Robust Localization of the Best Error with Finite Elements in the Reaction Diffusion Norm

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Robust Localization on Pairs of Elements

Let $\Omega \subset \mathbb{R}^d$, $u \in H^1(\Omega)$ and $\mathcal{T}$ a conforming triangulation. The best error with the space of continuous piecewise polynomial functions $S := S^{\ell,0}(\mathcal{T}) = \{v \in C^0(\Omega) : \ v \in \mathbb{P}^\ell(K), \forall\, K \in \mathcal{T}\}$ wrt the reaction-diffusion norm

$$|||\cdot|||^2 := \|\cdot\|^2 + \varepsilon \|\nabla \cdot\|^2$$

satisfies

$$\inf_{v \in S} |||u - v|||_\Omega \le C \left(\sum_{E \in \mathcal{E}} \inf_{P \in S|_{\omega_{\mathcal{T}}(E)}} |||u - P|||^2_{\omega_{\mathcal{T}}(E)} \right)^{1/2},$$

where $\omega_{\mathcal{T}}(E)$ is the pair of elements that share the face E, and $\mathcal{E}$ is the set of all faces. The constant C depends on the shape-parameter of $\mathcal{T}$, but it is independent of ε

A Counterexample

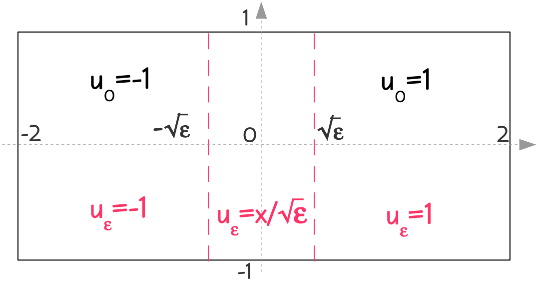
Let $\mathcal{P}_\mathcal{T}$ be the L^2-projection and $\mathcal{R}_\mathcal{T}$ the Ritz projection. On one hand

$$|||u_\varepsilon - \mathcal{R}_\mathcal{T} u_\varepsilon|||_\Omega \geq \|u_\varepsilon - \mathcal{P}_\mathcal{T} u_\varepsilon\|_{L^2(\Omega)} \overset{\varepsilon\to 0}{\to} \|u_0 - \mathcal{P}_\mathcal{T} u_0\|_{L^2(\Omega)} > 0$$

on the other-hand

$$\sum_{K\in\mathcal{T}} |||u_\varepsilon - \mathcal{R}_K u_\varepsilon|||_K^2 \leq \sum_{K\in\mathcal{T}} |||u_\varepsilon - u_0|||_K^2 \overset{\varepsilon\to 0}{\to} 0$$

Decoupling with local errors on elements implies that the constant grows with ε



Sketch of the Proof-1

We define a suitable interpolation operator $\Pi u := \sum_{z \in \mathcal{N}} u_z \phi_z$. For every K choose $E \subset \partial K$ and for every non-interior node $z \in \mathcal{N}$ choose an element K_z such that $z \in K_z$ and set

$$u_z = \begin{cases} \mathcal{R}_E u(z) & \text{if } z \in \mathcal{N}_{\overset{\circ}{K}}, \\ \int_{K_z} u \psi_z^{K_z} & \text{if } z \in \mathcal{N} \setminus (\cup_{K \in \mathcal{T}} \mathcal{N}_{\overset{\circ}{K}}). \end{cases}$$

where $\mathcal{R}_E$ is the Ritz-projection on $\omega_{\mathcal{T}}(E)$, and $\{\phi_z\}$, $\{\psi_z^K\}$ the nodal and the L^2-dual basis.

Sketch of the Proof-2

We split

$$|||u - \Pi u|||_\Omega^2 = \sum_{K \in \mathcal{T}} |||u - \Pi u|||_K^2$$

and insert $\mathcal{R}_E u$

$$|||u - \Pi u|||_K \leq |||u - \mathcal{R}_E u|||_K + |||\Pi u - \mathcal{R}_E u|||_K .$$

Since $\Pi u - \mathcal{R}_E u \in \mathbb{P}^\ell(K)$ we can write

$$|||\Pi u - \mathcal{R}_E u|||_K \leq \sum_{z \in \mathcal{N}_{\partial K}} \left| \mathcal{R}_E u(z) - \int_{K_z} u \psi_z^{K_z} \right| |||\phi_z|||_K .$$

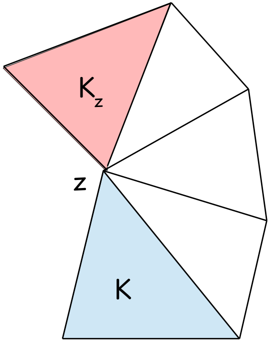
If $K_z = K$ we can bound separately

$$\left| \mathcal{R}_E u(z) - \int_K u \psi_z^K \right| \|\phi_z\|_K \lesssim \|\mathcal{R}_E u - u\|_{\omega_\mathcal{T}(E)} ,$$

$$\left| \mathcal{R}_E u(z) - \int_K u \psi_z^K \right| \sqrt{\varepsilon} \, \|\nabla \phi_z\|_K \lesssim \sqrt{\varepsilon} \, \|\nabla (\mathcal{R}_E u - u)\|_{\omega_\mathcal{T}(E)}$$

Sketch of the Proof-3

If $K_z \neq K$ we take a path that connects the two triangles. We need overlapping subdomains so that the part with the L^2-norm and the part with $\sqrt{\varepsilon}\,\|\nabla\cdot\|$ can still be bounded separately.



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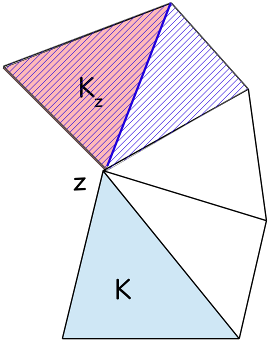

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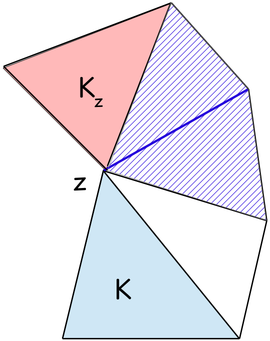

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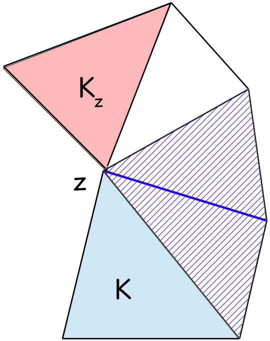

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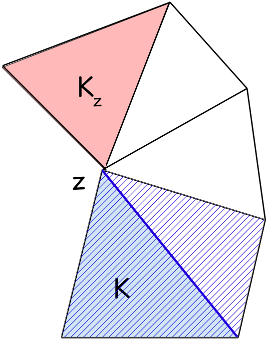


An Indicator for the Tree Approximation

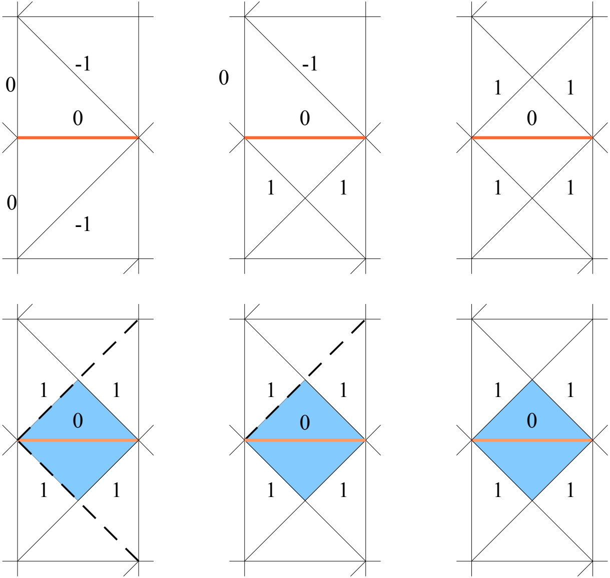
Our aim is to find an indicator suitable for the tree approximation algorithm of Binev-DeVore. It has to be independent of the triangulation. We take

$$e(K) := \sum_{E \in \partial K} \inf_{P \in S|_{\omega_\star(E)}} |||u - P|||^2_{\omega_\star(E)},$$

where $\omega_\star(E)$ is a minimal pair, defined as the intersection on all $\mathcal{T}_a$ admissible refinement of $\mathcal{T}_0$ of standard pairs

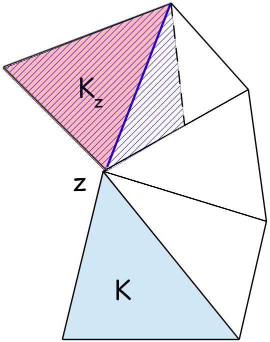
$$\omega_\star(E) := \bigcap_{\mathcal{T}_a} \omega_{\mathcal{T}_a}(E)$$

"Minimal" Pairs in the Context of Bisection



The minimal pairs satisfy:

- If $K \subseteq \omega_{\mathcal{T}}(E)$, we have that $\omega_\star(E) \cap K$ is an element of $\mathcal{T}$ or of some virtual refinement of $\mathcal{T}$, and $|\omega_\star(E) \cap K| \geq |K|/2$.
- For every $K \in \mathcal{T}$ there exists $E \in \mathcal{E}$, such that $\omega_\star(E) \supseteq K$.



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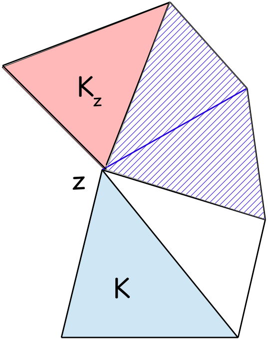

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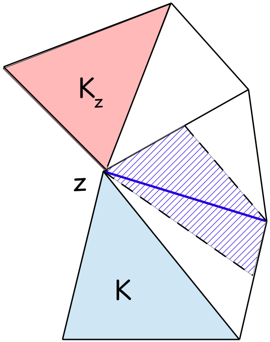

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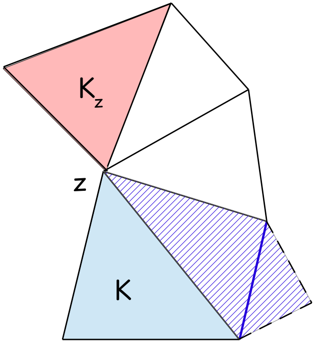

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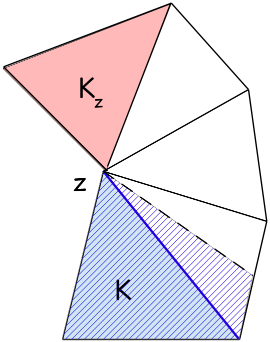

Robust Near-Best Approximations

Similarly as before we can prove

$$\inf_{v \in S} |||u - v|||_{\Omega} \leq C \left(\sum_{E \in \mathcal{E}} \inf_{P \in S|_{\omega_{\star}(E)}} |||u - P|||^2_{\omega_{\star}(E)} \right)^{\frac{1}{2}}$$

With the tree approximation we can thus compute robust near-best approximations, which may be used as benchmarks for e.g. Galerkin solutions.