A Locally Modified Parametric Finite Element Method for Interface Problems

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Overview

Fluid-Structure Interaction

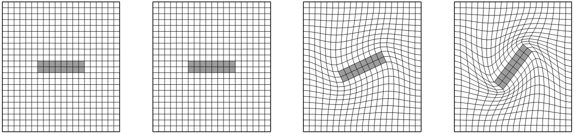
(FSI)

- Fluid governed by the incompressible Navier-Stokes Equations, typically formulated in Eulerian Coordinates
- Solid governed by a hyperelastic material law, typically formulated in Lagrangian Coordinates
- Moving interface

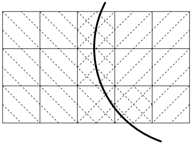
ALE method vs. Fully Eulerian Approach

Arbitrary Lagrangian Eulerian (ALE) method (Donea et al 1977, Hughes et al, 1981): Transform the fluid equations to an arbitrary reference frame that matches the solid's interface (*interface tracking*)

- Mesh degeneration for large deformation or movement of the solid
  $\Rightarrow$ Frequent remeshing required, contact impossible

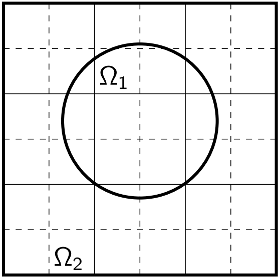


Fully Eulerian Approach (Dunne 2006: Transform the solid's equation to Eulerian Coordinates (*interface capturing*)



- Solid and interface move over mesh cells
- Problem: Interface is not resolved by the mesh
  $\Rightarrow$ Treatment of the interface cells?

Model Problem



Stationary model problem

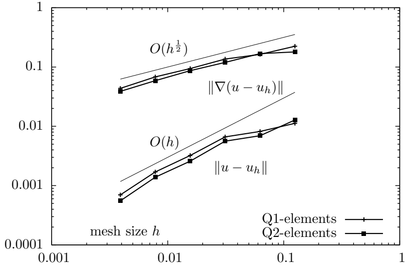
$$-\nabla \cdot (\kappa_i \nabla u) = f \text{ on } \Omega_i,\ i = 1, 2,$$
$$[u] = 0, \quad [\kappa \partial_n u] = 0 \text{ on } \Gamma,$$

where $\kappa_1 = 0.1$, $\kappa_2 = 1$.

∇u discontinuous across interface

Using Standard Finite Elements
$\nabla(u - u_h) = \mathcal{O}(1)$ in interface cells

$\Rightarrow$ Total energy norm error $\mathcal{O}(h^{1/2})$



Existing methods

Existing methods that recover optimal order

- **Harmonic Averaging** over diffusion constants: Tikhonov & Samarskii 1962, Shubin & Bell 1984

  ⇒ Not applicable for FSI problems

- **Fitted Finite Element Methods**: Babuška 1970, Feistauer & Sobotíková 1990, Ženíšek 1990, Bramble & King 1994, Xie et al 2008, Fang 2013

  ⇒ Remeshing required for moving interfaces

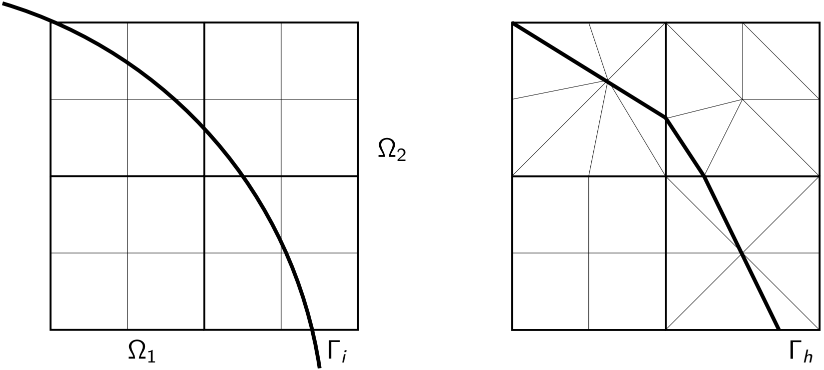
- Local Modification or Enrichment of the Finite Element Basis:
  Extended Finite Element Method (XFEM, Moës et al 1999, Hansbo & Hansbo 2002),
  Generalized Finite Element Method (GFEM, Babuška et al 1994, Strouboulis et al 2000)

  ⇒ Change in the number of unknowns and the structure/connectivity of the system matrix

Overview

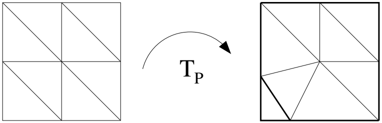
Idea

- Fixed patch mesh
- Split interface patches into eight triangles to resolve the interface



Locally Modified FEM

- Use patches as reference elements $T_P \in [Q_P]^2$

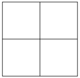


- Define a parametric finite element space on patches

$$V_h = \{\phi \in C(\bar{\Omega}),\ \phi \circ T_P^{-1} \in Q_P\}.$$

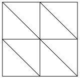
$\Rightarrow$ Inner mesh nodes are moved by means of the local transformation

- On regular patches we use piecewise Q_1 finite elements



$$Q_P = Q_1^{\text{patch}} = \{\phi \in C(\bar{P}),\ \phi\big|_{K_i} \in Q_1(K_i),\ i = 1, \dots, 4\}.$$

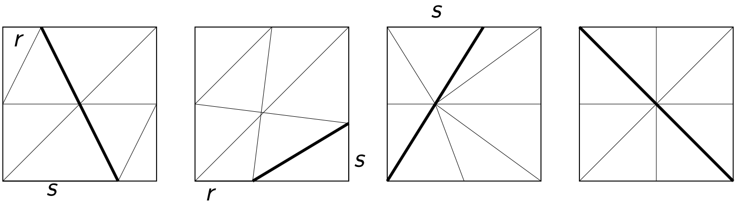
- On interface patches we use piecewise P_1 finite elements



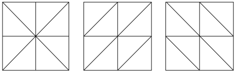
$$Q_P = P_1^{\text{tria}} = \{\phi \in C(\bar{P}),\ \phi\big|_{T_i} \in P_1(K_i),\ i = 1, \dots, 8\}.$$

Locally Modified FEM

- 4 different types of cut patches



- Two different types of underlying reference patches



- The subdivision can be anisotropic with $r, s \in (0, 1)$ arbitrary.
- Maximum angle condition: All interior angles smaller than 144°.

A priori error estimates

- Lagrange interpolation operator for $u \in H^1_0(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)$

$$\|u - L_h u\|_T + h_P \|\nabla(u - L_h u)\|_T \le C h_P^2 \|\nabla^2 u\|_T.$$

where h_P is the patch size.

Lemma 1: A priori estimate

Let Γ smooth with C^2-parametrization and assume

$$u \in H^1_0(\Omega) \cap H^2(\Omega_1 \cup \Omega_2), \quad \|u\|_{H^2(\Omega_1 \cup \Omega_2)} \le c_s \|f\|.$$

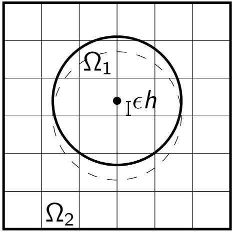
For the corresponding modified finite element solution $u_h \in V_h$ it holds

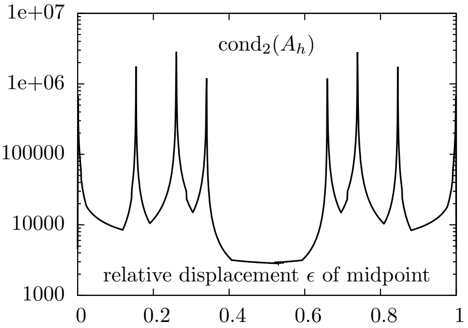
$$\|\nabla(u - u_h)\|_\Omega \le C h_P \|f\|, \quad \|u - u_h\|_\Omega \le C h_P^2 \|f\|$$

Overview

Condition Number

- Condition number of the stiffness matrix is unbounded for certain anisotropies
- Example: Model problem for different positions of the inner circle

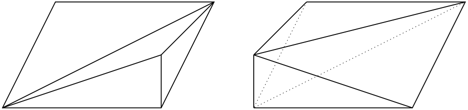




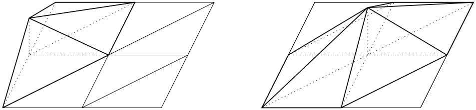
Condition Number

Modification of the Lagrange basis functions $\phi_1, ..., \phi_9$:

- (i) Split V_h in a hierarchical manner into V_{2h}



and $V_b = V_h \setminus V_{2h}$



- (ii) Scale each basis function $\tilde{\phi}_i$ such that

$$\|\tilde{\nabla}\phi_i\|_\Omega = \mathcal{O}(1)$$

In practice this can be achieved by a row- and columnwise scaling of the system matrix

Condition number

Lemma 2: Condition Number

Using the modified finite element basis described, the condition of the stiffness matrix remains bounded:

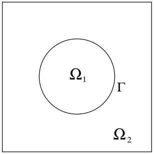
$$\mathrm{cond}_2(\mathbf{A}) \leq Ch^{-2},$$

with a constant $C > 0$ not depending on the interface location.

Proof: S.F., T.Richter: *A locally modified parametric finite element method for interface problems*, submitted (2013)

Overview

Model problem with analytical solution



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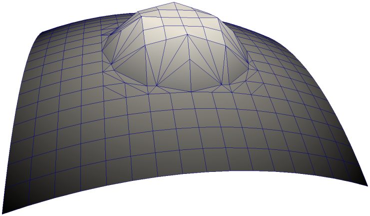
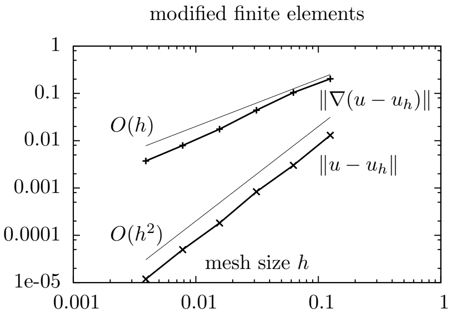
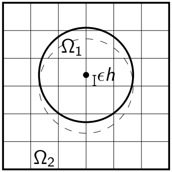
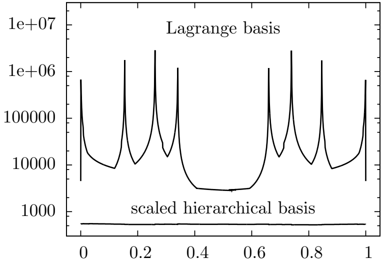




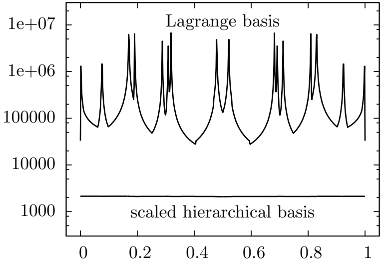
Condition Number

Condition number depending on the displacement of Ω_1 by ϵh for $\epsilon \in [0,1]$.



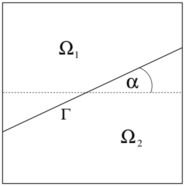


$h = 1/16$

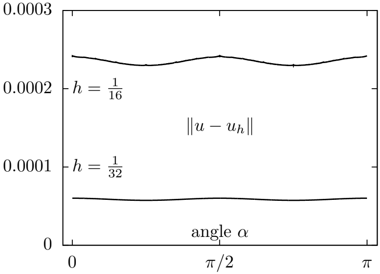


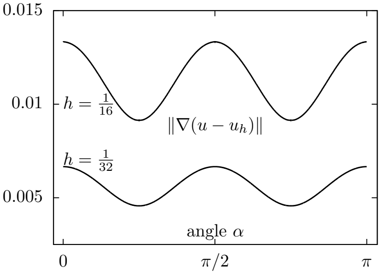
$h = 1/32$

Example 2: Dependence on Anisotropies



L^2- and H^1-norm errors for a tilted interface line depending on α for $0 < \alpha < \pi$





Application: Navier-Stokes Equations

(Navier-Stokes)

- Flow around a moving rigid obstacle
- Fluid governed by the incompressible Navier-Stokes Equations

Conclusion

We presented a new finite element scheme for interface problems that

- is of optimal order
- is easy to implement
- conserves the structure and connectivity of the system matrix
- avoids remeshing
- is suitable for moving interface problems.

- The condition of the system matrix is $\mathcal{O}(h^{-2})$.

Outlook

- Application to non-stationary problems with moving interfaces
- Nonstationary Fluid-Structure Interaction problems in Fully Eulerian Coordinates

Thank you for your attention !