Numerics for the stochastic heat equation based on a weak space-time variational formulation

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Outline

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0. \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = \dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

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- Stochastic heat equation
- Semigroup framework
- Weak space-time variational formulation

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Semigroup approach

Semigroup approach

Linear SPDE with additive noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \ \ t > 0 \\ X(0) = X_0 \end{cases}$$

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- $E(t) = e^{-tA}$, $t \geq 0$, C_0-semigroup of bounded linear operators on $\mathcal{H}$
- X_0 is an $\mathcal{F}_0$-measurable $\mathcal{H}$-valued random variable

Semigroup of bounded linear operators

Let $\{e^{-tA}\}_{t\geq 0}$ be the semigroup of bounded linear operators generated by $-A$, that is,

$$u(t) = e^{-tA}u_0$$

is the solution of the homogeneous evolution equation

$$\begin{cases} \dot{u} + Au = 0, & t > 0 \\ u(0) = u_0 \end{cases}$$

The solution of the non-homogeneous equation

$$\begin{cases} \dot{u} + Au = f, & t > 0 \\ u(0) = u_0 \end{cases}$$

is then given by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds$$

I also write $E(t) = e^{-tA}$.

Mild solution

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A weak solution satisfies the integral equation: for all $\eta \in D(A^*)$

$$\langle X(t), \eta \rangle + \int_0^t \langle X(s), A^* \eta \rangle \,\mathrm{d}s = \langle X_0, \eta \rangle + \int_0^t \langle \eta, B\,\mathrm{d}W(s) \rangle$$

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The unique solution is given by (mild solution)

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B\,\mathrm{d}W(s)$$

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Must give a rigorous meaning to W and the stochastic integral.

Q-Wiener process

- covariance operator $Q\colon \mathcal{U} \to \mathcal{U}$, self-adjoint, positive semidefinite, bounded, linear operator
- $Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty}$ ON basis
- $\beta_j(t)$, independent identically distributed, real-valued, Brownian motions
- $W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j$

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Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $\quad W(t)$ converges in $L_2(\Omega, \mathcal{U})$:
$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$

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- $Q = I$, “white noise”. $\quad W(t)$ is not $\mathcal{U}$-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense.

Q-Wiener process

If $\text{Tr}(Q) < \infty$:

- $W(0) = 0$
- continuous paths $t \mapsto W(t)$ in $\mathcal{U}$
- independent increments
- Gaussian law: $\mathbf{P} \circ (W(t) - W(s))^{-1} = \mathcal{N}(0, (t-s)Q), \quad s \le t$

$W(t)$ generates a filtration $\mathcal{F}_t$ so that it becomes a square integrable $\mathcal{U}$-valued martingale.

We can integrate with respect to W: $\displaystyle \int_0^t F(s)\,\mathrm{d}W(s)$.

The integral can be defined also when $\text{Tr}(Q) = \infty$.

Stochastic integral

$$W_A(t) = \int_0^t E(t-s)B\,\mathrm{d}W(s), \quad t \geq 0$$

- The stochastic integral
$$\int_0^t F(s)\,\mathrm{d}W(s) \approx \sum_{j=1}^{N} F(s_{j-1})(W(s_j) - W(s_{j-1}))$$
can be defined if $\mathbf{E}\int_0^t \|F(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty$.
- Isometry property:
$$\mathbf{E}\Big\|\int_0^t F(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|F(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s$$

Hilbert-Schmidt norm:

$$\|T\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \|T\varphi_j\|^2 < \infty, \qquad \{\varphi_j\} \text{ arbitrary ON basis in } \mathcal{U}$$

G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*
C. Prévôt and M. Röckner, *A Consise Course on Stochastic Partial Differential Equations*

Stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t>0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = L_2(\mathcal{D})$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H^1_0(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{H}$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-tA}$, analytic semigroup generated by $-A$

Weak solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$|v|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta (v,\phi_j)^2\Big)^{1/2}, \quad \dot H^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

For example: $\dot H^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}), \quad \dot H^1 = H_0^1(\mathcal{D})$

Mean square norm: $\|v\|_{L_2(\Omega,\dot H^\beta)}^2 = \mathbf{E}(|v|_\beta^2), \quad \beta \in \mathbf{R}$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then
$\|X(t)\|_{L_2(\Omega,\dot H^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}\Big)$

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Mean square norm: $\|v\|^2_{L_2(\Omega, \dot H^\beta)} = \mathbf{E}(|v|_\beta^2), \quad \beta \in \mathbf{R}$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then $\|X(t)\|_{L_2(\Omega, \dot H^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega, \dot H^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$

Two cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^\infty \|Q^{1/2} e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

Proof with $X_0 = 0$

$$
\begin{aligned}
\|X(t)\|^2_{L_2(\Omega,\dot H^\beta)} &= \mathbf{E}\Big(\Big\| \int_0^t \Lambda^{\beta/2} E(t-s)\,\mathrm{d}W(s) \Big\|^2 \Big) \\
&= \int_0^t \|\Lambda^{\beta/2} E(s) Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2} \phi_k\|^2\,\mathrm{d}s \\
&\le C \sum_{k=1}^{\infty} \|\Lambda^{(\beta-1)/2} Q^{1/2} \phi_k\|^2 \\
&= C \|\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}
$$

$$
\int_0^t \|\Lambda^{1/2} E(s) v\|^2\,\mathrm{d}s \le C\|v\|^2
$$

The finite element method

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$, $\quad S_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$
- S_h continuous piecewise linear functions
- $X_h(t) \in S_h;\ (\mathrm{d}X_h, \chi) + (\nabla X_h, \nabla \chi)\,\mathrm{d}t = (\mathrm{d}W, \chi)\ \forall \chi \in S_h,\ t > 0$
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $(\Lambda_h \psi, \chi) = (\nabla \psi, \nabla \chi)\ \forall \psi, \chi \in S_h$
- $A_h = \Lambda_h$
- $P_h \colon L_2 \to S_h,$ orthogonal projection, $\quad (P_h f, \chi) = (f, \chi)\ \forall \chi \in S_h$

$$\begin{cases} X_h(t) \in S_h, \quad X_h(0) = P_h X_0 \\ \mathrm{d}X_h + A_h X_h\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t > 0 \end{cases}$$

$P_h W(t)$ is Q_h-Wiener process with $Q_h = P_h Q P_h$.

Mild solution, with $E_h(t) = e^{-tA_h}$,

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h\,\mathrm{d}W(s)$$

Approximation of the semigroup

$$\begin{cases} u_t + Au = 0, \quad t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} u_{h,t} + A_h u_h = 0, \quad t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad\qquad u_h(t) = E_h(t)P_h v$$

Denote

$$F_h(t)v = E_h(t)P_h v - E(t)v, \quad |v|_\beta = \|A^{\beta/2}v\| = \|\Lambda^{\beta/2}v\|.$$

We have, for $0 \leq \beta \leq 2$,

- $\|F_h(t)v\| \leq Ch^\beta |v|_\beta, \quad t \geq 0$
- $\left(\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s\right)^{1/2} \leq Ch^\beta |v|_{\beta-1}, \quad t \geq 0$

V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*

Strong convergence in L_2 norm

Theorem. If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0,2]$, then
$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le Ch^\beta \Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} \Big)$

Recall: $\|X_h(t) - X(t)\|_{L_2(\Omega,H)} = \Big(\mathbf{E}(\|X_h(t) - X(t)\|^2) \Big)^{1/2}$

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then the rate is almost $O(h^{1/2})$.

No result for $Q = I$, $d \ge 2$.

Weak space-time variational framework

Function spaces:

$H_0^1(\mathcal{D}) \hookrightarrow L^2(\mathcal{D}) \hookrightarrow H^{-1}(\mathcal{D})$

Gelfand triple

$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$

$\mathcal{Y} = L^2([0,T];V)$, $\|y\|_{\mathcal{Y}}^2 = \int_0^T \|y(t)\|_V^2 \, dt$

$\mathcal{Y}_H = \mathcal{Y} \times H$

$\mathcal{X} = L^2([0,T];V) \cap H^1([0,T];V^*) \subset \mathcal{C}([0,T];H)$

$\|x\|_{\mathcal{X}}^2 = \|x\|_{L^2([0,T];V)}^2 + \|\dot{x}\|_{L^2([0,T];V^*)}^2$

Forward problem

$$\begin{cases} \dot{u}(t) + Au(t) = f(t) & \text{in } V^*,\ t \in (0, T], \\ u(0) = u_0 & \text{in } H. \end{cases}$$

Space-time variational formulation:

Find $u \in \mathcal{X}$ such that for all $y \in \mathcal{Y}_H$, $\mathscr{B}(u, y) = \mathscr{F}(y)$.

$\mathcal{X} = L^2([0, T]; V) \cap H^1([0, T]; V^*)$
$\mathcal{Y} = L^2([0, T]; V), \quad y = (y_1, y_2) \in \mathcal{Y}_H = \mathcal{Y} \times H$

$$\mathscr{B} : \mathcal{X} \times \mathcal{Y}_H \to \mathbb{R}$$

$$\mathscr{B}(x, y) = \int_0^T ({}_{V^*}\langle \dot{x}(t), y_1(t)\rangle_V + a(x(t), y_1(t)))dt + \langle x(0), y_2\rangle_H$$

$$\mathscr{F} \in \mathcal{Y}_H^*$$

$$\mathscr{F}(y) = \int_0^T {}_{V^*}\langle f(t), y_1(t)\rangle_V \, dt + \langle u_0, y_2\rangle_H$$

Adjoint problem

$$\begin{cases} -\dot{v}(t) + A^* v(t) = g(t) & \text{in } V^*, \forall t \in [0, T), \\ v(T) = v_T & \text{in } H, \end{cases}$$

Space-time variational formulation:

Find $v \in \mathcal{X}$ such that for all $y \in \mathcal{Y}_H$, $\mathscr{B}^*(y, v) = \mathscr{G}(y)$,

$$\mathscr{B}^* : \mathcal{Y}_H \times \mathcal{X} \to \mathbb{R}$$

$$\mathscr{B}^*(y, x) = \int_0^T ({}_{V^*}\langle y_1(t), -\dot{x}(t)\rangle_V + a(y_1(t), x(t)))dt + \langle y_2, x(T)\rangle_H$$

$$\mathscr{G} \in \mathcal{Y}_H^*$$

$$\mathscr{G}(y) = \int_0^T {}_{V^*}\langle g(t), y_1(t)\rangle_V dt + \langle y_2, v_T\rangle_H$$

Weak space-time formulation

Weak space-time variational formulation for the forward problem:

Find $u \in \mathcal{Y}_H$ such that for all $x \in \mathcal{X}$, $\mathscr{B}^*(u, x) = \mathscr{F}(x)$.

$$u = (u_1, u_2) \in \mathscr{B}^* : \mathcal{Y}_H \times \mathcal{X} \to \mathbb{R}$$

$$\mathscr{B}^*(u, x) = \int_0^T ({}_{V^*}\langle u_1(t), -\dot{x}(t)\rangle_V + a(u_1(t), x(t)))dt + \langle u_2, x(T)\rangle_H$$

$$\mathscr{F} \in \mathcal{X}^*$$

$$\mathscr{F}(x) = \int_0^T {}_{V^*}\langle f(t), x(t)\rangle_V dt + \langle u_0, x(0)\rangle_H$$

Unique solution and

$$\|u\|_{\mathcal{Y}_H} := \Big(\int_0^T \|u_1(t)\|_V^2 \, dt + \|u_2\|_H^2 \Big)^{\frac{1}{2}} \leq C\|\mathscr{F}\|_{\mathcal{X}^*}$$

Stochastic heat equation

$$\mathrm{d}U + AU\,\mathrm{d}t = f\,\mathrm{d}t + \mathrm{d}W; \quad U(0) = U_0$$

$$\mathbb{X} = L^2(\Omega; \mathcal{X}), \quad \mathbb{Y} = L^2(\Omega; \mathcal{Y}),$$
$$\mathbb{Y} = L^2(\Omega; V), \quad \mathbb{Y}_{\mathbb{H}} = \mathbb{Y} \times \mathbb{H} = L^2(\Omega; \mathcal{Y} \times H).$$

$$\mathfrak{B}^*(y,x) = \mathbb{E}\Big[\int_0^T [{}_{V^*}\langle y_1(t,\omega), -\dot{x}(t,\omega)\rangle_V + a(t;\omega; y_1(t,\omega), x(t,\omega))]dt + \langle y_2, x(T,\omega)\rangle_H\Big],$$
$$\mathfrak{F}(x) = \mathbb{E}\Big[\int_0^T {}_{V^*}\langle f(t,\omega), x(t,\omega)\rangle_V dt + \langle U_0, x(0,\omega)\rangle_H\Big],$$
$$\mathfrak{W}(x) = \mathbb{E}\Big[\int_0^T \langle dW(t,\omega), x(t,\omega)\rangle_H\Big],$$

Weak space-time variational formulation:

Find $U \in \mathbb{Y}_{\mathbb{H}}$ such that for all $x \in \mathbb{X}$, $\mathfrak{B}^*(U,x) = \mathfrak{F}(x) + \mathfrak{W}(x)$.

Stochastic heat equation

Weak space-time variational formulation:

Find $U \in \mathbb{Y}_{\mathbb{H}}$ such that for all $x \in \mathbb{X}$, $\mathfrak{B}^*(U, x) = \mathfrak{F}(x) + \mathfrak{W}(x)$.

Unique solution and

$$\|U\|_{\mathbb{Y}_H} := \left(\mathbb{E} \int_0^T \|U_1(t)\|_V^2 \, dt + \mathbb{E}\|U_2\|_H^2\right)^{\frac{1}{2}} \leq C\|\mathfrak{F}\|_{\mathbb{X}^*} + C\|\mathfrak{W}\|_{\mathbb{X}^*}$$

Machinery for proving additional regularity estimates and error estimates.