Recent Progress on the Search of 3D Euler Singularities

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1 Introduction
   - The Basic Problem and Previous Work
   - Our Discoveries: Existence of a Finite-Time Singularity

2 Numerical Results
   - Outline of the Method
   - Overview and First Sign of Singularity
   - Resolution Study
   - Confirming the Singularity I: Maximum Vorticity
   - Confirming the Singularity II: Vorticity Moments
   - Confirming the Singularity III: Vorticity Directions
   - Confirming the Singularity IV: Local Self-Similarity
Introduction

The Basic Problem and Previous Work

Our Discoveries: Existence of a Finite-Time Singularity

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Confirming the Singularity IV: Local Self-Similarity
The Basic Problem

- Problem to be studied: the 3D Euler equations for ideal incompressible flows:

\[ u_t + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0, \]

where
- \( u = (u_1, u_2, u_3)^T \): the 3D velocity vector
- \( p \): the scalar pressure

- The grand open problem: existence or nonexistence of global regular solutions from smooth initial data
- Closely related to one of the seven Clay Millennium Problems
Significance of the Problem

Why important?

- mathematically: problem remained open for more than 250 years
- physically: singularity in inviscid flows may (i) signify the onset of turbulence in viscous flows, and (ii) be a mechanism for energy transfer to small scales
- numerically: resolution of nearly singular flows presents a great challenge to computational fluid dynamics
Previous Work

On the theoretical side:
- Kato (1972): local well-posedness
- Constantin-Fefferman-Majda (1996): geometric constraints for blowup
- Deng-Hou-Yu (2005): Lagrangian localized geometric constraints

Other related work:
- Constantin-Majda-Tabak (1994): 2D quasi-geostrophic (QG) equations as a model for 3D Euler
- Cordoba (1998): no blowup of 2D QG near a hyperbolic saddle
Previous Work (Cont’d)

On the numerical search for singularity:

- Grauer and Sideris (1991): first numerical study of axisymmetric flows with swirl; blowup reported away from the axis
- Pumir and Siggia (1992): axisymmetric flows with swirl; blowup reported away from the axis
- Kerr (1993): antiparallel vortex tubes; blowup reported
- E and Shu (1994): 2D Boussinesq; no blowup observed
- Boratav and Pelz (1994): viscous simulations using Kida’s high-symmetry initial condition; blowup reported
- Grauer et al. (1998): perturbed vortex tube; blowup reported
- Hou and Li (2006): repetition of Kerr’s (1993) computation with higher resolution; no blowup observed
- Orlandi and Carnevale (2007): Lamb dipoles; blowup reported

Evidence for blowup is inconclusive and problem remains open
Main Difficulties

- On the theoretical side:
  - for global regularity: lack of appropriate controlling norms
  - for finite-time singularity: lack of effective lower-bound estimates
- On the numerical search for singularity:
  - generation of intense small scales from general initial data
  - lack of sufficient numerical resolution
  - lack of well-established procedure for confirmation of singularity
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Our Discoveries

After a series of careful numerical studies, we discover a class of initial data that lead to potentially singular solutions of the 3D Euler equations.

Main features of our study:
- focuses on solutions with axis-symmetry both for reduced computational complexity and for a better chance of singularity
- devises highly effective algorithms for adequate resolution
- employs rigorous criteria for confirmation of singularity
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The Equations

Equations being solved: the 3D axisymmetric Euler (Hou-Li-2008)

\[ u_{1,t} + u_r u_{1,r} + u_z u_{1,z} = 2 u_1 \psi_{1,z}, \]
\[ \omega_{1,t} + u_r \omega_{1,r} + u_z \omega_{1,z} = (u_1^2)_z, \]
\[-\left[ \partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2 \right] \psi_1 = \omega_1,\]

where

- \( u_1 = u^\theta / r, \ \omega_1 = \omega^\theta / r, \ \psi_1 = \psi^\theta / r \): the transformed angular velocity/vorticity/stream function
- \( u^r = -r \psi_{1,z}, \ \ u^z = 2 \psi_1 + r \psi_{1,r} \): the radial/axial velocity components
Computational Setup

- Equations numerically solved in the cylinder

\[ D = \left\{ (r, z) : 0 \leq r \leq 1, \ 0 \leq z \leq L \right\}, \]

with
- carefully chosen initial data
- periodic boundary conditions in \( z \)
- no-flow boundary condition at the wall \( r = 1 \)
Numerical Method

- Discretization in space: a hybrid 6th-order Galerkin and 6th-order finite difference method, on an adaptive (moving) mesh that is dynamically adjusted to the evolving solution.
- Discretization in time: an explicit 4th-order Runge-Kutta method, with an adaptively chosen time step.
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Overview of the Results

- Solution computed using 5 mesh resolutions, with mesh size ranging from $1024 \times 1024$ to $2048 \times 2048$
- Solution advanced indefinitely in time until either
  - the time step drops below $10^{-12}$, or
  - the minimum mesh spacing in $r$ drops below $\epsilon_r = 10^{-15}$, or
  - the minimum mesh spacing in $z$ drops below $\epsilon_z = 10^{-15}L$, whichever happens first
Stop Time

Table: Stop time $t_e$ and cause of termination for each computation, where $\delta_r$, $\delta_z$ denotes the minimum mesh spacing in $r$ and $z$, respectively.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t_e$</th>
<th>Cause of termination</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024 × 1024</td>
<td>0.00350556672</td>
<td>$\delta_r &lt; \epsilon_r$ and $\delta_z &lt; \epsilon_z$</td>
</tr>
<tr>
<td>1280 × 1280</td>
<td>0.00350555820</td>
<td>$\delta_z &lt; \epsilon_z$</td>
</tr>
<tr>
<td>1536 × 1536</td>
<td>0.00350555229</td>
<td>$\delta_z &lt; \epsilon_z$</td>
</tr>
<tr>
<td>1792 × 1792</td>
<td>0.00350555231</td>
<td>$\delta_r &lt; \epsilon_r$ and $\delta_z &lt; \epsilon_z$</td>
</tr>
<tr>
<td>2048 × 2048</td>
<td>0.00350554720</td>
<td>$\delta_r &lt; \epsilon_r$ and $\delta_z &lt; \epsilon_z$</td>
</tr>
</tbody>
</table>
Time Step

Table: Time step $\delta$ at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0^\dagger$</th>
<th>$t = 0.0034$</th>
<th>$t = 0.003505$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$1 \times 10^{-6}$</td>
<td>$4.9502 \times 10^{-8}$</td>
<td>$2.4240 \times 10^{-10}$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$1 \times 10^{-6}$</td>
<td>$3.9636 \times 10^{-8}$</td>
<td>$2.5772 \times 10^{-10}$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$1 \times 10^{-6}$</td>
<td>$3.2907 \times 10^{-8}$</td>
<td>$2.2223 \times 10^{-10}$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$1 \times 10^{-6}$</td>
<td>$2.8451 \times 10^{-8}$</td>
<td>$1.9122 \times 10^{-10}$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$1 \times 10^{-6}$</td>
<td>$2.4046 \times 10^{-8}$</td>
<td>$2.0272 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

$^\dagger$: The maximum time step allowed in our computations is $10^{-6}$. 
### Maximum Vorticity

**Table:** Maximum vorticity $\|\omega\|_\infty = \|\nabla \times u\|_\infty$ at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0$</th>
<th>$t = 0.0034$</th>
<th>$t = 0.003505$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$3.7699 \times 10^3$</td>
<td>$4.3127 \times 10^6$</td>
<td>$1.2416 \times 10^{12}$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$3.7699 \times 10^3$</td>
<td>$4.3127 \times 10^6$</td>
<td>$1.2407 \times 10^{12}$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$3.7699 \times 10^3$</td>
<td>$4.3127 \times 10^6$</td>
<td>$1.2403 \times 10^{12}$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$3.7699 \times 10^3$</td>
<td>$4.3127 \times 10^6$</td>
<td>$1.2401 \times 10^{12}$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$3.7699 \times 10^3$</td>
<td>$4.3127 \times 10^6$</td>
<td>$1.2401 \times 10^{12}$</td>
</tr>
</tbody>
</table>
Figure: The double logarithm of the maximum vorticity, $\log(\log \|\omega\|_\infty)$, computed on the $1024 \times 1024$ and the $2048 \times 2048$ mesh.
Observations

- Early signs of a looming singularity:
  - sharp decrease in time step and minimum mesh size in $r$ and $z$.
  - super-double-exponential growth of the maximum vorticity
- Singularity appears to be anisotropic in $(x, y, z)$, a thin tube-like singularity, but is isotropic as a function in the $(r, z)$-plane.
Effectiveness of the Adaptive Mesh

To assess the effectiveness of the adaptive mesh, define
- $p_\infty, q_\infty$: the mesh scaling ratios in $z$ and $r$
- $M_\infty, N_\infty$: the effective mesh resolutions in $z$ and $r$

near the location of the maximum vorticity
Mesh Scaling Ratios

**Table:** Mesh scaling ratios $p_\infty$, $q_\infty$ near the maximum vorticity at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.0034$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$6.7587 \times 10^2$ 7.6302 $\times 10^2$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$6.7632 \times 10^2$ 7.6316 $\times 10^2$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$6.7562 \times 10^2$ 7.6389 $\times 10^2$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$6.7621 \times 10^2$ 7.6391 $\times 10^2$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$6.7596 \times 10^2$ 7.6356 $\times 10^2$</td>
</tr>
</tbody>
</table>
Mesh Scaling Ratios (Cont’d)

**Table:** Mesh scaling ratios $p_\infty$, $q_\infty$ near the maximum vorticity at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.003505$</th>
<th>$p_\infty$</th>
<th>$q_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024 × 1024</td>
<td>1.9456 × 10^9</td>
<td>1.6316 × 10^9</td>
<td></td>
</tr>
<tr>
<td>1280 × 1280</td>
<td>1.9530 × 10^9</td>
<td>1.6285 × 10^9</td>
<td></td>
</tr>
<tr>
<td>1536 × 1536</td>
<td>1.9444 × 10^9</td>
<td>1.6328 × 10^9</td>
<td></td>
</tr>
<tr>
<td>1792 × 1792</td>
<td>1.9504 × 10^9</td>
<td>1.6344 × 10^9</td>
<td></td>
</tr>
<tr>
<td>2048 × 2048</td>
<td>1.9503 × 10^9</td>
<td>1.6330 × 10^9</td>
<td></td>
</tr>
</tbody>
</table>
Effective Mesh Resolutions

**Table:** Effective mesh resolutions $M_\infty$, $N_\infty$ near the maximum vorticity at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.0034$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_\infty$</td>
</tr>
<tr>
<td>1024 × 1024</td>
<td>$6.9210 \times 10^5$</td>
</tr>
<tr>
<td>1280 × 1280</td>
<td>$8.6569 \times 10^5$</td>
</tr>
<tr>
<td>1536 × 1536</td>
<td>$1.0378 \times 10^6$</td>
</tr>
<tr>
<td>1792 × 1792</td>
<td>$1.2118 \times 10^6$</td>
</tr>
<tr>
<td>2048 × 2048</td>
<td>$1.3844 \times 10^6$</td>
</tr>
</tbody>
</table>
Effective Mesh Resolutions (Cont’d)

**Table:** Effective mesh resolutions $M_\infty$, $N_\infty$ near the maximum vorticity at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.003505$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_\infty$</td>
</tr>
<tr>
<td>$1024 \times 1024$</td>
<td>$1.9923 \times 10^{12}$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$2.4999 \times 10^{12}$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$2.9866 \times 10^{12}$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$3.4951 \times 10^{12}$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$3.9942 \times 10^{12}$</td>
</tr>
</tbody>
</table>
Resolution Power of the Mesh

Figure: $\omega_1$ as a function of $r$ along $z = 2.22 \times 10^{-12}$ (the line passing through $\|\omega_1\|_\infty$) at $t = 0.003505$, computed on the $1024 \times 1024$ mesh.
Resolution Power of the Mesh (Cont’d)

$\omega_1(r, 2.22 \times 10^{-12})$ on 1024$^2$ mesh, $t = 0.003505$

Figure: A zoom-in view of $\omega_1(r, 2.22 \times 10^{-12})$. 
Resolution Power of the Mesh (Cont’d)

Figure: $\omega_1$ as a function of $z$ along $r = 1$ (the line passing through $\|\omega_1\|_\infty$) at $t = 0.003505$, computed on the $1024 \times 1024$ mesh.
Resolution Power of the Mesh (Cont’d)

Figure: A zoom-in view of $\omega_1(1, z)$.
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Resolution Study: Why and How

- Resolution study: critical for quality control
- "Traditional" error indicators used in Euler computations:
  - energy conservation
  - enstrophy and enstrophy production rate
  - energy spectra (for periodic problems)
  - conservation of circulation (recently proposed)
- None of them is adequate for singularity detection!
  - they are global error measures, but
  - singularity is typically a local property
The “Correct” Error Indicator

- Our opinion: the only error indicator suitable for singularity detection is the pointwise (sup-norm) error of vorticity.
- Computation of the sup-norm relative error of a numerical solution $v_N$, defined on an $N \times N$ mesh:
  1. compute a “reference solution” $\hat{v}$ on a finer mesh
  2. interpolate $\hat{v}$ to the (coarse) mesh on which $v_N$ is defined
  3. compute the maximum difference between the two solutions
  4. divide the result by the maximum of $|\hat{v}|$ to yield the error $e_N$
- Define the numerical order of convergence:

$$\beta_N = \log_k \left( \frac{e_N/k}{e_N} \right), \quad k > 1$$
Resolution Study for Our Computations

- We check the accuracy of our solutions in five steps:
  - code validation on test problems
  - resolution study on primitive variables (angular velocity/vorticity/stream function)
  - resolution study on vorticity vector
  - resolution study on global quantities (energy, enstrophy, enstrophy production, circulation, maximum vorticity)
  - resolution study in time

- Conclusion: solutions well resolved up to and including \( t = 0.003505 \) (recall \( t_e \approx 0.00350555 \))
Resolution Study on Vorticity Vector

![Graph showing sup-norm relative error of the vorticity vector $\omega$.](image)

**Figure:** Sup-norm relative error of the vorticity vector $\omega$. The last time instant shown in the figure is $t = 0.003504$. 

- **Resolution Study**
  - Considerations on the vorticity vector $\omega$.

**Numerical Results**

- Analysis of the sup-norm relative error $\epsilon$. 

- Data points for different resolutions:
  - $1024 \times 1024$
  - $1280 \times 1280$
  - $1536 \times 1536$
  - $1792 \times 1792$

- Graph illustrating the error over time $t$ in $10^{-3}$ units.
Resolution Study on Vorticity Vector (Cont’d)

Figure: Numerical order in sup-norm of the vorticity vector $\omega$. The last time instant shown in the figure is $t = 0.003504$. 
### Resolution Study on Primitive Variables

**Table:** Sup-norm relative error and numerical order of convergence of the transformed primitive variables $u_1$ at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.003505$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024 × 1024</td>
<td></td>
<td>9.4615 × 10^{-6}</td>
<td>–</td>
</tr>
<tr>
<td>1280 × 1280</td>
<td></td>
<td>3.6556 × 10^{-6}</td>
<td>4.2618</td>
</tr>
<tr>
<td>1536 × 1536</td>
<td></td>
<td>1.5939 × 10^{-6}</td>
<td>4.5526</td>
</tr>
<tr>
<td>1792 × 1792</td>
<td></td>
<td>7.5561 × 10^{-7}</td>
<td>4.8423</td>
</tr>
<tr>
<td>Sup-norm</td>
<td>1.0000 × 10^2</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
Resolution Study on Primitive Variables (Cont’d)

Table: Sup-norm relative error and numerical order of convergence of the transformed primitive variables $\omega_1$ at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.003505$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$6.4354 \times 10^{-4}$</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$2.4201 \times 10^{-4}$</td>
<td>$4.3829$</td>
<td></td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$1.1800 \times 10^{-4}$</td>
<td>$3.9396$</td>
<td></td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$6.4388 \times 10^{-5}$</td>
<td>$3.9297$</td>
<td></td>
</tr>
<tr>
<td>Sup-norm</td>
<td>$1.0877 \times 10^6$</td>
<td>$-$</td>
<td></td>
</tr>
</tbody>
</table>
Resolution Study on Primitive Variables (Cont’d)

Table: Sup-norm relative error and numerical order of convergence of the transformed primitive variables $\psi_1$ at selected time $t$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$t = 0.003505$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$2.8180 \times 10^{-10}$</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$4.7546 \times 10^{-11}$</td>
<td>$7.9746$</td>
<td></td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$1.0873 \times 10^{-11}$</td>
<td>$8.0925$</td>
<td></td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$2.9518 \times 10^{-12}$</td>
<td>$8.4583$</td>
<td></td>
</tr>
<tr>
<td>Sup-norm</td>
<td>$2.1610 \times 10^{-1}$</td>
<td>$-$</td>
<td></td>
</tr>
</tbody>
</table>
Conservation of Circulation

- Conservation of circulation: physically important but difficult to check (requires choice and tracking of material curves)
- In axisymmetric flows, however, circulations along the contours
  \[ C = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, \ z \ \text{a constant} \right\} \]
  have a particularly simple form: \( \Gamma(r, z) = 2\pi r^2 u_1 \)
- An alternative to conservation of circulation: monitoring the extrema \( \Gamma_1 = \min_{r,z} \Gamma \) and \( \Gamma_2 = \max_{r,z} \Gamma \), which must be conserved over time
Resolution Study on Conserved Quantities

**Table:** Kinetic energy $E$, minimum circulation $\Gamma_1$, maximum circulation $\Gamma_2$ and their maximum (relative) change over $[0, 0.003505]$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$|\delta E|_{\infty, t}$</th>
<th>$|\delta \Gamma_1|_{\infty, t}$</th>
<th>$|\delta \Gamma_2|_{\infty, t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>$1.53 \times 10^{-11}$</td>
<td>$4.35 \times 10^{-17}$</td>
<td>$1.25 \times 10^{-14}$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>$4.17 \times 10^{-12}$</td>
<td>$3.30 \times 10^{-17}$</td>
<td>$7.78 \times 10^{-15}$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>$2.08 \times 10^{-12}$</td>
<td>$3.13 \times 10^{-17}$</td>
<td>$9.95 \times 10^{-15}$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>$6.47 \times 10^{-13}$</td>
<td>$2.77 \times 10^{-17}$</td>
<td>$2.14 \times 10^{-14}$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$6.66 \times 10^{-13}$</td>
<td>$2.53 \times 10^{-17}$</td>
<td>$3.49 \times 10^{-14}$</td>
</tr>
<tr>
<td>Init. value</td>
<td>$55.93$</td>
<td>$0.00$</td>
<td>$628.32$</td>
</tr>
</tbody>
</table>
Outline

1 Introduction
- The Basic Problem and Previous Work
- Our Discoveries: Existence of a Finite-Time Singularity

2 Numerical Results
- Outline of the Method
- Overview and First Sign of Singularity
- Resolution Study
- Confirming the Singularity I: Maximum Vorticity
- Confirming the Singularity II: Vorticity Moments
- Confirming the Singularity III: Vorticity Directions
- Confirming the Singularity IV: Local Self-Similarity
The Beale-Kato-Majda (BKM) Criterion

- The main tool for studying blowup/non-blowup: the Beale-Kato-Majda (BKM) criterion (Beale et al. 1984)

**Theorem**

Let $u$ be a solution of the 3D Euler equations, and suppose there is a time $t_s$ such that the solution cannot be continued in the class

$$u \in C([0, t]; H^m) \cap C^1([0, t]; H^{m-1}), \quad m \geq 3$$

to $t = t_s$. Assume that $t_s$ is the first such time. Then

$$\int_0^{t_s} \| \omega(\cdot, t) \|_\infty \, dt = \infty, \quad \omega = \nabla \times u.$$
Applying the BKM Criterion

The “standard” approach to singularity detection:

1. assume the existence of an inverse power-law

\[ \|\omega(\cdot, t)\|_\infty \sim c(t_s - t)^{-\gamma}, \quad c, \gamma > 0 \]

2. estimate \( t_s \) and \( \gamma \) using a line fitting:

\[ \left[ \frac{d}{dt} \log \|\omega(\cdot, t)\|_\infty \right]^{-1} \sim \frac{1}{\gamma} (t_s - t) \]

3. estimate \( c \) using another line fitting:

\[ \log \|\omega(\cdot, t)\|_\infty \sim -\gamma \log (\hat{t}_s - t) + \log c, \]

where \( \hat{t}_s \) is the singularity time estimated in step 2.
Applying the BKM Criterion

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\log \|\omega(\cdot, t)\|_\infty \sim -\gamma \log (\hat{t}_s - t) + \log c,
\]

where \( \hat{t}_s \) is the singularity time estimated in step 2
The Key to Success

- The key to the line fitting: the choice of the fitting interval
- Must be placed within the asymptotic regime of the power-law, if such a law exists
- Selection has been discretionary in previous studies
- Inadvertent choice of the fitting interval has led to false predictions of finite-time singularity
Our Criteria

- Our criteria for choosing the fitting interval $[\tau_1, \tau_2]$:
  - $\tau_2$ is the last time at which the solution is still “accurate”
  - the line fitting computed on $[\tau_1, \tau_2]$ is “optimal”

- These ideas have been implemented in a subroutine with user tunable parameters.

- Our criteria for a successful line fitting:
  - both $\tau_2$ and the line-fitting predicted singularity time $\hat{t}_s$ converge to the same finite value as the mesh is refined; the convergence should be monotone, i.e. $\tau_2 \uparrow t_s$, $\hat{t}_s \downarrow t_s$
  - $\tau_1$ converges to a finite value that is strictly less than $t_s$ as the mesh is refined
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  - $\tau_1$ converges to a finite value that is strictly less than $t_s$ as the mesh is refined
Numerical Results

Confirming the Singularity I: Maximum Vorticity

Applying the Ideas: Indication of a Power-Law

\[ y = \left[ \frac{d}{dt} \log \| \omega \|_\infty \right]^{-1} \] on 2048\(^2\) mesh

Figure: Inverse logarithmic time derivative of the maximum vorticity, \( \left[ \frac{d}{dt} \log \| \omega \|_\infty \right]^{-1} \), computed on the 2048 \( \times \) 2048 mesh.
Applying the Ideas: Computing the Line Fitting

Figure: Inverse logarithmic time derivative \( \left[ \frac{d}{dt} \log \| \omega \|_\infty \right]^{-1} \) and its line fitting \( \hat{\gamma}_1^{-1} (\hat{t}_s - t) \), computed on the 2048 \( \times \) 2048 mesh.
Applying the Ideas: Computing the Line Fitting

Figure: A zoom-in view of the line fitting $\hat{\gamma}_1^{-1}(\hat{t}_s - t)$. 

Numerical Results
Confirming the Singularity I: Maximum Vorticity

$3.5048 \times 10^{-3}$

$3.505 \times 10^{-3}$

$3.5052 \times 10^{-3}$

$3.5054 \times 10^{-3}$

$3.5056 \times 10^{-3}$

$y = \left[ \frac{d}{dt} \log \| \omega \|_{\infty} \right]^{-1}$ on $2048^2$ mesh

$y(t)$

$\hat{\gamma}^{-1}(\hat{t}_s - t)$
Applying the Ideas: Computing the Line Fitting

Figure: Maximum vorticity $\|\omega\|_{\infty}$ and its inverse power-law fitting $\hat{c} (\hat{t}_s - t)^{-\hat{\gamma}_2}$, computed on the 2048 $\times$ 2048 mesh.
Applying the Ideas: Computing the Line Fitting

Figure: A zoom-in view of the inverse power-law fitting $\hat{c}(\hat{t}_s - t)^{-\hat{\gamma}_2}$.
Applying the Ideas: the “Best” Fitting Interval

**Table:** The “best” fitting interval $[\tau_1, \tau_2]$ and the estimated singularity time $\hat{t_s}$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\hat{t_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>0.003306</td>
<td>0.003410</td>
<td>0.0035070</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>0.003407</td>
<td>0.003453</td>
<td>0.0035063</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>0.003486</td>
<td>0.003505</td>
<td>0.0035056</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>0.003479</td>
<td>0.003505</td>
<td>0.0035056</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>0.003474</td>
<td>0.003505</td>
<td>0.0035056</td>
</tr>
</tbody>
</table>
## Numerical Results

Confirming the Singularity I: Maximum Vorticity

### Applying the Ideas: Results of the Line Fitting

**Table:** The best line fittings for $\|\omega\|_\infty$ computed on $[\tau_1, \tau_2]$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$\hat{\gamma}_1$</th>
<th>$\hat{\gamma}_2$</th>
<th>$\hat{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>2.5041</td>
<td>2.5062</td>
<td>$4.8293 \times 10^{-4}$</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>2.4866</td>
<td>2.4894</td>
<td>$5.5362 \times 10^{-4}$</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>2.4544</td>
<td>2.4559</td>
<td>$7.4912 \times 10^{-4}$</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>2.4557</td>
<td>2.4566</td>
<td>$7.4333 \times 10^{-4}$</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>2.4568</td>
<td>2.4579</td>
<td>$7.3273 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

$\dagger$: $\hat{\gamma}_1$ is computed from $\left[ \frac{d}{dt} \log(\|\omega\|_\infty) \right]^{-1} \sim \gamma^{-1}(t_s - t)$.

$\ddagger$: $\hat{\gamma}_2$ is computed from $\log(\|\omega\|_\infty) \sim -\gamma \log(t_s - t) + \log \hat{c}$.

**Conclusion:** solution develops a singularity at $t_s \approx 0.0035056$ (recall $t_e \approx 0.00350555$)
Comparison with Other Numerical Studies

**Table:** Comparison of our results with other numerical studies. K: Kerr (1993); BP: Boratav and Pelz (1994); GMG: Grauer et al. (1998); OC: Orlandi and Carnevale (2007); $\tau_2$: the last time at which the solution is deemed “well resolved”.

<table>
<thead>
<tr>
<th>Studies</th>
<th>$\tau_2$</th>
<th>$t_s$</th>
<th>Effec. res.</th>
<th>Vort. amp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>17</td>
<td>18.7</td>
<td>$\leq 512^3$</td>
<td>23</td>
</tr>
<tr>
<td>BP</td>
<td>1.6$^\dagger$</td>
<td>2.06</td>
<td>$1024^3$</td>
<td>180</td>
</tr>
<tr>
<td>GMG</td>
<td>1.32</td>
<td>1.355</td>
<td>$2048^3$</td>
<td>21</td>
</tr>
<tr>
<td>OC</td>
<td>2.72</td>
<td>2.75</td>
<td>$1024^3$</td>
<td>55</td>
</tr>
<tr>
<td>Ours</td>
<td>0.003505</td>
<td>0.0035056</td>
<td>$(3 \times 10^{12})^2$</td>
<td>$3 \times 10^8$</td>
</tr>
</tbody>
</table>

$^\dagger$: According to Hou and Li (2008).
Outline

1 Introduction
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2 Numerical Results
   - Outline of the Method
   - Overview and First Sign of Singularity
   - Resolution Study
   - Confirming the Singularity I: Maximum Vorticity
   - Confirming the Singularity II: Vorticity Moments
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Vorticity Moments

- Another quantity of interest: vorticity moment integrals

\[
\Omega_{2m} = \left[ \int_D |\omega|^{2m} \, dx \right]^{1/2m}, \quad m = 1, 2, \ldots
\]

- By Hölder’s inequality,

\[
\Omega_{2m} \leq \Omega_{2n} |D|^{(n-m)/(2mn)}, \quad 1 \leq m < n
\]

- In particular, the blowup of any \( \Omega_{2m} \) implies the blowup of \( \|\omega\|_\infty = \Omega_\infty \).
Higher Vorticity Moments

All vorticity moments of order \( \geq 2 \) seem to blow up.

- We first check if \( \Omega_{2m} \ (m > 1) \) blows up like an inverse power-law,

\[
\Omega_{2m}(t) \sim c(t_s - t)^{-\gamma}, \quad c, \gamma > 0?
\]

- We found that it has nearly linear inverse logarithmic time derivatives, thus
- obey an inverse power-law, and
- are amenable to a line fitting
Results of the Line Fitting

Table: The line fitting of the $2m$-th vorticity moment $\Omega_{2m}$, computed on the interval $[\tau_1, \tau_2]$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$\hat{t}_{2m,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m = 2$</td>
</tr>
<tr>
<td>$1024 \times 1024$</td>
<td>0.0035231</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>0.0035115</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>0.0035056</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>0.0035057</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>0.0035057</td>
</tr>
</tbody>
</table>

Recall $t_s \approx 0.0035056$
Results of the Line Fitting (Cont’d)

Table: The line fitting of the $2^m$-th vorticity moment $\Omega_{2^m}$, computed on the interval $[\tau_1, \tau_2]$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$\hat{\gamma}_{2^m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m = 2$</td>
</tr>
<tr>
<td>1024 × 1024</td>
<td>1.2542</td>
</tr>
<tr>
<td>1280 × 1280</td>
<td>1.1306</td>
</tr>
<tr>
<td>1536 × 1536</td>
<td>1.0019</td>
</tr>
<tr>
<td>1792 × 1792</td>
<td>1.0039</td>
</tr>
<tr>
<td>2048 × 2048</td>
<td>1.0062</td>
</tr>
</tbody>
</table>

Conclusion: higher vorticity moments blow up at $t_s$
Outline

1 Introduction
   - The Basic Problem and Previous Work
   - Our Discoveries: Existence of a Finite-Time Singularity

2 Numerical Results
   - Outline of the Method
   - Overview and First Sign of Singularity
   - Resolution Study
   - Confirming the Singularity I: Maximum Vorticity
   - Confirming the Singularity II: Vorticity Moments
   - **Confirming the Singularity III: Vorticity Directions**
   - Confirming the Singularity IV: Local Self-Similarity
Key quantity in the BKM criterion: the sup-norm of the vorticity magnitude

The vorticity direction $\xi = \omega/|\omega|$ could also play a role!

Recall the vorticity equation

$$|\omega|_t + u \cdot \nabla |\omega| = \alpha |\omega|,$$

where $\alpha = \xi \cdot \nabla u \cdot \xi$ is the vorticity amplification factor
Vorticity Amplification Factor

- Direct computation shows

\[ \alpha = \frac{3}{4\pi} P.V. \int D(\hat{y}, \xi(x + y), \xi(x))|\omega(x + y)| \frac{dy}{|y|^3}, \]

where \( \hat{y} = y / |y| \) and

\[ D(e_1, e_2, e_3) = (e_1 \cdot e_3) \det(e_1, e_2, e_3) \]

- Note the formal quadratic nonlinearity \( \alpha |\omega| \sim |\omega|^2 \)
Key observation: “smoothly varying” vorticity direction $\xi$
- makes $D(\hat{y}, \xi(x + y), \xi(x))$ “small near $y = 0$”, and
- prevents $\alpha$ from growing like some power of $|\omega|$.

The most well-known (non)blowup criteria in this direction:
- Constantin-Fefferman-Majda (CFM), 1996
- Deng-Hou-Yu (DHY), 2005
The CFM Criterion

- Essential ideas of CFM: no blowup if, among other things,
  - \( \xi \) is locally Lipschitz, or
  - \( \sin \phi_x(y) \) is locally Lipschitz in \( y \), where \( \phi_x(y) \) is the angle between \( \xi(x) \) and \( \xi(x + y) \)

- Note that
  - both conditions ensure the Lipschitz continuity of \( D(\hat{y}, \xi(x + y), \xi(x)) \) at \( y = 0 \), in view of the estimate:

\[
|D(\hat{y}, \xi(x + y), \xi(x))| \leq |\sin \phi_x(y)| \leq |\xi(x + y) - \xi(x)|
\]

- the second condition allows for antiparallel vortex lines but the first one doesn’t
The DHY Criterion

- Essential ideas of DHY: no blowup if, among other things,
  - the divergence of $\xi$, $\nabla \cdot \xi$, and
  - the curvature $\kappa = |\xi \cdot \nabla \xi|$, 
  along a vortex line do not grow “too fast” compared with the “diminishing rate” of the length of the vortex line
- Similar in spirit to CFM but more localized
Checking Against the CFM Criterion

Figure: The local Lipschitz constants of $\xi$ and $D(\hat{y}, \xi(x + y), \xi(x))$ at the location of the maximum vorticity, $\tilde{q}_0$. 
Checking Against the DHY Criterion

**Figure:** The maximum/minimum of $\nabla \cdot \xi$ and $\kappa$ in a local neighborhood $D_\infty(t)$ of the maximum vorticity.
Observations

- The rapid growth of the “smoothness indicators” shown in the previous plots
  - implies the breakdown of both CFM and DHY (can be more rigorously checked using line fitting)
  - is a result of the “densely packed” vortex lines near the location of the maximum vorticity
Geometry of the Vorticity Direction

\[ \tilde{\xi} = (\xi^r, \xi^z) \] near \( \tilde{q}_0 \) on 1024\(^2\) mesh, \( t = 0.003505 \)

**Figure:** The 2D vorticity direction \( \tilde{\xi} = (\xi^r, \xi^z)^T \) near the maximum vorticity. The maximum of \( |\xi^\theta| \) in this region is \( 2.1874 \times 10^{-6} \) and hence is negligible.
Figure: The $z$-component $ξ^z$ of the vorticity direction $ξ$ near the maximum vorticity. Note the rapid variation of $ξ^z$ in $z$. 
An Algebraic Point of View

- The previous analysis suggests the growth of $\alpha$ depend on the smoothness of $\xi$ (a geometric point of view).
- From an algebraic point of view,

$$\alpha = \xi \cdot \nabla u \cdot \xi = \xi \cdot S\xi, \quad S = \frac{1}{2}(\nabla u + \nabla u^T),$$

thus the growth of $\alpha$ depends on the eigenstructure of $S$. 
Spectral Dynamics

- Due to symmetry, $S$ has
  - 3 real eigenvalues $\{\lambda_i\}_{i=1}^3$ (assuming $\lambda_1 \geq \lambda_2 \geq \lambda_3$), and
  - a complete set of orthogonal eigenvectors $\{w_i\}_{i=1}^3$
- We discover, at the location of the maximum vorticity, that:
  - the vorticity direction $\xi$ is perfectly aligned with $w_2$, i.e.

$$\lambda_2 = \alpha = \frac{d}{dt} \log \|\omega\|_\infty \sim c_2 (t_s - t)^{-1}$$

- the largest positive/negative eigenvalues satisfy

$$\lambda_{1,3} \sim \pm \frac{1}{2} \|\omega\|_\infty \sim \pm c(t_s - t)^{-2.457}$$
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   - The Basic Problem and Previous Work
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2 Numerical Results
   - Outline of the Method
   - Overview and First Sign of Singularity
   - Resolution Study
   - Confirming the Singularity I: Maximum Vorticity
   - Confirming the Singularity II: Vorticity Moments
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Locally Self-Similar Solutions

Solutions of the 3D Euler equations in $\mathbb{R}^3$ have special scaling properties:

$$u(x, t) \rightarrow \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t), \quad \lambda > 0, \ \alpha \in \mathbb{R}$$

Can this give rise to a (locally) self-similar blowup?
Nonexistence Results of Chae

- Essential ideas: no blowup of the form

\[ \nabla u(x, t) \sim \frac{1}{t_s - t} \nabla U \left( \frac{x - x_0}{[t_s - t]^\beta} \right), \quad x \in \mathbb{R}^3 \]

if, among other things,

- \( \limsup_{t \to t_s} (t_s - t) \| \nabla u(\cdot, t) \|_\infty < \infty \) and is “not large enough”, or
- \( \Omega := \nabla \times U \) decays “sufficiently fast” at \( \infty \)

- The scaling \( \nabla u = O(t_s - t)^{-1} \) is dictated by **dimensional analysis**.
Self-Similar Solutions with Axis-Symmetry

In axisymmetric flows, self-similar solutions naturally take the form

\[
\begin{align*}
    u_1(\tilde{x}, t) &\sim (t_s - t)^{\gamma_u} U \left( \frac{\tilde{x} - \tilde{x}_0}{\ell(t)} \right), \\
    \omega_1(\tilde{x}, t) &\sim (t_s - t)^{\gamma_\omega} \Omega \left( \frac{\tilde{x} - \tilde{x}_0}{\ell(t)} \right), \\
    \psi_1(\tilde{x}, t) &\sim (t_s - t)^{\gamma_\psi} \Psi \left( \frac{\tilde{x} - \tilde{x}_0}{\ell(t)} \right),
\end{align*}
\]

where \( \tilde{x} = (r, z)^T \) and \( \ell(t) \sim [\delta^{-1}(t_s - t)]^{\gamma_\ell} \) is a length scale.

Not a “conventional” self-similar solution when viewed in \( \mathbb{R}^3 \)! It is a tube-like anisotropic singularity due to the axi-symmetry.
The New Scaling Laws

- Substituting the self-similar ansatz into the axisymmetric Euler equations yields

\[
\begin{align*}
\gamma_u - 1 &= \gamma_u + \gamma_\psi - 2\gamma_\ell, \\
\gamma_\omega - 1 &= \gamma_\omega + \gamma_\psi - 2\gamma_\ell = 2\gamma_u - \gamma_\ell, \\
\gamma_\psi - 2\gamma_\ell &= \gamma_\omega,
\end{align*}
\]

or, after simplification,

\[
\begin{align*}
\gamma_\omega &= -1, \\
\gamma_\psi &= -1 + 2\gamma_\ell, \\
\gamma_u &= -1 + \frac{1}{2}\gamma_\ell.
\end{align*}
\]
This gives rise to a scaling law

\[ \| \nabla u(\cdot, t) \|_\infty \sim c(t_s - t)^{\gamma_u - \gamma_\ell}, \]

which

- may be very different from the “standard” law \( O(t_s - t)^{-1} \), and which
- gives new hope for the existence of a self-similar solution
Identifying a Self-Similar Solution

- Recall the three elements of a locally self-similar solution:
  - the center of self-similarity, \( \tilde{x}_0 \)
  - a neighborhood of \( \tilde{x}_0 \) in which self-similarity is observed
  - a self-similar profile

- In our computation: \( \tilde{x}_0 \) must be the location of the maximum vorticity, the point of blowup

- To identify a “self-similar neighborhood”, consider

\[
C_\infty(t) = \left\{ (r, z) \in D : |\omega(r, z, t)| = \frac{1}{2} \|\omega(\cdot, t)\|_\infty \right\}
\]
Existence of Self-Similar Neighborhood

Figure: The level curves of $\frac{1}{2}\|\omega\|_\infty$ in linear-linear scale at various time instants.
Existence of Self-Similar Neighborhood (Cont’d)

Figure: The level curves of $\frac{1}{2} \| \omega \|_\infty$ in log-log scale (against the variables $1 - r$ and $z$) at various time instants. Note the similar shapes of all curves.
Existence of Self-Similar Neighborhood (Cont’d)

Figure: The rescaled level curves of $\frac{1}{2} \|\omega\|_\infty$ on $2048^2$ mesh.
Existence of Self-Similar Neighborhood (Cont’d)

Figure: A zoom-in view of the rescaled level curves of $\frac{1}{2}\|\omega\|_{\infty}$ on $2048^2$ mesh.
The Length Scale

- A natural choice of the length scale:

\[
\ell(t) = \max_{\tilde{x} \in D_\infty(t)} |\tilde{x} - \tilde{x}_0|,
\]

where

\[
D_\infty(t) = \left\{ (r, z) : |\omega(r, z, t)| \geq \frac{1}{2} \|\omega(\cdot, t)\|_\infty \right\}
\]

- Similar study suggests the existence of self-similar profiles: shall use \( \omega_1 \) to illustrate
Existence of Self-Similar Profiles along $r$

Figure: The $r$-cross section of $\omega_1$ near $z = 0$ at various time instants; plot shown in linear-linear scale.
Existence of Self-Similar Profiles along $r$ (Cont’d)

Figure: The $r$-cross section of $\omega_1$ near $z = 0$ at various time instants; plot shown in log-log scale (against the variables $1 - r$ and $z$).
Figure: The rescaled solution $\tilde{\omega}_1$ near $z = 0$. 
Existence of Self-Similar Profiles along $r$ (Cont’d)

Figure: A zoom-in view of the rescaled solution $\tilde{\omega}_1$ near $z = 0$. 
Existence of Self-Similar Profiles along $z$

Figure: The $z$-cross section of $\omega_1$ along $r = 1$ at various time instants; plot shown in linear-linear scale.
Existence of Self-Similar Profiles along $z$ (Cont’d)

Figure: The $z$-cross section of $\omega_1$ along $r = 1$ at various time instants; plot shown in log-log scale (against the variables $1 - r$ and $z$).
Existence of Self-Similar Profiles along $z$ (Cont’d)

Figure: The rescaled solution $\tilde{\omega}_1$ along $r = 1$. 
Figure: A zoom-in view of the rescaled solution $\tilde{\omega}_1$ along $r = 1$. 

rescaled $\tilde{\omega}_1$ along $r = 1$ on $2048^2$ mesh

$t = 0.00347$

$t = 0.003505$
Indication of Self-Similarity in 2D

Figure: The contour plot of $\omega_1$ near the maximum vorticity at $t = 0.003$. 
Figure: The contour plot of $\omega_1$ near the maximum vorticity at $t = 0.0034$. 

Thomas Y. Hou (ACM, Caltech)  Finite-Time Singularity of 3D Euler  Crete 2013  94 / 101
Figure: The contour plot of $\omega_1$ near the maximum vorticity at $t = 0.0035$. 

\[ r = 1 - 1.51 \times 10^{-7} \]

\[ z = 6.71 \times 10^{-9} \]
Figure: The contour plot of $\omega_1$ near the maximum vorticity at $t = 0.003505$. 
Determining the Self-Similar Solution

- Given the existence of a self-similar solution, the scaling laws can be estimated from a line fitting.
- The estimates are then checked against the relations

\[
\gamma_\omega = -1, \quad \gamma_\psi = -1 + 2\gamma_\ell, \quad \gamma_u = -1 + \frac{1}{2}\gamma_\ell,
\]

\[
\|\omega(\cdot, t)\|_\infty \sim c(t_s - t)^{\gamma_u - \gamma_\ell},
\]

for consistency.
The Scaling Exponents

Table: Scaling exponents of $\ell$, $u_1$, $\omega_1$, and $\psi_1$.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$\hat{\gamma}_\ell$</th>
<th>$\hat{\gamma}_u$</th>
<th>$\hat{\gamma}_\omega$</th>
<th>$\hat{\gamma}_\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1024 \times 1024$</td>
<td>2.7359</td>
<td>0.4614</td>
<td>$-0.9478$</td>
<td>4.7399</td>
</tr>
<tr>
<td>$1280 \times 1280$</td>
<td>2.9059</td>
<td>0.4629</td>
<td>$-0.9952$</td>
<td>4.8683</td>
</tr>
<tr>
<td>$1536 \times 1536$</td>
<td>2.9108</td>
<td>0.4600</td>
<td>$-0.9964$</td>
<td>4.8280</td>
</tr>
<tr>
<td>$1792 \times 1792$</td>
<td>2.9116</td>
<td>0.4602</td>
<td>$-0.9966$</td>
<td>4.8294</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>2.9133</td>
<td>0.4604</td>
<td>$-0.9972$</td>
<td>4.8322</td>
</tr>
</tbody>
</table>

$\gamma_\ell \geq 1$: consistent with the \textit{a posteriori} bound $\|u\|_{\infty} \leq C$
Consistency Check

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>$-1 + \frac{1}{2} \hat{\gamma}_\ell$</th>
<th>$-1 + 2 \hat{\gamma}_\ell$</th>
<th>$\hat{\gamma}<em>u - \hat{\gamma}</em>\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024 $\times$ 1024</td>
<td>0.3679</td>
<td>4.4717</td>
<td>$-2.2745$</td>
</tr>
<tr>
<td>1280 $\times$ 1280</td>
<td>0.4530</td>
<td>4.8118</td>
<td>$-2.4430$</td>
</tr>
<tr>
<td>1536 $\times$ 1536</td>
<td>0.4554</td>
<td>4.8215</td>
<td>$-2.4508$</td>
</tr>
<tr>
<td>1792 $\times$ 1792</td>
<td>0.4558</td>
<td>4.8232</td>
<td>$-2.4514$</td>
</tr>
<tr>
<td>2048 $\times$ 2048</td>
<td>0.4567</td>
<td>4.8266</td>
<td>$-2.4529$</td>
</tr>
</tbody>
</table>

Ref. value

$\hat{\gamma}_u : 0.4604$  $\hat{\gamma}_\psi : 4.8322$  $\hat{\gamma}_1 : 2.4568$

$\|\omega\|_\infty \sim c(t_s - t)^{-2.45}$: consistent with Chae’s nonexistence results
Main contributions of our study: discovery of potentially singular solutions of the 3D Euler equations

Similar singularity formation also observed in 2D Boussinesq equations for stratified flows

Outlook
- Understanding the physical mechanism for blow-up.
- More mathematical analysis (partial results obtained on a simplified model problem).
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- the **Brutus cluster** at ETH Zürich (ETHZ), where part of the preliminary computations in this study were done

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