Convergence of inexact adaptive finite element solvers

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Joint work with M. Arioli (RAL, UK) and D. Loghin (Birmingham, UK)

EFEF, ACMAC, Ηράκλειο, 31.05 – 01.06.2013

Overview

1. The setting
2. Inexact SOLVE – theory
3. Adaptive stopping criteria for CG
4. Numerical Experiments

M. Arioli, E.H. Georgoulis , and D. Loghin
Stopping criteria for adaptive finite element solvers
SIAM J. Sci. Comput., to appear.

The setting

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Consider

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $a \in [L^\infty(\Omega)]^{d\times d}$ SPD, and $f \in L^2(\Omega)$.

Denote the energy norm by $||\cdot||_a := \|\sqrt{a}\nabla(\cdot)\|$

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Finite element space $\mathcal{V} := \{V \in H_0^1(\Omega) : \ V|_\kappa \circ F_\kappa \in \mathcal{P}_r(\hat{\kappa}), \ \kappa \in \mathcal{T}\}$.

The finite element method reads:

$$\text{find } U \in \mathcal{V} \text{ such that} \quad a(U, V) = l(V) \quad \forall V \in \mathcal{V}$$

where

$$a(w, v) := \int_\Omega a \nabla w \cdot \nabla v \, \mathrm{d}x \qquad \text{and} \qquad l(v) := \int_\Omega f v \, \mathrm{d}x$$

The setting

Standard Adaptive FEM algorithm

SOLVE – ESTIMATE – MARK – REFINE

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Question: Guaranteed convergence of AFEM with inexactness in “SOLVE”?

AFEM Convergence with exact SOLVE

SOLVE:

$$\text{find } U_m^e \in \mathcal{V}_m \quad \text{such that} \quad a(U_m^e, V_m) = l(V_m) \quad \forall V_m \in \mathcal{V}_m.$$

ESTIMATE:

$$\eta^2_{\mathcal{T}_m}(U_m^e, \kappa) := h_\kappa^2 \|f + \nabla \cdot (a \nabla U_m^e)\|_\kappa^2 + \sum_{e \in \Gamma_{\mathrm{int},m} \cap \partial\kappa} h_e \|[a \nabla U_m^e \cdot \mathbf{n}]\|_e^2.$$

MARK:
For some $0 < \theta \leq 1$, find $\mathcal{M}_m \subset \mathcal{T}_m$:

$$\eta^2_{\mathcal{T}_m}(U_m^e, \mathcal{M}_m) \geq \theta \; \eta^2_{\mathcal{T}_m}(U_m^e, \mathcal{T}_m).$$

REFINE:
Standard mesh refinement (newest vertex bisection)

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Theorem (Cascon, Kreuzer, Nochetto & Siebert '08)

There exist constants $\xi > 0$ and $0 < \alpha < 1$ such that

$$\|u - U_{m+1}^e\|_a^2 + \xi \eta_{\mathcal{T}_{m+1}}^2(U_{m+1}^e, \mathcal{T}_{m+1}) \leq \alpha \left(\|u - U_m^e\|_a^2 + \xi \eta_{\mathcal{T}_m}^2(U_m^e, \mathcal{T}_m) \right).$$

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with inexact solution satisfying a given criterion.

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} with U_m if exact and with $\tilde{U}_m$ if inexact.

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Remark: In general, we arrive to different meshes for each case.

AFEM Convergence with inexact SOLVE

Theorem

Let u, θ, $\tilde{U}_m$ and $\tilde{U}_{m+1}$, $m \geq 1$ as above, be such that

$$\mu_1 ||\tilde{U}_m - U_m||_a^2 + \mu_2 ||\tilde{U}_{m+1} - U_{m+1}||_a^2 \leq \nu_1 \eta_m^2(\tilde{U}_m) + \nu_2 \eta_{m+1}^2(\tilde{U}_{m+1}),$$

for suitably chosen $\mu_1, \mu_2 > 0$ and $\nu_1, \nu_2 \geq 0$. Then, there exist a constant $0 < \alpha < 1$, depending only on the shape regularity of $\tilde{\mathcal{T}}_1$ and on the marking parameter θ, such that

$$||u - \tilde{U}_{m+1}||_a^2 + \zeta \eta_{m+1}^2(\tilde{U}_{m+1}) \leq \alpha \big(||u - \tilde{U}_m||_a^2 + \zeta \eta_m^2(\tilde{U}_m) \big),$$

for some $\zeta \equiv \zeta(\mu_1, \mu_2, \nu_1, \nu_2) > 0$.

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Extends known results.

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Corollary

Let u, θ, $\tilde{U}_m$ and $\tilde{U}_{m+1}$, $m \geq 1$ as above, be such that

$$||\tilde{U}_m - U_m||_a^2 + \mu||\tilde{U}_{m+1} - U_{m+1}||_a^2 \leq \nu\eta_m^2(\tilde{U}_m),$$

for suitably chosen $\mu > 0$ and $\nu > 0$. Then, there exist a constant $0 < \alpha < 1$, depending only on the shape regularity of $\tilde{\mathcal{T}}_1$ and on the marking parameter θ, such that

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Remark: We estimate inexactness using information from previous iteration only!

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Practical inexactness estimation:

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Focus on iterative linear solvers: Conjugate Gradient

Stopping criteria for linear solvers

Linear system

$$A_m \mathbf{u}_m = \mathbf{b}_m$$

and let $\mathbf{u}_m^k$ the k-th CG iterate at step m of AFEM corresponding to $U_m^k \in \tilde{\mathcal{V}}_m$.

Then:

$$\|U_m - U_m^k\|_a = \|\mathbf{u}_m - \mathbf{u}_m^k\|_{A_m} = \|\mathbf{r}_m^k\|_{A_m^{-1}}, \qquad \mathbf{r}_m^k := \mathbf{b}_m - A_m \mathbf{u}_m^k$$

where $\langle x, y \rangle_A := x^T A y$, $x, y \in \mathbb{R}^N$,

Estimation of the residual – Golub-Meurant bound

Recalling that for CG, we have $(\mathbf{r}_m^k)^T \mathbf{u}_m^k = 0$, then

$$\|\mathbf{u}_m - \mathbf{u}_m^k\|_{A_m}^2 = \|\mathbf{r}_m^k\|_{A_m^{-1}}^2 = \mathbf{b}_m^T A_m^{-1} \mathbf{b}_m - \mathbf{b}_m^T \mathbf{u}_m^k.$$

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Crucially, we can have

$$F(A_m) = \mathbf{b}_m^T A_m^{-1} \mathbf{b}_m = \int_{\lambda_{\min}(A_m)}^{\lambda_{\max}(A_m)} \lambda^{-1} \mathrm{d}\omega(\lambda),$$

where the measure ω is a non-decreasing step function with jump discontinuities depending on the Fourier coefficients of $\mathbf{b}_m$ at the eigenvalues of A_m.

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Challenge: Find guaranteed lower bound $\lambda < \lambda_{\min}(A_m)$.

Lower bounds for $\lambda_{\min}(A_m)$

- Poincaré inequality $\|v\|_{L^2(\Omega)} \leq C_P |v|_{H^1(\Omega)}$ + (Wathen '87) $\longrightarrow$

$$\lambda_{\mathsf{min}}(A_m) \geq \frac{a_{\mathsf{min}}}{C_P^2} \min_{\kappa \in \mathcal{T}_m} \lambda_{\mathsf{min}}(M_\kappa)$$

where M_κ elemental mass matrices.

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- We can use "adaptive" Poincaré estimate:

Lemma (Loghin '13?)

Let $\mathcal{D}_m \supseteq \tilde{\mathcal{M}}_m$ region of refined elements, s.t. $\partial\Omega \neq \mathcal{D}_m \cap \partial\Omega \neq \emptyset$. Let A_m, A_{m+1} denote the matrices assembled on consecutive subdivisions in the inexact AFEM algorithm. Then

$$\lambda_{\min}(A_{m+1}) \ge \min\left\{ \lambda_{\min}(A_m), \frac{a_{\min}}{(C_P^m)^2} \min_{\kappa \in \mathcal{D}_m} \lambda_{\min}(M_\kappa) \right\}.$$

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- Estimates using Lanczos algorithm

Numerical Experiments

1. DNR: ideal bound using the exact dual norm of the residual;
2. GM0: ideal Golub-Meurant upper bound using the exact $\lambda_{\min}(A_m)$;

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3. GM1: Golub-Meurant upper bound with adaptive bounds for $\lambda_{\min}(A_m)$;
4. GM2: Golub-Meurant upper bound with global Poincaré bound for $\lambda_{\min}(A_m)$;

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5. GM3: Golub-Meurant with Lanczos estimate for $\lambda_{\min}(A_m)$;
6. HS: the Hestenes-Stiefel estimator (delay $d = 5$ steps);
7. AG: the anti-Gauss estimator;
8. ER($|\log tol|$): the standard Euclidean residual with various stopping tolerances tol.

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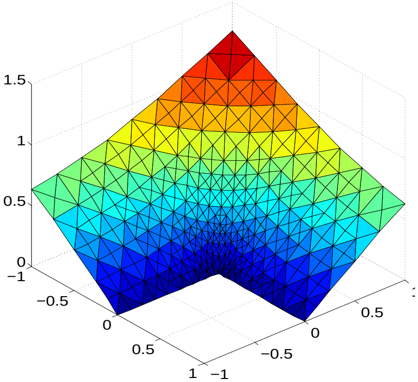
CG iteration has a cost proportional to the number of nonzeros, the cost formula employed is

$$\text{mv} := \text{matvecs}(m) = \sum_{k=1}^{m} \frac{\text{nnz}(A_k)}{\text{nnz}(A_m)} \cdot \text{its}(k),$$

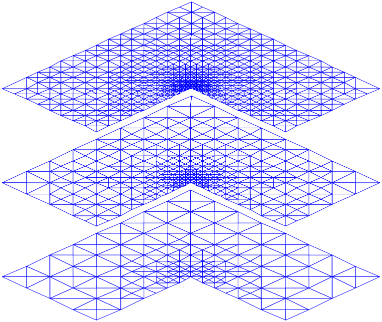
where $\text{nnz}(A_k)$ number of nonzero elements of A_k

Test problem 1

$\theta = 0.75$



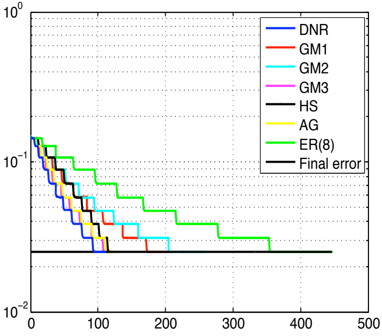
(a) Solution of test problem 1



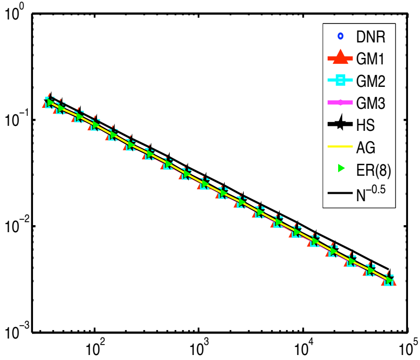
(b) Adaptive meshes: $m = 4, 6, 8$; $N_0 = 83$.

Figure: Solution and adaptive meshes for Test problem 1

Test problem 1



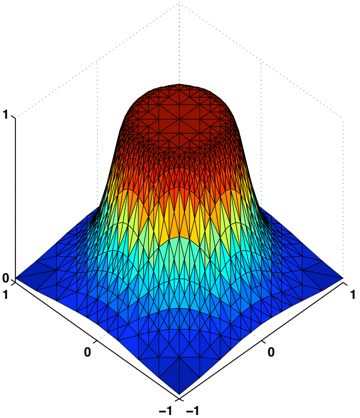
(a) Errors $\|u - U_m^k\|_a$ for $m = 10$.



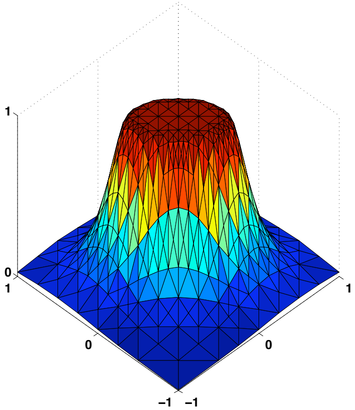
(b) Convergence rates, inexact AFEM, $m = 20$.

Test problem 2

$$a = \frac{1}{\epsilon}\begin{pmatrix} 1 & \epsilon - 1 \\ \epsilon - 1 & 1 \end{pmatrix}$$

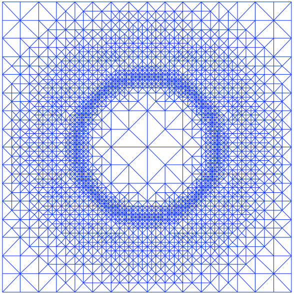


(c) $\epsilon = 1$.

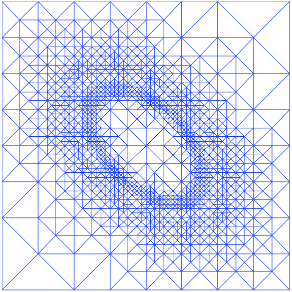


(d) $\epsilon = 1/2$.

Test problem 2



(e) $\epsilon = 1$.



(f) $\epsilon = 1/2$.

Figure: Solutions and adaptive meshes ($m = 10$) for Test problem 2

Test problem 2 ($m = 20$)

	$\epsilon = 1$			$\epsilon = 1/2$		
method	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	549,047	7.1750e–3	–	264,033	2.3346e–2	–
DNR	546,902	7.2040e–3	415	264,301	2.3416e–2	396
GM1	545,386	7.2009e–3	478	263,867	2.3359e–2	494
GM2	549,035	7.1750e–3	796	264,121	2.3343e–2	1,104
GM3	544,814	8.9772e–3	205	264,261	2.9244e–2	292
HS	539,606	1.4612e–2	11	264,814	1.1781e–1	13
AG	548,663	8.2482e–3	13	260,995	4.3329e–2	15
ER(6)	548,540	7.1782e–3	1,639	264,033	2.3346e–2	1,754
ER(8	549,047	7.1750e–3	2,075	264,033	2.3346e–2	2,356
ER(10)	549,047	7.1750e–3	3,003	264,033	2.3346e–2	3,007

Test problem 3

$a = 1, u = exp(-10r)$

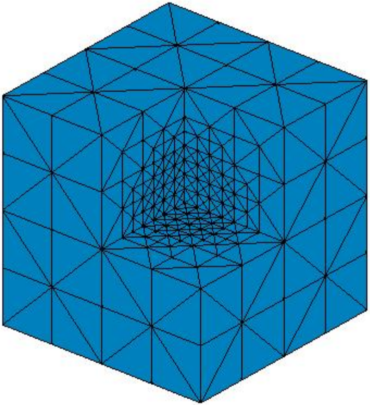


Figure: Adaptive meshing for test problem 3

Test problem 3

method	$N_0 = 142$			$N_0 = 779$			$N_0 = 5,191$		
	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	19,579	8.9670e-2	–	131,250	4.7497e-2	–	950,961	†	–
DNR	19,507	8.9617e-2	120	131,452	4.7744e-2	302	†	†	†
GM1	19,573	8.9665e-2	179	131,243	4.7498e-2	447	951,057	2.4695e-2	1,065
GM2	19,582	8.9672e-2	250	131,232	4.7497e-2	570	950,988	2.4696e-2	1,292
GM3	19,510	8.9682e-2	151	131,251	4.7484e-2	353	951,077	2.4695e-2	1,018
HS	19,648	9.0596e-2	120	131,606	4.9477e-2	291	958,982	2.6542e-2	676
AG	*	*	*	*	*	*	*	*	*
ER(6)	19,587	8.9677e-2	238	131,246	4.7497e-2	465	951,239	2.4692e-2	958
ER(8)	19,579	8.9670e-2	331	131,250	4.7497e-2	649	950,954	2.4697e-2	1,505
ER(10)	19,579	8.9670e-2	412	131,250	4.7497e-2	840	950,932	2.4697e-2	2,001

Table: m=10. Legend: †: out of memory; −: does not apply; ∗: does not exist.