Role of scattering in correlation-based imaging in random media

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- Principle of sensor array imaging:
  - probe an unknown medium with waves,
  - record the waves transmitted through or reflected by the medium,
  - process the recorded data to extract relevant information about some features of the medium.
Sensor array imaging through a homogeneous medium

- Sensor array imaging of a reflector located at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

  Measured data: $\{u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s\}$.

- Mathematical model:

  $$\left(\frac{1}{c_0^2} + \frac{1}{c_{\text{ref}}^2} 1_{B_{\text{ref}}} (\vec{x} - \vec{y})\right) \frac{\partial^2 u}{\partial t^2} (t, \vec{x}; \vec{x}_s) - \Delta_{\vec{x}} u(t, \vec{x}; \vec{x}_s) = f(t) \delta(\vec{x} - \vec{x}_s)$$

- Purpose of imaging: using the measured data, build an imaging function $I(\vec{y}^S)$ that would ideally look like $\frac{1}{c_{\text{ref}}^2} 1_{B_{\text{ref}}} (\vec{y}^S - \vec{y})$, in order to extract the relevant information $(\vec{y}, B_{\text{ref}}, c_{\text{ref}})$ about the reflector.
• Classical imaging functions:

1) Least-Squares imaging: minimize the quadratic misfit between measured data and synthetic data obtained by solving the wave equation with a candidate \((\vec{y}_{test}, B_{test}, c_{test})\).

2) Reverse Time imaging: simplify Least-Squares imaging by “linearization” of the forward problem.

3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.
• Classical imaging functions:

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2) Reverse Time imaging: simplify Least-Squares imaging by “linearization” of the forward problem.

3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

• Kirchhoff Migration function:

\[
I_{\text{KM}}(\vec{y}^S) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} u(T(\vec{x}_s, \vec{y}^S) + T(\vec{y}^S, \vec{x}_r), \vec{x}_r; \vec{x}_s)
\]

It forms the image with the superposition of the backpropagated traces.

\(T(\vec{y}^S, \vec{x})\) is the travel time from \(\vec{x}\) to \(\vec{y}^S\), i.e. \(T(\vec{y}^S, \vec{x}) = |\vec{y}^S - \vec{x}|/c_0\).

- Very robust with respect to measurement noise [1].
- Sensitive to clutter noise (scattering medium): If the medium is scattering, then Kirchhoff Migration (usually) does not work.

Imaging through a scattering medium

- Sensor array imaging of a reflector located at \( \vec{y} \). \( \vec{x}_s \) is a source, \( \vec{x}_r \) is a receiver.

Data: \( \{ u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s \} \).

\[
\left( \frac{1}{c^2(\vec{x})} + \frac{1}{c^2_{\text{ref}}} 1_{B_{\text{ref}}}(\vec{x} - \vec{y}) \right) \frac{\partial^2 u}{\partial t^2}(t, \vec{x}; \vec{x}_s) - \Delta \vec{x} u(t, \vec{x}; \vec{x}_s) = f(t) \delta(\vec{x} - \vec{x}_s)
\]

- Random medium model:

\[
\frac{1}{c^2(\vec{x})} = \frac{1}{c^2_0} \left( 1 + \mu(\vec{x}) \right)
\]

c_0 is a reference speed,

\( \mu(\vec{x}) \) is a zero-mean random process.
Imaging through a randomly scattering medium: strategy

- A detailed stochastic analysis is possible in different asymptotic regimes (small wavelength, large propagation distance, small correlation length, ...).
- In the limit the wave equation with random coefficients is replaced by a stochastic partial differential equation driven by Brownian fields; for instance, an Itô-Schrödinger equation in the paraxial regime.
- Stochastic calculus can then be used.

- Compute the mean and variance of an imaging function $I(\vec{y}^S)$.
  $\Rightarrow$ resolution and stability analysis.
- The mean imaging function $\vec{y}^S \rightarrow \mathbb{E}[I(\vec{y}^S)]$ characterizes the precision in the localization and characterization of the reflector (resolution).
- Criterium for statistical stability:

  $$\text{SNR} := \frac{\mathbb{E}[I(\vec{y}^S)]}{\text{Var}(I(\vec{y}^S))^{1/2}} > 1$$

  $\Rightarrow$ design the imaging function to get good trade-off between stability and resolution.
• General results obtained by a multiscale analysis.

• The mean wave is small while the wave fluctuations are large.
  \[ \Rightarrow \text{The Kirchhoff Migration function (or Reverse Time imaging function) is unstable in randomly scattering media.} \]

• The wave fluctuations at nearby points and nearby frequencies are correlated. The wave correlations carry information about the medium.
  \[ \Rightarrow \text{One can use local cross correlations for imaging.} \]

• More detailed results depend on the scattering regime.
Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{x} = (x, z)$)

$$
(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}(1 + \mu(x, z))\hat{u} = 0.
$$

Consider the paraxial regime “$\lambda \ll l_c \ll L$”:

$$
\omega \rightarrow \frac{\omega}{\varepsilon^4}, \quad \mu(x, z) \rightarrow \varepsilon^3 \mu\left(\frac{x}{\varepsilon^2}, \frac{z}{\varepsilon^2}\right).
$$

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$
\hat{u}^\varepsilon(\omega, x, z) = e^{i\frac{\omega z}{c_0}}\hat{\phi}^\varepsilon(\omega, \frac{x}{\varepsilon^2}, z)
$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0} \partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2} \frac{1}{\varepsilon} \mu\left(\frac{x}{\varepsilon^2}, \frac{z}{\varepsilon^2}\right) \hat{\phi}^\varepsilon\right) = 0.$$
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\]

- In the regime \( \varepsilon \ll 1 \), the forward-scattering approximation in direction \( z \) is valid and \( \hat{\phi} = \lim_{\varepsilon \rightarrow 0} \hat{\phi}^\varepsilon \) satisfies the Itô-Schrödinger equation [1]

\[
2i \frac{\omega}{c_0} \partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2} \dot{B}(x, z) \hat{\phi} = 0
\]

with \( B(x, z) \) Brownian field \( \mathbb{E}[B(x, z)B(x', z')] = \gamma(x - x') \min(z, z') \),

\[
\gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(x, z)] dz.
\]

Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{x} = (x, z)$)

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Consider the paraxial regime "$\lambda \ll l_c \ll L$":

\[ \omega \to \omega \varepsilon^4, \quad \mu(x, z) \to \varepsilon^3 \mu(\frac{x}{\varepsilon^2}, \frac{z}{\varepsilon^2}). \]

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

\[ \hat{u}^\varepsilon(\omega, x, z) = e^{i \frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon(\omega, \frac{x}{\varepsilon^2}, z) \]

satisfies

\[ \varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left( 2i \frac{\omega}{c_0} \partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2} \frac{1}{\varepsilon} \mu(x, \frac{z}{\varepsilon^2}) \hat{\phi}^\varepsilon \right) = 0. \]

- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction $z$ is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

\[ d\hat{\phi} = \frac{ic_0}{2\omega} \Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0} \hat{\phi} \circ dB(x, z) \]

with $B(x, z)$ Brownian field $\mathbb{E}[B(x, z)B(x', z')] = \gamma(x - x') \min(z, z')$, $\gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(x, z)] dz$.

Wave propagation in the random paraxial regime

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• In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction $z$ is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$
d\hat{\phi} = \frac{ic_0}{2\omega} \Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0} \hat{\phi} dB(x, z) - \frac{\omega^2 \gamma(0)}{8c_0^2} \hat{\phi} dz
$$

with $B(x, z)$ Brownian field $\mathbb{E}[B(x, z)B(x', z')] = \gamma(x - x') \min(z, z')$, $
\gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(x, z)]dz$.

• We introduce the fundamental solution \( \hat{G}(\omega, (x, z), (x_0, z_0)) \):

\[
d\hat{G} = \frac{ic_0}{2\omega} \Delta_{\perp} \hat{G} \, dz + \frac{i\omega}{2c_0} \hat{G} \circ dB(x, z)
\]

starting from \( \hat{G}(\omega, (x, z = z_0), (x_0, z_0)) = \delta(x - x_0) \).

• In a homogeneous medium \( (B \equiv 0) \) the fundamental solution is

\[
\hat{G}_0(\omega, (x, z), (x_0, z_0)) = \exp \left( \frac{i\omega |x - x_0|^2}{2c_0 |z - z_0|} \right)
\]

\[
\sigma \frac{2i\pi c_0 |z - z_0|}{\omega}
\]
We introduce the fundamental solution \( \hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) \):

\[
d\hat{G} = \frac{ic_0}{2\omega} \Delta_\perp \hat{G} dz + \frac{i\omega}{2c_0} \hat{G} \circ dB(\mathbf{x}, z)
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\[
\hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) = \frac{\exp\left(\frac{i\omega|\mathbf{x} - \mathbf{x}_0|^2}{2c_0|z - z_0|}\right)}{2i\pi c_0 |z - z_0| \omega}.
\]

In a random medium, by Itô’s formula

\[
\mathbb{E}[\hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0))] = \hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) \exp\left(-\frac{\gamma(\mathbf{0})\omega^2|z - z_0|}{8c_0^2} \right),
\]

where \( \gamma(\mathbf{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(\mathbf{x}, z)] dz \).

Strong damping of the mean wave.

\( \implies \) Reverse Time imaging and Kirchhoff migration fail.
• In a random medium, by Itô’s formula

\[
\mathbb{E}\left[ \hat{G}'(\omega, (x, z), (x_0, z_0)) \hat{G}(\omega, (x', z), (x_0, z_0)) \right] = \hat{G}_0(\omega, (x, z), (x_0, z_0)) \hat{G}_0(\omega, (x', z), (x_0, z_0)) \exp\left( - \frac{\gamma_2(x - x') \omega^2 |z - z_0|}{4c_0^2} \right),
\]

where \( \gamma_2(x) = \int_0^1 \gamma(0) - \gamma(xs) ds \) (note \( \gamma_2(0) = 0 \)).

• The fields at nearby points are correlated.

• Same results in frequency: The fields at nearby frequencies are correlated.

\( \implies \) One should migrate local cross correlations for imaging.
• In a random medium, by Itô’s formula
\[
\mathbb{E} [ \hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) \hat{G}(\omega, (\mathbf{x}', z), (\mathbf{x}_0, z_0)) ] \\
= \hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) \hat{G}_0(\omega, (\mathbf{x}', z), (\mathbf{x}_0, z_0)) \exp \left(- \frac{\gamma^2(\mathbf{x} - \mathbf{x}') \omega^2 |z - z_0|}{4c_0^2} \right),
\]
where \( \gamma^2(\mathbf{x}) = \int_0^1 \gamma(0) - \gamma(\mathbf{x}s) ds \) (note \( \gamma^2(0) = 0 \)).

• The fields at nearby points are correlated.

• Same results in frequency: The fields at nearby frequencies are correlated.
  \( \implies \) One should migrate local cross correlations for imaging.

• In a random medium, by Itô’s formula, one can write a closed-form equation for the \( n \)-th order moment.

  Depending on the statistics of the random medium, the wave fluctuations may have Gaussian statistics or not [1].

Application: Imaging below an “overburden”

Imaging below an “overburden”
From van der Neut and Bakulin (2009)
Array imaging of a reflector at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

Data: \( \{ u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s \} \).

If the “overburden” is scattering, then **Kirchhoff Migration** does not work:

\[
I_{KM}(\vec{y}^S) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} u(\mathcal{T}(\vec{x}_s, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_r), \vec{x}_r; \vec{x}_s)
\]
Numerical simulations

Computational setup

Image obtained with Kirchhoff Migration

(simulations carried out by Adrien Semin and Chrysoula Tsogka)
Imaging below an overburden

\( \vec{x}_s \) is a source, \( \vec{x}_r \) is a receiver. Data: \( \{ u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s \} \).

Image with Kirchhoff Migration of the cross correlation matrix:

\[
\mathcal{I}(y^S) = \sum_{r,r'=1}^{N_r} C(\mathcal{T}(\vec{x}_r, y^S) + \mathcal{T}(y^S, \vec{x}_{r'}), \vec{x}_r, \vec{x}_{r'}),
\]

with

\[
C(\tau, \vec{x}_r, \vec{x}_{r'}) = \sum_{s=1}^{N_s} \int u(t, \vec{x}_r; \vec{x}_s) u(t + \tau, \vec{x}_{r'}; \vec{x}_s) dt , \quad r, r' = 1, \ldots, N_r
\]
Numerical simulations

Kirchhoff Migration

cross correlation Migration

Heraklion September 19, 2013
Analysis in randomly scattering media

• Does the cross correlation imaging function give good images in scattering media?

→ It is possible to analyze the resolution and stability of the imaging function in randomly scattering media:
- analysis in the random paraxial regime,
- analysis in the randomly layered regime,
- analysis in the radiative transfer regime.

• General results:
Imaging function is stable provided the bandwidth is large enough and/or the source array is large enough.
Resolution is essentially independent of the size of the source array.

• Detailed results: Clarify the role of scattering.
- in the random paraxial regime, scattering helps (it enhances the angular diversity of the illumination).
- in the randomly layered regime, scattering does not help (it reduces the angular diversity of the illumination).

Assume that:

- the source aperture is $b$ and the receiver aperture is $a$.
- there is a point reflector at $\bar{y} = (y, -L_y)$.
- the covariance function $\gamma(x) = \int \mathbb{E}[\mu(0,0)\mu(x, z)] dz$ can be expanded as $\gamma(x) = \gamma(0) - \gamma_2 |x|^2 + o(|x|^2)$ for small $|x|$.
- scattering is strong: $\frac{\gamma(0) \omega_0^2 L}{c_0^2} > 1$ ($\rightarrow$ mean wave is damped).
Imaging below an overburden: analysis in the paraxial regime

Effective source aperture:

Homogeneous medium

$$b_{\text{eff}} = b$$

Random medium

$$b_{\text{eff}} = \left( b^2 + \bar{\gamma}_2 L^3 \right)^{1/2}$$
Imaging below an overburden: analysis in the paraxial regime

Homogeneous medium

Effective source aperture:

\[ b_{\text{eff}} = b \]

Effective receiver aperture:

\[ a_{\text{eff}} = b \frac{L_y - L}{L_y} \]

Random medium

Effective source aperture:

\[ b_{\text{eff}} = \left( b^2 + \frac{\bar{\gamma}_2 L^3}{3} \right)^{1/2} \]

Effective receiver aperture:

\[ a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y} \]
Imaging below an overburden: analysis in the paraxial regime

- The imaging function for the search point $\bar{y}^S$ is

$$I(\bar{y}^S) = \frac{1}{N_r^2} \sum_{r,r' = 1}^{N_r} C(\mathcal{T}(\bar{x}_r, \bar{y}^S) + \mathcal{T}(\bar{y}^S, \bar{x}_r'), \bar{x}_r, \bar{x}_r')$$

- The imaging function is statistically stable ($\lambda \ll b \ll L$).

- The lateral resolution is $\frac{\lambda_0(L_y - L)}{a_{\text{eff}}}$. The range resolution is $\frac{c_0}{B}$.

- Since $a_{\text{eff}} |_{\text{rand}} > a_{\text{eff}} |_{\text{homo}}$, this shows that scattering helps.
  - physical reason: scattering enhances the angular diversity of the illumination.
  - effect already noticed for time-reversal experiments, in which the recorded waves are time-reversed and sent back in the real medium.

Heraklion

September 19, 2013
Randomly layered medium

• Random medium model \( (\vec{x} = (x, z)) \):
  \[
  \frac{1}{c^2(\vec{x})} = \frac{1}{c_0^2}(1 + \mu(z))
  \]
  
  \( c_0 \) is a reference speed,
  \( \mu(z) \) is a zero-mean random process.

• Consider the time-harmonic form of the scalar wave equation \( (\vec{x} = (x, z)) \)
  \[
  (\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}(1 + \mu(z))\hat{u} = 0
  \]

Consider the scaled regime “\( l_c \ll \lambda \ll L \)”: 
  \[
  \omega \to \frac{\omega}{\varepsilon}, \quad \mu(z) \to \mu\left(\frac{z}{\varepsilon^2}\right)
  \]

The moments of the random Green’s function are known in the limit \( \varepsilon \to 0 \) [1].

Imaging below an overburden: analysis in the layered regime

- Assume that:
  - the source aperture is $b$ and the receiver aperture is $a$.
  - there is a point reflector at $\vec{y} = (\vec{y}, -L_y)$.
  - the localization length $L_{loc}$ is smaller than $L$ (strong scattering):

\[
L_{loc} = \frac{4c_0^2}{\gamma \omega_0^2}, \quad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz
\]
Imaging below an overburden: analysis in the layered regime

Effective source aperture:

Homogeneous medium

\[ b_{\text{eff}} = b \]

Randomly layered medium

\[ b_{\text{eff}}^2 = 4L_{\text{loc}}L \quad (\ll b^2) \]
Imaging below an overburden: analysis in the layered regime

Homogeneous medium

Randomly layered medium

Effective source aperture:

\[ b_{\text{eff}} = b \]

\[ b_{\text{eff}}^2 = 4L_{\text{loc}}L \ (\ll b^2) \]

Effective receiver aperture:

\[ a_{\text{eff}} = b \frac{L_y - L}{L_y} \]

\[ a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y} \]
Imaging below an overburden: analysis in the layered regime

- The imaging function for the search point $\vec{y}^S$ is

$$I(\vec{y}^S) = \frac{1}{N^2_r} \sum_{r,r'} C(T(\vec{x}_r, \vec{y}^S) + T(\vec{y}^S, \vec{x}_{r'}), \vec{x}_r, \vec{x}_{r'})$$

- The imaging function is statistically stable ($\lambda \ll b, L$).

- The lateral resolution is $\frac{\lambda_0 (L_y - L)}{a_{\text{eff}}}$. The range resolution is $\frac{c_0}{B} \left(1 + \frac{B^2 L}{4\omega_0^2 L_{\text{loc}}}ight)^{1/2}$.

- Since $a_{\text{eff}} |_{\text{rand}} < a_{\text{eff}} |_{\text{homo}}$, this shows that scattering does not help.
  - physical reason: scattering reduces the angular (and frequency) diversity of the illumination.
Further results

• Use of other imaging functions based on cross-correlations (or Wigner distribution functions).

• Use of ambient noise sources.
One can apply correlation-based imaging techniques to signals emitted by ambient noise sources.
→ Volcano monitoring (early warning of the eruption of Le Piton de la Fournaise in October 2010).
→ Passive reflector imaging.

• Use of higher-order correlations.
One can apply imaging techniques based on special fourth-order cross correlations. Useful when the statistics of the wave fluctuations is not Gaussian.
Passive sensor imaging of a reflector

- Ambient noise sources (○) emit stationary random signals.
- The signals \( u(t, \vec{x}_r) \) are recorded by the receivers \( (\vec{x}_r)_{r=1,...,N_r} \) (▲).
- The cross correlation matrix is computed and migrated:

\[
\mathcal{I}(\bar{y}^S) = \sum_{r, r' = 1}^{N_r} C_T(\mathcal{T}(\bar{x}_{r'}, \bar{y}^S) + \mathcal{T}(\bar{x}_r, \bar{y}^S), \bar{x}_r, \bar{x}_{r'})
\]

with \( C_T(\tau, \bar{x}_r, \bar{x}_{r'}) = \frac{1}{T} \int_0^T u(t + \tau, \bar{x}_{r'})u(t, \bar{x}_r)dt \)

Provided the ambient noise illumination is long (in time) and diversified (in angle): good stability [1].

Conclusions

- In scattering media one should migrate *well chosen* cross correlations of data, not data themselves.
- Method can be applied with ambient noise sources instead of controlled sources.
- Scattering can help! Already noticed for time-reversal experiments, but far from clear in imaging problems.