\mathcal{N}$ and edges $\mathcal{S}$.

$$U_\mathcal{T} \in \mathbb{V}(\mathcal{T}): \quad \int_\Omega \nabla U_\mathcal{T} \cdot \nabla V\,\mathrm{d}x = \int_\Omega fV\,\mathrm{d}x \quad \text{for all } V \in \mathbb{V}(\mathcal{T}).$$

Error Estimation

Edge Based Residual Indicators

For $S \in \mathcal{S}$ define
$$\mathcal{E}_{\mathcal{T}}^2(S) := \int_S h_S \left|[\![\nabla U_{\mathcal{T}}]\!]\right|^2 \, \mathrm{d}s + \sum_{\partial T \supset S} \int_T h_T^2 f^2 \, \mathrm{d}x$$

Error Bounds

$$\|u - U_{\mathcal{T}}\|_{\Omega}^2 \lesssim \mathcal{E}_{\mathcal{T}}^2(\mathcal{S}) := \sum_{S \in \mathcal{S}} \mathcal{E}_{\mathcal{T}}^2(S) \lesssim \|u - U_{\mathcal{T}}\|_{\Omega}^2 + \operatorname{osc}^2(\mathcal{T})$$

Discrete Error Bounds

Let $\mathcal{T}_* \geq \mathcal{T}$ be a refinement of $\mathcal{T}$. Then

$$\|U_{\mathcal{T}} - U_{\mathcal{T}_*}\|_{\Omega}^2 \lesssim \mathcal{E}_{\mathcal{T}}^2(\mathcal{S} \setminus \mathcal{S}_*) \lesssim \|U_{\mathcal{T}} - U_{\mathcal{T}_*}\|_{\Omega}^2 + \sum_{T \in \mathcal{T} \setminus \mathcal{T}_*} \int_T h_T^2 f^2 \, \mathrm{d}x$$

Refinement Framework

Admissible Triangulations

- **Initial conforming Refinement** $\mathcal{T}_0$ of Ω with refinement edges labeled as in [Binev, Dahmen, DeVore '04] or [Stevenson, 08].
- $\mathbb{T} := \{\mathcal{T} : \text{conforming refinement of } \mathcal{T}_0 \text{ using NVB}\}$
- **Refinement Routine:** Let $\mathcal{T} \in \mathbb{T}$ with edges $\mathcal{S}$ then, for $S \in \mathcal{S}$, let

$$\mathcal{T} \leq \mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, S) \in \mathbb{T},$$

be the **smallest** refinement of $\mathcal{T}$ s.t. $S \notin \mathcal{S}_*$.

Complexity of REFINE [BDD '04], [Stevenson, 08]

Let $\mathcal{T}_0 = \mathcal{T}_0 \leq \mathcal{T}_1 \leq \mathcal{T}_2 \leq \ldots \subset \mathbb{T}$ such that for some $\mathcal{M}_k \subseteq \mathcal{S}_k$ we have

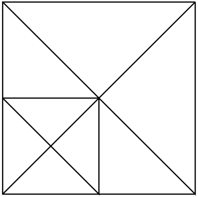
$$\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k) \quad \Rightarrow \quad \#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{n<k} \#\mathcal{M}_n$$

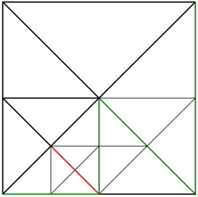
Modified Framework

For $S \in \mathcal{S} = \mathcal{S}(\mathcal{T}), \mathcal{T} \in \mathbb{T}$, define refined$(\mathcal{T}, S) := \mathcal{S} \setminus$ REFINE$(\mathcal{T}, S)$.

Accumulated Error Indicators

$$\bar{\mathcal{E}}^2_{\mathcal{T}}(S) := \mathcal{E}^2_{\mathcal{T}}(\text{refined}(\mathcal{T}, S)) \;\Rightarrow\; \|U_{\mathcal{T}} - U_{\mathcal{T}_*}\|^2_\Omega + \sum_{T \in \mathcal{T} \setminus \mathcal{T}_*} \int_T h_T^2 f^2 \,\mathrm{d}x \simeq \bar{\mathcal{E}}^2_{\mathcal{T}}(S)$$





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The Adaptive Finite Element Method (AFEM)

Choose $\mu \in (0,1]$ and set $k = 0$.

SOLVE: compute $U_k \in \mathbb{V}(\mathcal{T}_k)$;

ESTIMATE: compute $\bar{\mathcal{E}}^2_{\max}(k) := \max\left\{\mathcal{E}^2_k\big(\text{refined}(\mathcal{T}_k,\, S)\big) : S \in \mathcal{S}_k\right\}$;

MARK: $\mathcal{M}_k := \emptyset$, $\mathcal{C}_k := \mathcal{S}(\mathcal{T}_k)$, $\widetilde{\mathcal{M}}_k := \emptyset$;

while $\mathcal{C}_k \neq \emptyset$ do

select $S \in \mathcal{C}_k$;

if $\mathcal{E}^2_k(\text{refined}(\mathcal{T}_k,\, S) \setminus \widetilde{\mathcal{M}}_k) \geq \mu\, \bar{\mathcal{E}}^2_{\max}(k)$;

then $\mathcal{M}_k := \mathcal{M}_k \cup \{S\}$;

$\widetilde{\mathcal{M}}_k := \widetilde{\mathcal{M}}_k \cup \text{refined}(\mathcal{T}_k,\, S)$;

end if;

$\mathcal{C}_k := \mathcal{C}_k \setminus \text{refined}(\mathcal{T}_k,\, S)$;

end while;

REFINE: compute $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$ and increment k.

The Main Result: Instant Optimality

Theorem [Diening, K., Stevenson 2013]

There exist constants $C = C(\mathcal{T}_0)$, $\tilde{C} = \tilde{C}(\mathcal{T}_0, \mu)$, such that for $k, m \in \mathbb{N}$ with

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \geq \tilde{C}\, m,$$

we have

$$C\left(|\!|\!| u - U_{\mathcal{T}} |\!|\!|_\Omega^2 + \operatorname{osc}^2(\mathcal{T}) \right) \geq |\!|\!| u - U_{\mathcal{T}_k} |\!|\!|_\Omega^2 + \operatorname{osc}^2(\mathcal{T}_k)$$

for all $\mathcal{T} \in \mathbb{T}$ with $\#\mathcal{T} - \#\mathcal{T}_0 \leq m$.

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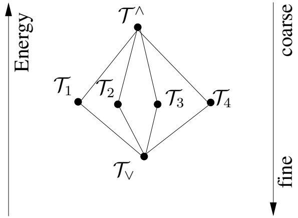
Proof: Lower Diamond Estimate



Minimal Diamond

A set $(\mathcal{T}^\wedge, \mathcal{T}_\vee; \mathcal{T}^1, \ldots, \mathcal{T}^m)$ is called *minimal diamond*, if

- $\mathcal{T}^\wedge$ is the finest coarsening and
- $\mathcal{T}_\vee$ is the coarsest refinement

of $\mathcal{T}^1, \ldots, \mathcal{T}^m$.

Lower Diamond Estimate

Let $(\mathcal{T}^\wedge, \mathcal{T}_\vee; \mathcal{T}^1, \ldots, \mathcal{T}^m)$ be a minimal diamond, such that the sets $\Omega(\mathcal{T}^j \setminus \mathcal{T}_\vee) := \operatorname{interior} \bigcup \{T \in \mathcal{T}^j \setminus \mathcal{T}_\vee\}$ are pairwise disjoint. Then

$$\mathcal{J}(U_\wedge) - \mathcal{J}(U_\vee) = \frac{1}{2}\, |U_\wedge - U_\vee|^2_{H^1(\Omega)}$$
$$\simeq \frac{1}{2} \sum_{j=1}^{m} |U_j - U_\vee|^2_{H^1(\Omega_j)} = \sum_{j=1}^{m} \mathcal{J}(U_j) - \mathcal{J}(U_\vee).$$

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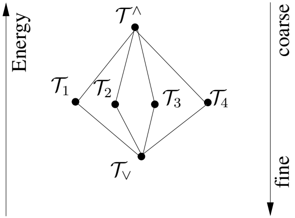

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Proof: Total Energy

For $\mathcal{T} \in \mathbb{T}$, we set: $$\mathcal{G}(\mathcal{T}) := \mathcal{J}(U_{\mathcal{T}}) + \sum_{T\in\mathcal{T}} h_T^2 \|f\|_{L^2(T)}^2.$$

Properties

- $\mathcal{G}(\mathcal{T}) - \mathcal{J}(u) \simeq |U_{\mathcal{T}} - u|_{H^1(\Omega)}^2 + \operatorname{osc}^2(\mathcal{T})$.
- For $\mathcal{T}_*$ being a refinement of $\mathcal{T}$, we have
$$\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_*) \simeq \mathcal{E}_{\mathcal{T}}^2(\mathcal{S} \setminus \mathcal{S}_*).$$
- The energy $\mathcal{G} : \mathbb{T} \to \mathbb{R}$ satisfies the lower diamond estimate.

For $m \in \mathbb{N}$, we define
$$\mathcal{T}_m^{\text{opt}} \in \mathbb{T} : \quad \mathcal{G}(\mathcal{T}_m^{\text{opt}}) = \min\left\{\mathcal{G}(\mathcal{T}) \colon \mathcal{T} \in \mathbb{T} \text{ mit } \#\mathcal{T} - \#\mathcal{T}_0 \le m\right\}$$

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RUB

Proof: How to choose $\mathcal{C}$ and $\mathcal{U}$?

Population

For a triangulation $\mathcal{T}$, we define the corresponding *population* as $\mathcal{P} := \mathcal{N}(\mathcal{T})$.

Populations have a tree structure. Thanks to the initial labeling we can assign each person to a generation:

$$\mathrm{gen}(child) = \mathrm{gen}(parent) + 1.$$

A population is conforming if for all $P \in \mathcal{P}$ all ancestors of P are in $\mathcal{P}$.

Limited Genetic Diversity

$$\sup_{\mathcal{P}} \sup_{P\in\mathcal{P}} \sup_{k\in\mathbb{N}} \#\big(\mathrm{ancestors}(P) \cap \mathrm{gen}^{-1}(k)\big) =: c_{\mathrm{GD}} < \infty.$$

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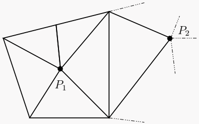

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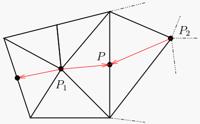

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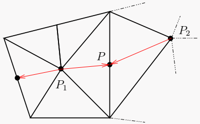

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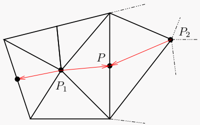

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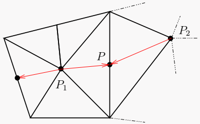

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Proof: Main Idea of Optimality

Assume for simplicity: In each iteration, AFEM marks only one largest element.

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\text{opt}}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})$.

$$\begin{aligned}
\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq \bar{\mathcal{E}}_{\max}^2(k) \\
&\geq \frac{1}{\#\mathcal{U}} \sum_{\mathcal{T} \in \mathcal{U}} \mathcal{E}_k^2(\mathcal{S}(\mathcal{T}) \setminus \mathcal{S}_k) \\
&\gtrsim \frac{1}{\#\mathcal{U}} \left(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee) \right) \\
&\gtrsim \frac{1}{\#\mathcal{U}} \left(\mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \right) \\
&\gtrsim \frac{1}{\#\mathcal{U}} \sum_{\mathcal{T} \in \mathcal{C}} \left(\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \right) \\
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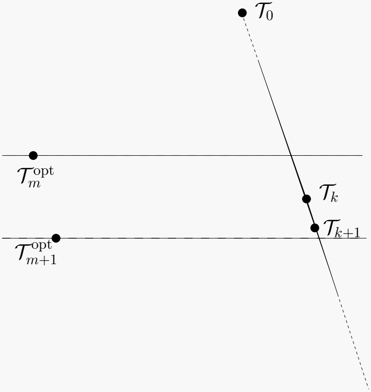

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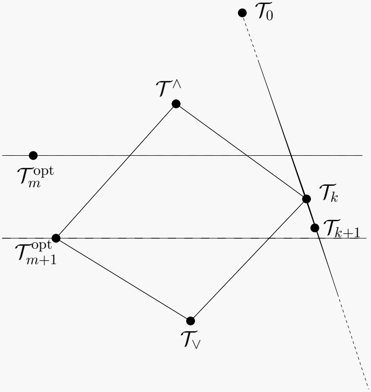

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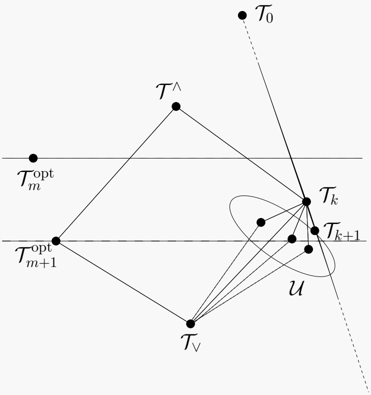

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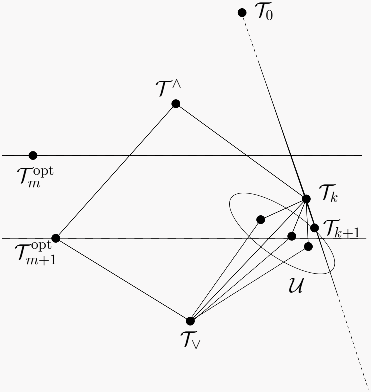
Proof: Main Idea of Optimality

Assume for simplicity: In each iteration, AFEM marks only one largest element.

Induction:

For fixed $k \in \mathbb{N}$ let $m \in \mathbb{N}$, such that $\mathcal{G}(\mathcal{T}_m^{\rm opt}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\rm opt})$.



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\begin{aligned}
\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq \bar{\mathcal{E}}^2_{\max}(k) \\
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RUB

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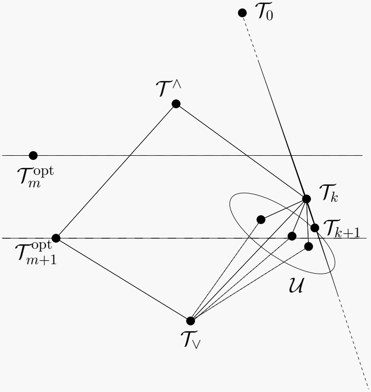

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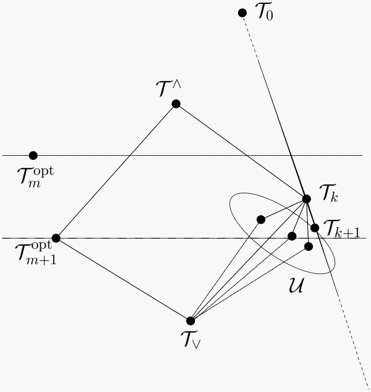

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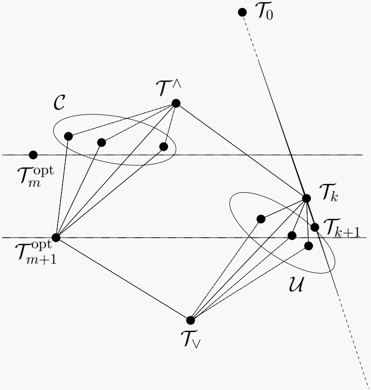

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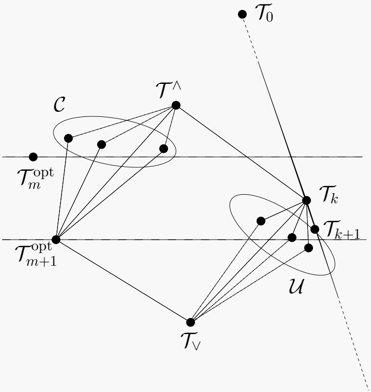

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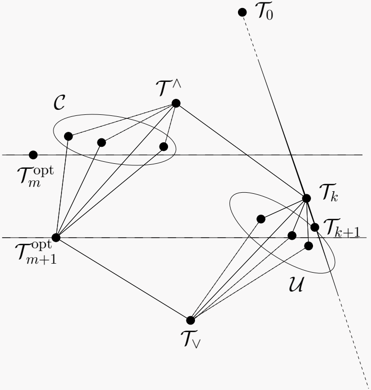

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\end{aligned}
$$

Proper choice of $\mathcal{U}$ and $\mathcal{C}$: $\quad \frac{\#\mathcal{C}}{\#\mathcal{U}} \geq c_{\text{GD}}$.

Induction yields $C \in \mathbb{N}$, such that

$$\mathcal{G}(\mathcal{T}_{Cm}) - \mathcal{J}(u) \leq \mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{J}(u)$$

for all $m \in \mathbb{N}_0$.

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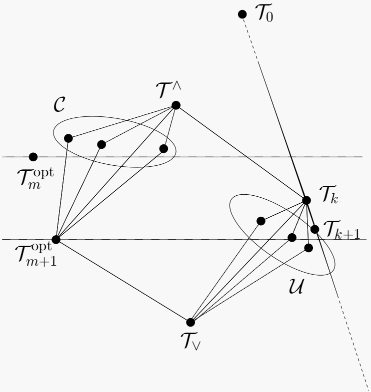

$$\begin{aligned}\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\simeq \bar{\mathcal{E}}^2_{\max}(k)\\ &\gtrsim \frac{\#\mathcal{C}}{\#\mathcal{U}} \left(\mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\right)\\ &\gtrsim \mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\end{aligned}$$

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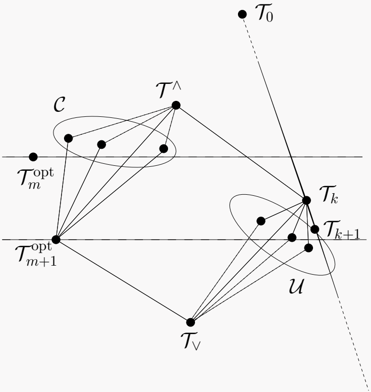

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Last Slide

Thank you for your attention!