Trefftz-Discontinuous Galerkin Methods for Acoustic Scattering - Recent Advances

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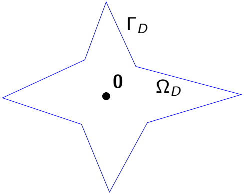
Outline

- Model problem
- Trefftz-discontinuous Galerkin methods
- Approximation properties of Trefftz FE spaces (Vekua's theory)
- Exponential convergence

Acoustic scattering problem

Exterior boundary value problem

- sound-soft scatterer: $\Omega_D \subset \mathbb{R}^3$
  - Lipschitz
  - polygonal
  - star-shaped w.r.t. the origin
- time-harmonic incident field with complex amplitude u^i and wave number $k = \omega/c$
- $u = u^i + u^s$



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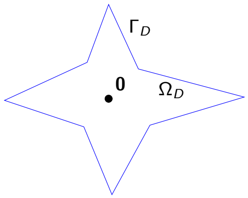


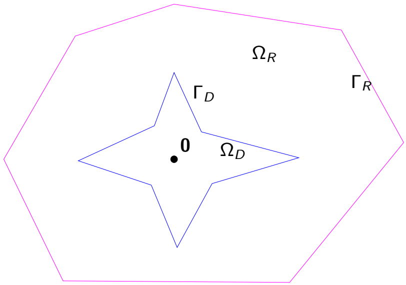
$$\begin{cases} -\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}_D & \text{(Helmholtz equation)} \\ u = 0 \quad \text{on } \Gamma_D := \partial\Omega_D & \text{(sound-soft obstacle)} \\ \lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left(\frac{\partial u^s(\boldsymbol{x})}{\partial |\boldsymbol{x}|} - iku^s(\boldsymbol{x}) \right) = 0 & \text{(radiation condition)} \end{cases}$$

Acoustic scattering problem

Artificial bounded domain

$\Omega_R \supset \overline{\Omega}_D$ with boundary Γ_R:

- $\text{dist}(\Gamma_D, \Gamma_R) > 0$
- either Γ_R smooth or Ω_R Lipschitz polyhedron
- Ω_R star-shaped w.r.t. a ball centered at the origin

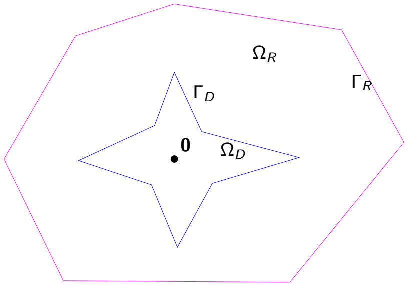


Acoustic scattering problem

Model problem

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega := \Omega_R \setminus \overline{\Omega}_D \\ u = 0 & \text{on } \Gamma_D \\ \nabla u \cdot n - ik\vartheta u = g_R & \text{in } \Gamma_R \end{cases}$$

($g_R \in L^2(\Gamma_R)$ impedance trace of u^i)



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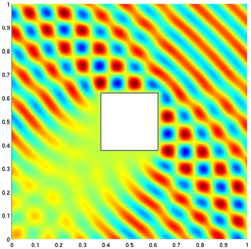

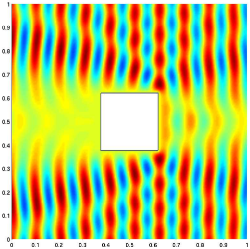

Trefftz-finite element methods

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \qquad (\mathcal{L} \text{ elliptic operator}) \\ + \text{ b.c.} \end{cases}$$

Erich Immanuel Trefftz (1888-1937)

Trefftz spaces for $\mathcal{L}$

Given a mesh $\mathcal{T}_h$ of Ω, for all $K \in \mathcal{T}_h$ we define the local Trefftz spaces

$$T(K) = \{v|_K \ : \ \mathcal{L}v = 0\}$$

and set

$$T(\mathcal{T}_h) = \{v \in L^2(\Omega) \ : \ v|_K \in T(K) \quad \forall K \in \mathcal{T}_h\}$$

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Trefftz finite element spaces

Let $V_p(K) \subset T(K)$ be finite dimensional local spaces; we set

$$V_p(\mathcal{T}_h) = \{v \in L^2(\Omega) \ : \ v|_K \in V_p(K) \quad \forall K \in \mathcal{T}_h\}$$

Trefftz-finite element methods

Examples of finite element Trefftz functions

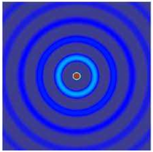
- Laplace operator: harmonic polynomials

Trefftz-finite element methods

Examples of finite element Trefftz functions

- Laplace operator: harmonic polynomials
- Helmholtz operator: plane waves, circular/spherical waves

$\boldsymbol{x} \mapsto e^{ik\boldsymbol{d}\cdot\boldsymbol{x}}$



Trefftz-finite element methods

How to "match" traces across interelement boundaries?

- Partition of unity [Babuška & Melenk, 1995-1997]
- Least squares [Stojek, 1998], [Monk & Wang, 1999], [Barnett & Betcke, 2010]
- Lagrange multipliers [Farhat, Harari & Franca, 2001], [Farhat, Tezaur, Hetmaniuk, Harari,..., 2003-]

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- Discontinuous Galerkin methods (UWVF) [Cessenat & Després, 1996-2003], [Huttunen, Kaipio, Luostari, Malinen & Monk, 2002-], [Gabard, 2007], [Buffa & Monk, 2008], [Gittelson, Hiptmair, Moiola & Perugia, 2009-]

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In this talk:

Trefftz discontinuous Galerkin (TDG) methods

TDG methods

Helmholtz equation: $-\Delta u - k^2 u = 0 \quad \text{in } \Omega$

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- Introduce a mesh $\mathcal{T}_h = \{K\}$; multiply by test functions and integrate by parts element by element twice (ultra weak formulation)

$$\int_K u\,(-\Delta\overline{v} - k^2\overline{v})\,\mathrm{d}V + \int_{\partial K} u\,\nabla\overline{v}\cdot n_K\,\mathrm{d}S - \int_{\partial K} \nabla u\cdot n_K\,\overline{v}\,\mathrm{d}S = 0$$

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- Replace continuous by discrete functions, traces by numerical fluxes

$$u, v \ \rightarrow\ u_{hp}, v_{hp}, \qquad \text{on } \partial K: \quad u \rightarrow \widehat{u}_{hp}, \quad \nabla u \rightarrow ik\widehat{\boldsymbol{\sigma}}_{hp}$$

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TDG formulation

For every $K \in \mathcal{T}_h$, $\displaystyle \int_{\partial K} \widehat{u}_{hp}\,\nabla \overline{v}_{hp}\cdot n_K\,\mathrm{d}S - \int_{\partial K} ik\widehat{\boldsymbol{\sigma}}_{hp}\cdot n_K\,\overline{v}_{hp}\,\mathrm{d}S = 0$

TDG methods

Numerical fluxes

$$ik\widehat{\boldsymbol{\sigma}}_{hp} = \begin{cases} \{\!\{\nabla_h u_{hp}\}\!\} - \alpha\, ik\, [\![u_{hp}]\!]_N & \text{on interior faces} \\ \nabla_h u_{hp} - \alpha\, ik\, u_{hp} n & \text{on faces on } \Gamma_D \\ \nabla_h u_{hp} - (1-\delta)\,(\nabla_h u_{hp} + ik\vartheta u_{hp} n - g_R n) & \text{on faces on } \Gamma_R \end{cases}$$

$$\widehat{u}_{hp} = \begin{cases} \{\!\{u_{hp}\}\!\} - \beta\,(ik)^{-1}[\![\nabla_h u_{hp}]\!]_N & \text{on interior faces} \\ 0 & \text{on faces on } \Gamma_D \\ u_{hp} - \delta\left((ik\vartheta)^{-1}\nabla_h u_{hp}\cdot n + u_{hp} - (ik\vartheta)^{-1} g_R\right) & \text{on faces on } \Gamma_R \end{cases}$$

with $\alpha, \beta > 0,\ \ 0 < \delta \le 1/2$

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- The choice $\alpha = \beta = \delta = 1/2$ gives the UWVF by Cessenat & Després (see [Gabard, 2007], [Buffa & Monk, 2008], [Gittelson, Hiptmair & Perugia, 2009]).

Trefftz-discontinuous Galerkin (TDG) methods

A priori error estimates

[Hiptmair, Moiola & Perugia, Appl. Num. Math. (2013)]

- regularity/stability of the dual solution ($\varphi \in H^{\frac{3}{2}+s}(\Omega)$ with $0 < s \leq 1/2$) in non star-shaped domains, with explicit dependence of the stability constant on k (extending the results of [Melenk, 1995], [Cummings & Feng, 2006], [Hetmaniuk, 2007])

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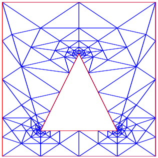
- flux parameters depending on the local meshsize
- *hp*-error analysis allowing for meshes which are locally refined near the scatterer



Trefftz-discontinuous Galerkin (TDG) methods

L^2-norm abstract error estimates

$$\|u - u_{hp}\|_{0,\Omega} \leq C(\Omega, k, h) \inf_{v_{hp} \in V_p(\mathcal{T}_h)} \|u - v_{hp}\|_{DG^+}$$

with $C(\Omega, k, h) = d_\Omega[(kh)^{-\frac{1}{2}} + (d_\Omega k)^{\frac{1}{2}}(d_\Omega^{-1} h)^s]$

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(best approximation)

Approximation properties of Trefftz FE spaces to be investigated

Approximation properties of Trefftz FE spaces

Vekua's theory for a 2nd order elliptic operator $\mathcal{L}$

$$\text{harmonic functions} \quad \underset{V_1}{\overset{V_2}{\rightleftarrows}} \quad \{v : \mathcal{L}v = 0\}$$

$$\text{harmonic polynomials} \quad \rightleftarrows \quad \text{generalized harmonic polynomials}$$

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Then: approximation of generalized harmonic polynomials in $V_p(\mathcal{T}_h)$

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Then: approximation of generalized harmonic polynomials in $V_p(\mathcal{T}_h)$

$V_p(\mathcal{T}_h)$ plane wave or circular/spherical wave spaces: sharp best approximation estimates of Helmholtz solutions in weighted H^r-norms

[Moiola, Hiptmair & Perugia, ZAMP (2011)]

Exponential convergence

hp-TDG on geometric meshes

We aim at proving that, if u is a solution of the acoustic scattering problem, then

$$\inf_{v_{hp} \in V_p(\mathcal{T}_h)} \|u - v_{hp}\|_{DG^+} \leq C\, exp(-b\sqrt{\#DOFs})$$

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Vekua's theory: transfer approximation properties of harmonic polynomials for harmonic functions to generalized harmonic polynomials for Helmholtz solutions

Exponential convergence

Approximation of harmonic functions by harmonic polynomials

[Hiptmair, Moiola, Perugia & Schwab, Tech. Rep. 2012]

Assuming

- $D \subset \mathbb{R}^2$
- $\exists 0 < \rho \leq 1/2$ s.t. D contains B_ρ
- $\exists 0 < \rho_0 < \rho$ s.t. D is star-shaped w.r.t. B_{ρ_0}

if u is harmonic and belongs to $W^{1,\infty}(D_\delta)$, then there is a sequence of harmonic polynomials $\{Q_p\}_p$ of degree at most p such that

$$|u - Q_p|_{W^{j,\infty}(D)} \leq C \exp(-bp) \|u\|_{W^{1,\infty}(D_\delta)}$$

Exponential convergence

TDG approximations of the Laplace equation on geometric meshes

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$\#DOFs$ piecewise polynomials of degree $\leq p$ on geometric meshes with ℓ layers

- full polynomials: $O(p^2\ell)$
- harmonic polynomials: $O(p\,\ell)$

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Exponential convergence for the Laplace equation

TDG for the Laplace equation on geometric meshes: exponential convergence with rate

$$\exp(-b\sqrt{\#DOFs})$$

(instead of $\exp(-b\sqrt[3]{\#DOFs})$)