# A Sparse Grid DG method for the Vlasov-Poisson System

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+ fruitful & inspiring discussions: B. Makridakis & M. Georgoulis

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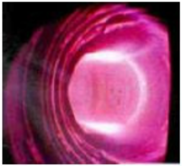
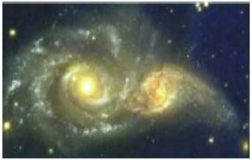
# Outline

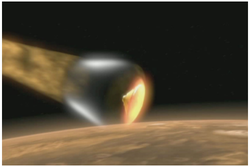
# Motivation: Kinetic Equations

**Aim:** Description of "dilute particle gases" at an *intermediate scale* between microscopic scale and the hydrodynamical scale

- "Dilute gases": Huge system of particles (each particle: $x$ and $v$)
  - ▷ *Statistical Description*







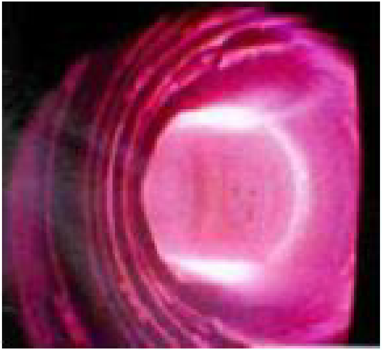
**Examples of "dilute gases":** Plasma Physics (electrons, neutrons, ions), Astrophysics (galaxies, stars), Economy (individuals), Re-entry of aircraft: (gas molecules), ...

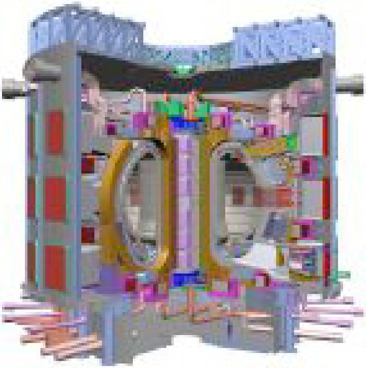
**Main object:** $f(t, x, v)$

▷ gives "probability" of finding particles in volume $dxdv$ at time $t$, around $(x, v)$

# Motivation: Plasma & ITER-project

- Fusion: can we produce clean & huge energy?
- transport of charged particles in Plasmas





**Tokomak**

(toroidal chamber with magnetic coils)

# Vlasov-Poisson system

## Mean field approximation

$$
\begin{cases}
\dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_x \Phi(t,x) \cdot \nabla_v f = 0 \\
-\Delta_x \Phi = \pm\rho \qquad (-\mathrm{div}_x F = \pm\rho) \\
f(0,x,v) = f_0(x,v)
\end{cases}
$$

$(x, v, t) \ \in \ \Omega_x \times \mathbb{R}^3 \times [0, T]$

- Electrostatic Vlasov-Poisson: transport of charged particles in plasmas $U(x) = -\frac{1}{4\pi}\frac{1}{|x|}$.
- Gravitating Vlasov-Poisson : $U(x) = \frac{1}{4\pi}\frac{1}{|x|}$
- Vlasov-Maxwell: $F(t,x) = E(t,x) + \frac{v}{c} \wedge B(t,x)$

- **Remark:** Real Simulation require solving 3D VP–VM (6 +1 dimensional)!!

# 1D Vlasov-Poisson System

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \Phi_x f_v = 0 \qquad (x,v) \in [0,1]\times[-L,L],\ t\in[0,T],$$

$$-\Phi_{xx} = \rho(x,t) - 1 \qquad (x,t) \in [0,1]\times[0,T],$$

$+$ initial condition $f(x,v,0) = f_0 \geq 0$ +bc: periodic in $x$

- [Cooper-Klimax (80)]: $\exists\ L > 0\ :\ f(x,v,t) \equiv 0$ for all $v \notin [-L,L]$

$$\rho(x,t) := \int_{[-L,L]} f(x,v,t)dv \quad : \quad \int_0^1 \rho(x,t)dx = 1$$

$$\boxed{\Omega = [0,1]\times[-L,L]} \quad \Omega_t = \Omega\times[0,t]$$

- Existence, Uniqueness & Regularity: ✓

[Jordanski (64), Cooper-Klimax (80), Glassey (96), . . .]

# Properties of Vlasov-Poisson System

- $f = f(x, v, t)$,
- $E := \Phi_x = \Phi_x(x, t)$

$$
\text{(VP-1D)} \quad \begin{cases} \bullet \quad \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} - E f_v = 0 & \text{in } \Omega_t, \\ \bullet \quad -\Phi_{xx} = \rho - 1 & \text{in } [0, 1] \times [0, T], \end{cases}
$$

- Positivity $f(x, v, t) \geq 0, \quad \forall x, v, t$
- mass conservation: $\displaystyle\int_\Omega f(x, v, t)dxdv = 1, \quad \forall t \qquad [\int_0^1 \rho dx = 1]$
- $L^p$-conservation: $\|f(t)\|_{L^p(\Omega)} = \|f(0)\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty.$
- Energy conservation: $\frac{d}{dt}\left(\int_\Omega f(x, v, t)|v|^2 dxdv + \int_{\mathcal{I}} [E(x, t)]^2 dx\right) = 0$

- Remark: A Good Numerical scheme should *enjoy* these properties

# Numerical Simulation - Approximation for VP

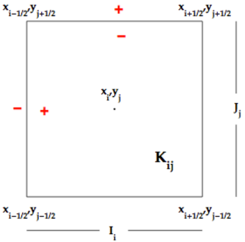
- Particle Methods: [Cottet & Raviart (84), Birdsall&Langdon (85),...]
  - **PIC** (and related) methods *noise in phase space*
- Eulerian Methods:
  - **Finite Volumes (FV):** [Filbet (2001),Crouseilles & Filbet (2004)]
    - *low order, restrictions on the mesh, ...*
  - Semi-Lagrangian Methods: [Sonnendrücker *et. al* (98-...), Besse (08-09)..]
    - *need of high order interpolation for origin of characteristic* $\longrightarrow$ *loss of local character of reconstruction, ..., expensive*
  - Discontinuous Galerkin:
    - Simulation: [Gamba et al(12), Rossmanith& Seal (12), BA&Hajian(...)]
    - Analysis: [B.A, Carrillo & Shu (11) & (12)]

- Our Aim: Tackle high dimensionality (if possible within DG)

# Basic Notation

- Partition $\mathcal{I}_h = \{I_i\}_{1\leq i\leq N_x}$ of $\mathcal{I} = (0,1)$ with $I_i = (x_{i-1/2}, x_{i+1/2})$
  $\mathcal{J}_h = \{J_i\}_{1\leq i\leq N_v}$ of $\mathcal{J} = (-L, L)$ with $J_j = (v_{j-1/2}, v_{j+1/2})$

- $\mathscr{T}_h = \mathcal{I}_h \otimes \mathcal{J}_h$ cartesian partition of $\Omega$ into elements $K$
- $\mathcal{Z}_h^k := \mathbb{Q}^k(\mathscr{T}_h)$




- Traces: $(\phi_h)^{\pm}_{i+1/2,v} := \phi_h(x^{\pm}_{i+1/2}, v)$, $\quad (\phi_h)^{\pm}_{x,j+1/2} := \phi_h(x, v^{\pm}_{i+1/2})$

- Average : $\{\phi_h\}_{i+1/2,v} := \frac{1}{2}((\phi_h)^{+}_{i+1/2,v} + (\phi_h)^{-}_{i+1/2,v})$
- Jump: $[\![\phi_h]\!]_{i+1/2,v} := (\phi_h)^{+}_{i+1/2,v} - (\phi_h)^{-}_{i+1/2,v}$