Discontinuous Galerkin methods and discrete variational calculus

with applications to discrete curvature problems

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[1] EPSRC grant "Group actions on function approximation spaces"

1 **Variational problems**

2 The p–biharmonic equation

3 The Monge–Ampère problem

4 The affine maximal surface equation

5 Conclusions and outlook

Underlying ideas of variational calculus

- Variational calculus was developed to aid in the solution of a certain class of nonlinear problems, the *variational problems*, $\mathscr{L}[\cdot] = 0$.
- The underlying idea is to suppose the the nonlinear operator $\mathscr{L}[\cdot]$ is the (Frechet) derivative of an action functional $\mathscr{J}[\cdot]$,i.e.,

$$\mathscr{L}[u] = \mathrm{D}\mathscr{J}[u].$$

- Solutions to this problem are now critical points of $\mathscr{J}[\cdot]$ which may be easier to find or at least to characterise in some way.

Variational problems

The problems we are concerned with take the form of finding a minima (extremal in general) of a variational problem over an appropriate function space, i.e., find $u \in \mathscr{X}$ such that

$$\mathscr{J}[u] = \inf_{v \in \mathscr{X}} \mathscr{J}[v]$$

where

$$\mathscr{J}[u] := \int_{\Omega} L(u, \mathrm{D}^2 u)$$

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Euler–Lagrange equations

It's well known that if u is sufficiently smooth, the extremal to the variational problem satisfies the Euler–Lagrange equation to this problem.

$$\mathscr{L}[u] := \mathrm{D}^2 \colon \big(\partial_2 L(u, \mathrm{D}^2 u)\big) + \partial_1 L(u, \mathrm{D}^2 u) = 0$$

Example – the biharmonic problem

The (2–)biharmonic problem arises from the following variational problem: Seek $u \in \overset{\circ}{\mathrm{W}}{}^2_2(\Omega)$ such that

$$\mathscr{J}[u] = \int_\Omega \frac{1}{2} |\Delta u|^2 - fu \to \min .$$

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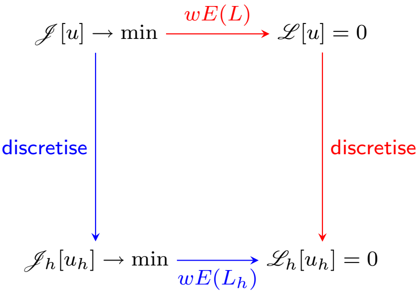
Euler–Lagrange equations

The associated Euler–Lagrange equations are

$$\begin{aligned} 0 = \mathscr{L}[u] &:= \mathrm{D}^2 \!:\! \big(\partial_2 L(u, \mathrm{D}^2 u)\big) + \partial_1 L(u, \mathrm{D}^2 u) \\ &= \mathrm{D}^2 \!:\! (\Delta u \boldsymbol{I}) - f \\ &= \Delta^2 u - f \end{aligned}$$

Figure: A commuting diagram showing the two different paths for successfully discretising a variational problem.



BIG FAT NOTE!

The results I will present for the p–biharmonic is based on ideas from [Buffa and Ortner, 2009, Di Pietro and Ern, 2010, Burman and Ern, 2008].

The p–biharmonic problem

The p–biharmonic problem is a fourth order variational PDE over $\mathring{W}^2_p(\Omega)$

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where for this problem

$$\int_\Omega L(u, \nabla u, \mathrm{D}^2 u) = \int_\Omega \frac{1}{p}\,|\Delta u|^p - fu$$

properties of the problem

Written in weak form, the Euler Lagrange equation reads: Find $u \in \mathring{\mathrm{W}}_p^2(\Omega)$ such that

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta v = \int_\Omega f v \quad \forall\, v \in \mathring{\mathrm{W}}_p^2(\Omega).$$

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- it is lower weakly semicontinuous

numerical methods for the case $p = 2$

If $p = 2$ the problem reduces to the linear fourth order biharmonic equation: Find $u \in \mathrm{H}^2_0(\Omega) := \mathring{\mathrm{W}}^2_p(\Omega)$ such that

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A conforming method to this class of problem would require the use of C^1 FEs, for example the Argyris element. These are computationally costly and nasty to implement! Motivating the use of nonconforming methods.

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what we do

We want to treat the problem for general p.

discrete Hessian

The main idea is to introduce an auxilliary variable into the formulation, i.e., transform the problem into a mixed method, to do this we require a discrete notion of Hessian of a piecewise discontinuous object.

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- [Lakkis and Pryer, 2013] used in the context of approximating the solution of fully nonlinear problems

discrete Hessian in a dG setting

It's not too hard to derive a general discrete Hessian, one can follow the arguments in [Arnold et al., 0102] for an extremely general setup.

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"interior penalty" dG Hessian

An example of the finite element Hessian is the unique (Reisz representation) element $\boldsymbol{H}[v_h] \in \mathbb{V}^{d\times d}$

$$\begin{aligned}\int_\Omega \boldsymbol{H}[v_h]\ \Phi &= -\int_\Omega \nabla_h v_h \otimes \nabla_h \Phi + \int_{\mathscr{E}\cup\partial\Omega} [\![v_h]\!] \otimes \{\!\!\{ \nabla_h \Phi \}\!\!\} \\ &\quad + \int_{\mathscr{E}\cup\partial\Omega} [\![\Phi]\!] \otimes \{\!\!\{ \nabla_h v_h \}\!\!\} . \\ &= \int_\Omega \mathrm{D}_h^2 v_h \Phi - \int_{\mathscr{E}\cup\partial\Omega} [\![\nabla_h v_h]\!]_\otimes \{\!\!\{ \Phi \}\!\!\} \\ &\quad + \int_{\mathscr{E}\cup\partial\Omega} [\![v_h]\!] \otimes \{\!\!\{ \nabla_h \Phi \}\!\!\}\end{aligned}$$

back to the continuous variational problem

Recall, we were trying to find $u \in \mathring{\mathrm{W}}^2_p(\Omega)$ such that

$$\mathscr{J}[u] = \min_{v \in \mathring{\mathrm{W}}^2_p(\Omega)} \mathscr{J}[v],$$

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a discrete version

We need to construct a discrete action functional. We want to find a $u_h \in \mathbb{V}$ such that

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choice of discrete action functional

We choose

$$\mathscr{J}_h[u_h] := \int_\Omega \frac{1}{p} |\operatorname{trace} \boldsymbol{H}[u_h]|^p + f u_h + \sigma \left(\int_{\mathscr{E} \cup \partial\Omega} h^{1-p} |[\![\nabla_h u_h]\!]|^p + h^{1-2p} |[\![u_h]\!]|^p \right)$$

assumptions on the FE space

Define

$$\|v_h\|_{dG,p}^p := \left\|\mathrm{D}_h^2 u_h\right\|_{\mathrm{L}_p(\Omega)}^p + h^{1-p}\left\|[\![\nabla_h u_h]\!]\right\|_{\mathrm{L}_p(\mathscr{E})}^p + h^{1-2p}\left\|[\![u_h]\!]\right\|_{\mathrm{L}_p(\mathscr{E})}^p$$

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We will assume the finite element space $\mathbb{V}$ is chosen such that the $\mathrm{L}_2(\Omega)$ orthogonal projection operator satisfies:

$$\lim_{h\to 0}\|v - \mathrm{P}_{\mathbb{V}}\, v\|_{\mathrm{L}_p(\Omega)} = 0$$

$$\lim_{h\to 0}\|\nabla v - \nabla_h(\mathrm{P}_{\mathbb{V}}\, v)\|_{\mathrm{L}_p(\Omega)} = 0 \text{ and}$$

$$\lim_{h\to 0}\|v - \mathrm{P}_{\mathbb{V}}\, v\|_{dG,p} = 0.$$

Piecewise discontinuous quadratic FEs satisfy this.

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- The discrete Hessian converges, i.e.,

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outline of the proof(s)

- Prove discrete Sobolev embeddings, a la, Gargliardo–Nirenberg–Sobolev

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- Coercivity of the variational problem follows from this
- The weak lower semicontinuity can then be used to show convergence

provable rates for the 2–biharmonic problem

In the papers [Süli and Mozolevski, 2007, Georgoulis and Houston, 2009] rates of convergence are given for the 2–biharmonic problem, these are

$$\|u - u_h\| = \mathrm{O}(h^2) \text{ for } k = 2 \qquad \|u - u_h\| = \mathrm{O}(h^{k+1}) \text{ for } k > 2$$

$$\|u - u_h\|_{dG,2} = \mathrm{O}(h^{k-1}).$$

Note that for piecewise quadratic finite elements the convergence rate is suboptimal in $\mathrm{L}_2(\Omega)$.

computational observations for the p–biharmonic

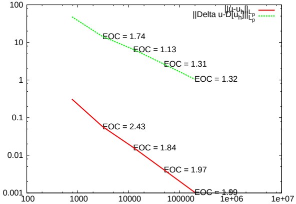
During the experiments we noted a superconvergence for both the $\mathrm{L}_p(\Omega)$ errors of the solution as well as the auxiliary variable for odd values of p and even values of k. Computationally we observed that

$$\|u - u_h\|_{\mathrm{L}_p(\Omega)} = \begin{cases} \mathrm{O}(h^2) \text{ when } k = 2 \\ \mathrm{O}(h^{k+1+\frac{1}{p}}) \text{ when } k > 2, \ k \text{ is even and } p \text{ is odd} \\ \mathrm{O}(h^{k+1}) \text{ otherwise} \end{cases}$$

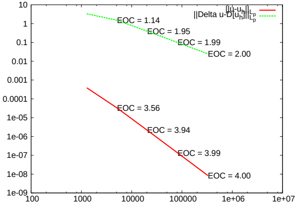
and that

$$\left\|\mathrm{D}^2 u - \boldsymbol{H}[u_h]\right\|_{\mathrm{L}_p(\Omega)} = \begin{cases} \mathrm{O}(h^{k-1+\frac{1}{p}}) \text{ when } k \text{ is even and } p \text{ is odd} \\ \mathrm{O}(h^{k-1}) \text{ otherwise} . \end{cases}$$

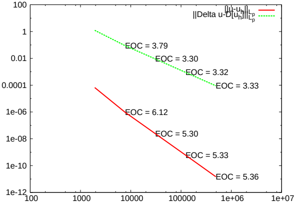
Figure: Benchmark test for the 3–biharmonic problem.



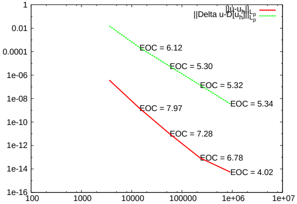
(a) $k = 2$, piecewise quadratic FEs.



(b) $k = 3$, piecewise cubic FEs.

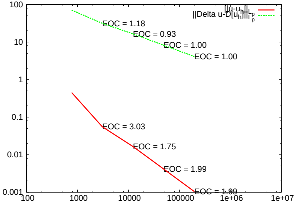


(c) $k = 4$, piecewise quartic FEs.



(d) $k = 6$, piecewise sextic FEs.

Figure: Benchmark test for the 4–biharmonic problem.



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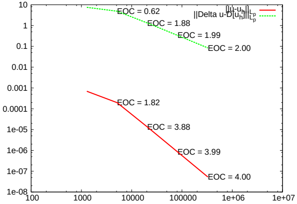

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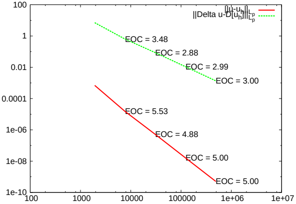

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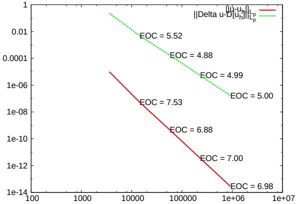

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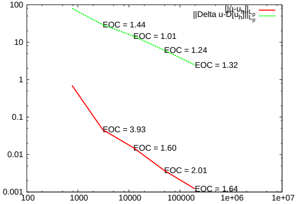
Figure: Benchmark test for the 5–biharmonic problem.



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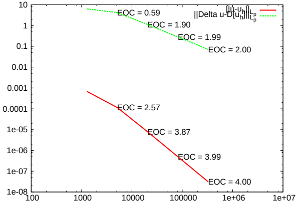

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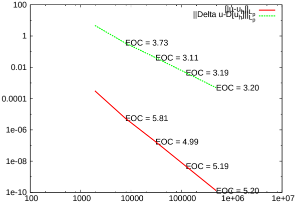

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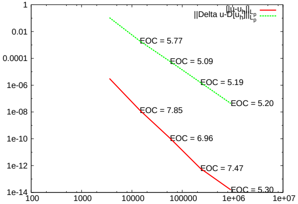

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Monge–Ampère equation

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lack of unique solution

Another is that they generally do not admit a unique solution. The Monge–Ampère equation admits two, one convex and one concave.

Monge–Ampère equation

The Monge–Ampère equation is a special example of a fully nonlinear PDE: Find $u \in \overset{\circ}{\mathrm{W}}{}^1_2(\Omega) \cap \mathrm{W}^2_2(\Omega)$ such that

$$\det \mathrm{D}^2 u = f$$

fully nonlinear PDEs are nonvariational

In general a fully nonlinear PDE is nonvariational in nature. This is one of the reasons why they are so difficult to construct numerical methods for.

lack of unique solution

Another is that they generally do not admit a unique solution. The Monge–Ampère equation admits two, one convex and one concave.

lack of classical solution

Even if the problem data is smooth the solution may need to be sought in a generalised sense (viscosity solution).

The Monge–Ampère equation IS variational

In fact, it is a degenerate variational problem. Let u be the unique convex (strong) solution to the Monge–Ampère equation, then it is also the unique convex minimiser to the variational problem

$$\mathscr{J}[u] = \inf_{v \in \mathring{\mathrm{W}}^1_2(\Omega) \cap \mathrm{W}^2_2(\Omega)} \mathscr{J}[v]$$

with

$$\mathscr{J}[u] = \int_\Omega \frac{u}{d+1} \left(\det \mathrm{D}^2 u - f\right)$$

It falls under the framework developed in [Lakkis and Pryer, 2013]

The discrete variational problem is to find $(u_h, \boldsymbol{H}[u_h]) \in \mathbb{V} \times \mathbb{V}^{d\times d}$ such that

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$$\mathscr{J}_h[u_h] := \int_\Omega \frac{u_h}{d+1}(\det \boldsymbol{H}[u_h] - f) + \sigma\left(\int_{\mathscr{E}\cup\partial\Omega} h^{-1} \left|[\![\nabla_h u_h]\!]\right|^2 + h^{-3} \left|[\![u_h]\!]\right|^2\right)$$

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Existing method

The weak Euler Lagrange equations to this problem coincide with a dG formulation of a method already constructed in [Lakkis and Pryer, 2013].

The affine maximal surface equation

Let u be a strictly convex function then it follows that $\det \mathrm{D}^2 u > 0$ and we may introduce a uniformly convex hypersurface $\mathscr{M}$ with Gaussian curvature κ. The affine area functional typically denoted in the literature as

$$\mathscr{A}[u] := \int_\Omega \left(\det \mathrm{D}^2 u\right)^{\frac{1}{d+2}} = \int_{\mathscr{M}_\Omega} \kappa^{\frac{1}{d+2}},$$

where

$$\mathscr{M}_\Omega := \{(\boldsymbol{x}, u(\boldsymbol{x})) \in \mathscr{M} | \boldsymbol{x} \in \Omega\}$$

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Euler Lagrange equations

The associated Euler–Lagrange equation is a nonlinear fourth order PDE

$$\mathrm{D}^2 \colon \left(\frac{\mathrm{Cof}\left(\mathrm{D}^2 u\right)}{\det(\mathrm{D}^2 u)^{\frac{(d+1)}{d+2}}} \right) = 0,$$

where $\mathrm{Cof}(\boldsymbol{B})$ denotes the matrix of cofactors of $\boldsymbol{B}$.

Properties of $\mathscr{A}[\cdot]$

- The affine area functional is (strictly) concave over the cone of convex functions, i.e.,

$$\mathscr{A}[\theta u + (1-\theta)\, v] \geq \theta \mathscr{A}[u] + (1-\theta)\, \mathscr{A}[v]$$

- The affine area functional is upper weakly semicontinuous, i.e.,, let $\{u_j\}_j$ be a sequence of convex functions converging uniformly to u then

$$\limsup_{j\to\infty} \mathscr{A}[u_j] \leq \mathscr{A}[u].$$

- The (weak) Euler Lagrange equation: Find $u \in \mathring{\mathrm{W}}^2_2(\Omega)$ such that

$$0 = \int_\Omega \left(\det \mathrm{D}^2 u\right)^{\frac{-(d+1)}{d+2}} \operatorname{Cof}\left(\mathrm{D}^2 u\right) : \mathrm{D}^2 \phi \quad \forall\, \phi \in \mathring{\mathrm{W}}^2_2(\Omega)$$

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Remark

Actually for surfaces representable as graphs the only smooth solutions to this problem are quadratic. We add a source term and the functional becomes

$$\mathscr{A}[u] = \int_\Omega \left(\det \mathrm{D}^2 u\right)^{\frac{1}{d+2}} - f u$$

The discrete variational problem

The properties of $\mathscr{A}[\cdot]$ allow us to set up the scheme as for the p–biharmonic problem: Find $u_h \in \mathbb{V}$ such that it is the maximiser of the discrete functional

$$\mathscr{A}_h[u_h] := \int_\Omega (\det \boldsymbol{H}[u_h])^{\frac{1}{d+2}} - f u_h + \sigma\left(\int_{\mathscr{E}\cup\partial\Omega} h^{-1} \left|[\![\nabla_h u_h]\!]\right|^2 + h^{-3} \left|[\![u_h]\!]\right|^2\right)$$

properties of the scheme

Analysis is almost complete: We have

- The discrete minimisation problem is coercive in

$$\|u_h\|_{dG,2}^2 := \left\|\mathrm{D}_h^2 u_h\right\|_{\mathrm{L}_2(\Omega)}^2 + h^{-1} \left\|[\![\nabla_h u_h]\!]\right\|_{\mathrm{L}_2(\mathscr{E})}^2 + h^{-3} \left\|[\![u_h]\!]\right\|_{\mathrm{L}_2(\mathscr{E})}^2$$

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- The discrete problem is well posed (for σ big enough)
- The following stability bound holds

$$\|\boldsymbol{H}[v_h]\|_{\mathrm{L}_2(\Omega)}^2 \leq C\,\|v_h\|_{dG,2}^2 \qquad \forall\, v_h \in \mathbb{V}.$$

Convergence

- Since the problem is well posed we have $\boldsymbol{H}[u_h]$ is strictly positive definite
- In this case we call $\{u_h, \boldsymbol{H}[u_h]\}$ a *finite element convex* sequence
- The limit of a finite element convex sequnce is convex
- We have the discrete solution converges to the convex solution of the problem strongly in $\mathrm{L}_2(\Omega)$

$$\lim_{h \to 0} u_h = u$$

- The discrete Hessian converges strongly in $\mathrm{L}_2(\Omega)$

$$\lim_{h \to 0} \boldsymbol{H}[u_h] = \mathrm{D}^2 u$$

computational observations

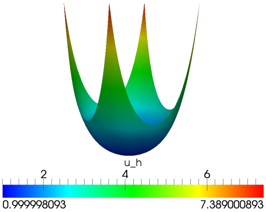
Computationally we observed that for smooth solutions

$$\|u - u_h\|_{L_2(\Omega)} = O(h^{k+1})$$

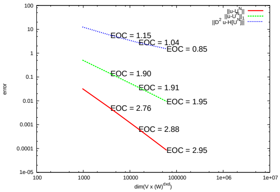
$$\|u - u_h\|_{W_2^1(\mathscr{T})} = O(h^k) \text{ and}$$

$$\left\|D^2 u - \boldsymbol{H}[u_h]\right\|_{L_2(\Omega)} = O(h^{k-1}).$$

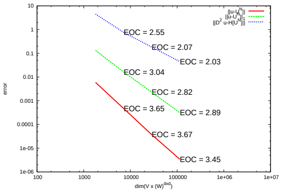
Figure: Benchmark test for the affine maximal surface problem. Smooth convex solution.



(a) Smooth convex solution $u \in \mathrm{C}^{\infty}(\Omega)$.



(b) $k = 2$, piecewise quadratic FEs.



(c) $k = 3$, piecewise cubic FEs.

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Outlook

1. A general framework for a certain class of 4th order problems?

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1. A general framework for a certain class of 4th order problems?

$$\mathrm{D}^2 \colon \left(\boldsymbol{A}(u, \mathrm{D}^2 u) \mathrm{D}^2 u \right) = f$$

2. Extending the method for hypersurfaces not representable by graphs. Using the method in a surface FEM context.

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