A Kac Model for Fermions

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The homogeneous fermionic Boltzmann equation

\[ \frac{\partial f}{\partial t} (v \rightarrow t) = \int S^2 dv_1 B(|v - v_1| \rightarrow !) [f_0 f_1 (1 \leftrightarrow f_1) (1 \leftrightarrow f_0) | \{z \} ] \]

\[ f(v \rightarrow 0) = f_{\text{in}} (v) \]

\[ v_2 = R_3, t_2 = (0 \rightarrow 1) \rightarrow (\sim 3) \]

"INFINITE" system of weakly interacting fermions [Uehling - Uhlenbeck 1933]

If \( f_{\text{in}} (v) \): velocity distribution at time \( t = 0 \)

\[ \int S^2 dv f_{\text{in}} (v) = 1 \]

\[ \mathbf{I} \mathbf{v}_0 = \mathbf{v}_! \cdot \left[ (\mathbf{v}_! \cdot !) \rightarrow \mathbf{v}_1 \right] \]

\[ \mathbf{I} \mathbf{v}_0 = \mathbf{v}_1 + ! \cdot \left[ (\mathbf{v}_! \cdot !) \rightarrow \mathbf{v}_1 \right] \]

\[ \mathbf{I} f_0 := f(v \rightarrow t) \rightarrow f_0 \]

\[ \mathbf{I} f_1 := f(v_0 \rightarrow t) \rightarrow f_1 \]

\[ \mathbf{I} B(|v - v_1| \rightarrow !) \] cross-section associated with the transition \( v \rightarrow v_1 \)

\[ \mathbf{I} \mathbf{v}_2 + \mathbf{v}_2 = (\mathbf{v}_! \mathbf{v}_! \rightarrow \mathbf{v}_0) \]

energy conservation

\[ |v - v_1| = |v_0 - v_0| \] relative velocity preserved

\[ \mathbf{I} \mathbf{v}_1 + \mathbf{v}_1 = \mathbf{v}_0 + \mathbf{v}_0 \] momentum conservation
The homogeneous fermionic Boltzmann equation

\[
\begin{aligned}
\frac{\partial}{\partial t} f(v, t) &= \int_{S^2} d\omega \int_{\mathbb{R}^3} d\nu_1 B(|v - \nu_1|, \omega) [f' f_1' (1 - \alpha f)(1 - \alpha f_1) - f f_1 (1 - \alpha f')(1 - \alpha f_1')] \\
f(v, 0) &= f_{in}(v)
\end{aligned}
\]

\(v \in \mathbb{R}^3, t \in (0, +\infty), S^2 = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}, \quad \alpha \in (0, 1), \quad (\alpha \sim \hbar^3)\)

"INFINITE" system of weakly interacting fermions \cite{Uehling - Uhlenbeck 1933}

- \(f_{in}(v)\): velocity distribution at time \(t = 0 \Rightarrow f_{in} \in L^1_+ (\mathbb{R}^3), \int d\nu f_{in}(\nu) = 1\)
- \(v' = v - \omega \cdot [(v - \nu_1) \cdot \omega], \quad v'_1 = \nu_1 + \omega \cdot [(v - \nu_1) \cdot \omega]\),
- \(f' := f(v', t), \quad f'_1 := f(v'_1, t), \quad f := f(v, t), \quad f_1 := f(\nu_1, t)\)
- \(B(|v - \nu_1|, \omega)\) cross-section associated with the transition \(v, \nu_1 \rightarrow v', \nu'_1\) \([\rightarrow B ' 'GOOD' ']\)
- \(v^2 + v_1^2 = (v')^2 + (v'_1)^2\) energy conservation
- \(|v - \nu_1| = |v' - \nu'_1|\) relative velocity preserved
- \(v + \nu_1 = v' + \nu'_1\) momentum conservation
The Kac Model
The Kac Model

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t W^N(V_N, t) = \frac{1}{N} \sum_{1 \leq i < j \leq N} \int d\omega B(|v_i - v_j|, \omega) \left[ W^N(V_{i:j}^N, t) - W^N(V_N, t) \right] \\
W^N(V_N, 0) = f_{in}^N(V_N)
\end{array} \right.
\end{align*}
\]

\[V_N = (v_1, \ldots, v_N) \in \mathbb{R}^{3N}, \quad V_{i:j}^N := (v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N) \in \mathbb{R}^{3N}, \quad t \in (0, +\infty)\]

- stochastic \(N\) particle dynamics, \(W^N(V_N, t)\): velocity distribution \([M. \ Kac \ 1956]\)

- \(W^N(t) \in L^1_+(\mathbb{R}^3)\) for all \(t \geq 0\), \(\int dV_N \ W^N(V_N, t) = 1\)

- momentum and energy conservation hold
The Kac Model

\[
\begin{aligned}
\partial_t W^N(V_N, t) &= \frac{1}{N} \sum_{1 \leq i < j \leq N} \int d\omega B(|v_i - v_j|, \omega) \left[ W^N(V^i:j_N, t) - W^N(V_N, t) \right] \\
W^N(V_N, 0) &= f_{in}^\otimes N(V_N)
\end{aligned}
\]

\(V_N = (v_1, \ldots, v_N) \in \mathbb{R}^{3N}, V^i:j_N := (v_1, \ldots, v'_i, \ldots, v'_j, \ldots, v_N) \in \mathbb{R}^{3N}, t \in (0, +\infty)\)

- stochastic N particle dynamics, \(W^N(V_N, t)\): velocity distribution \([\text{M. Kac 1956}]\)

- \(W^N(t) \in L_+^1(\mathbb{R}^3)\) for all \(t \geq 0\), \(\int dV_N W^N(V_N, t) = 1\)

- momentum and energy conservation hold

- \(f^N_k(V_k, t) := \int dv_{k+1} \ldots dv_N W^N(V_N, t), k = 1, \ldots, N\) marginal distributions

\[
\Downarrow
\]

\(B(|\nu|, \omega)\) bounded wrt \(\nu\): \(f^N_k(t) \to (f(t))^{\otimes k}\) as \(N \to \infty\) \((L^1 - \text{PoC})\)

where \(f(t)\) solves the homogeneous Boltzmann eqn with initial datum \(f_{in}(\nu)\).
Kac Model for Fermions

GOAL:
analogous model (and result) for the homogeneous fermionic Boltzmann equation

IDEA:
define a stochastic $N$-particle dynamics that "simulates" the Pauli exclusion principle

Fix $\epsilon > 0$ and consider a partition of the one-particle phase-space $\mathbb{R}^3$ made of cells of side $\epsilon^3$

Define:

$\implies (v \rightarrow v_m) = 1 \implies$ if $v \rightarrow v_m$ are in different cells

$\implies (v \rightarrow v_m) = 0 \implies$ otherwise

and

$(V_N) = \sum_{1 \leq \tau < m \leq N} (v \rightarrow v_m) \cdot \text{An}$

$N$-particle configuration $V_N$ is said to be admissible $(V_N) = 1$.

We work on $A_N := \{ V_N \in \mathbb{R}^{3N} : (V_N) = 1 \}$.
**GOAL**: analogous model (and result) for the homogeneous fermionic Boltzmann equation
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**IDEA**: define a stochastic $N$-particle dynamics that "simulates" the Pauli exclusion principle
**Kac Model for Fermions**

**GOAL**: analogous model (and result) for the homogeneous fermionic Boltzmann equation

**IDEA**: define a stochastic $N$-particle dynamics that "simulates" the Pauli exclusion principle

\[
\begin{align*}
\text{Fix } \delta > 0 \text{ and consider a partition of the one-particle phase-space } \mathbb{R}^3 \text{ made of cells } \Delta \\
of \text{side } \delta \text{ (volume } \delta^3) 
\end{align*}
\]
Kac Model for Fermions

**GOAL:** analogous model (and result) for the homogeneous fermionic Boltzmann equation

**IDEA:** define a stochastic $N$-particle dynamics that "simulates" the Pauli exclusion principle

\[ \downarrow \]

Fix $\delta > 0$ and consider a partition of the one-particle phase-space $\mathbb{R}^3$ made of cells $\Delta$ of side $\delta$ (volume $\delta^3$)

Define:

\[ \chi_\delta(v_\ell, v_m) = \begin{cases} 1, & \text{if } v_\ell \text{ and } v_m \text{ are in different cells} \\ 0, & \text{otherwise} \end{cases} \]

and

\[ \chi_\delta(V_N) = \prod_{1 \leq \ell < m \leq N} \chi_\delta(v_\ell, v_m) \]

• An $N$-particle configuration $V_N$ is said to be **admissible** iff $\chi_\delta(V_N) = 1$.

• We work on $\mathcal{A}_\delta^N := \{ V_N \in \mathbb{R}^{3N} \text{ s.t. } \chi_\delta(V_N) = 1 \}$
Kac Model for Fermions

\[ \mathcal{W}(V_N \hookrightarrow t) = \sum_{\{X_i, j = 1, i < j \}} \{Z, B_{ij}(V_i, j \hookrightarrow t) \} \]

the transition \( V_N \hookrightarrow V_i, j \) is allowed if and only if both the departure and arrival configuration are admissible.
Kac Model for Fermions

\[ \begin{align*}
\partial_t W^N(V_N, t) &= \frac{1}{N} \sum_{i,j=1 \atop i<j}^N \int d\omega \ B_{ij}^\omega \left[ \bar{\chi}_\delta(V_N) W^N(V_{N}^{i:j}, t) - \bar{\chi}_\delta(V_{N}^{i:j}) W^N(V_N, t) \right] \\
W^N(V_N, 0) &= W_0^N(V_N) \text{ s.t. } \text{supp } W_0^N \subseteq \mathcal{A}_\delta^N \quad (\text{ME})
\end{align*} \]

where \( B_{i,j}^\omega := B(|v_i - v_j|, \omega) \)

\[ \Rightarrow \quad W^N(t) \in L^1_+(\mathbb{R}^3) \text{ for all } t \geq 0, \quad \int dV_N \ W^N(V_N, t) = 1 \]

\[ \Rightarrow \quad \text{the transition } V_N \to V_N^{i:j} \text{ is allowed if and only if both the departure and arrival configuration are admissible} \]

\[ \Downarrow \]

\[ \text{supp } W^N(t) \subseteq \mathcal{A}_\delta^N, \quad \forall \ t > 0 \]
Propagation of Chaos

$t = 0$: \(N\) "almost" uncorrelated particles (hyp. of molecular chaos)

$m > 0$: many-body dynamics ruled by correlations arise

Definition of \(\{f_{Nk}(t)\}_{Nk=1}^{\infty}\) (marginal distributions)

For any \(k\):

\[
\lim_{N \to \infty} f_{Nk}(t), \quad \text{suitable hyp. on } B, W_{N=0}
\]

Limiting distributions \(\{f_{1k}(t)\}_{k=1}^{\infty}\)

For \(k = 1\):

Solve the homogeneous U-U Equation

For \(k \geq 2\):

\(f_{1k}(t) = (f_{11}(t))^{\otimes k}\) (propagation of chaos)
Propagation of Chaos

\[ t = 0: \ \text{N "almost" uncorrelated particles (hyp. of molecular chaos)} \]
Propagation of Chaos

\[ t = 0: \ N \ "almost" \ uncorrelated \ particles \ (hyp. \ of \ molecular \ chaos) \]

\[ \downarrow \]

\[ t > 0: \ \text{many-body dynamics ruled by (ME) \rightarrow correlations arise} \]
Propagation of Chaos

\( t = 0 \): N "almost" uncorrelated particles (hyp. of molecular chaos)

\( \downarrow \)

\( t > 0 \): many-body dynamics ruled by (ME) \( \rightarrow \) correlations arise

\( \downarrow \)

definition of \( \{ f^N_k(t) \}_{k=1}^N \) (marginal distributions)
Propagations of Chaos

$t = 0$: $N$ "almost" uncorrelated particles (hyp. of molecular chaos)

\[ \downarrow \]

$t > 0$: many-body dynamics ruled by (ME) $\rightarrow$ correlations arise

\[ \downarrow \]

definition of $\{ f^N_k(t) \}_{k=1}^N$ (marginal distributions)

\[ \downarrow \]

for any $k$: $\lim_{N \rightarrow +\infty, \delta \rightarrow 0} f^N_k(t)$ $\rightarrow$ suitable hyp. on $B$ and $W_0^N$
Propagation of Chaos

\[ t = 0: \text{N "almost" uncorrelated particles (hyp. of molecular chaos)} \]

\[ \downarrow \]

\[ t > 0: \text{many-body dynamics ruled by (ME) \rightarrow correlations arise} \]

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definition of \( \{f^N_k(t)\}_{k=1}^N \) \( (\text{marginal distributions}) \)

\[ \downarrow \]

for any \( k: \lim_{N \to +\infty, \delta \to 0} f^N_k(t) \)

\[ \leftarrow \text{suitable hyp. on } B \text{ and } W_0^N \]

\[ \downarrow \]

limiting distributions \( \{f^\infty_k(t)\}_{k=1}^{+\infty} \)
Propagation of Chaos

\[ t = 0: \text{N "almost" uncorrelated particles (hyp. of molecular chaos)} \]

\[ \downarrow \]

\[ t > 0: \text{many-body dynamics ruled by (ME) } \rightarrow \text{correlations arise} \]

\[ \downarrow \]

definition of \( \{f^N_k(t)\}_{k=1}^N \) (marginal distributions)

\[ \downarrow \]

for any \( k \):

\[
\lim_{N \to +\infty, \delta \to 0} f^N_k(t) \quad \rightarrow \text{suitable hyp. on } B \text{ and } W_0^N
\]

\[ \downarrow \]

limiting distributions \( \{f^\infty_k(t)\}_{k=1}^{+\infty} \)

\[ \downarrow \]

for \( k = 1 \): \( f^\infty_1(t) \) solves the homogeneous U-U Equation
Propagation of Chaos

\[ t = 0: \ N \ "almost" \ uncorrelated \ particles \ (\textit{hyp. \ of \ molecular \ chaos}) \]

\[ \downarrow \]

\[ t > 0: \ \text{many-body \ dynamics \ ruled \ by} \ (\textit{ME}) \rightarrow \text{correlations \ arise} \]

\[ \downarrow \]

\text{definition \ of} \ \{f_{N}^{N}(t)\}^{N}_{k=1} \ (\textit{marginal \ distributions})

\[ \downarrow \]

\[ \text{for \ any} \ k: \ \lim_{N \to +\infty, \delta \to 0} f_{k}^{N}(t) \leftrightarrow \text{suitable \ hyp. \ on} \ B \ \text{and} \ W_{0}^{N} \]

\[ \downarrow \]

\text{limiting \ distributions} \ \{f_{k}^{\infty}(t)\}^{+\infty}_{k=1}

\[ \downarrow \]

\[ \text{for} \ k = 1: \ f_{1}^{\infty}(t) \text{\ solves \ the} \ \textit{homogeneous} \ \textit{U-U Equation} \]

\[ \downarrow \]

\[ \text{for} \ k \geq 2: \ f_{k}^{\infty}(t) = (f_{1}^{\infty}(t))^{\otimes k} \ (\textit{propagation \ of \ chaos}) \]
Choice of the scaling

To recover the fermionic equation we need to keep track of the "exclusion mechanism" even for large $N$, this is the "origin" of the "fermionic" part of the collision kernel ($\ldots$)

The "exclusion mechanism" is implemented by the admissibility requirement and that can be seen as a constraint on the volume occupied by the $N$ particles constituting the system, ($V_N = 1$)

The regime we HAVE TO consider is $N \to 0$ with $N^3 > 0$

Remark 1: if $N^3 \neq 0$ we go back to the classical homogeneous Boltzmann eqn

NOTE: support $W_N \in \mathbb{A}_N$

We will assume asymptotic factorization at $t = 0$: $f_N(k)(0) \approx f_k(\infty)$.
Choice of the scaling

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Choice of the scaling

- To recover the fermionic equation we need to keep track of the "exclusion mechanism" even for large $N \mapsto$ this is the "origin" of the "fermionic" part of the collision kernel (... $\rightarrow \alpha f_{1}...$)
Choice of the scaling

• To recover the fermionic equation we need to keep track of the "exclusion mechanism" even for large $N$ $\rightarrow$ this is the "origin" of the "fermionic" part of the collision kernel ($\ldots - \alpha ff_1\ldots$)

• The "exclusion mechanism" is implemented by the admissibility requirement and that can be seen as a constraint on the volume occupied by the $N$ particles constituting the system $\rightarrow$ $\bar{\chi}_\delta(V_N) = 1$ $\Rightarrow$ occupied volume $\sim N\delta^3$
Choice of the scaling

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\[ \Downarrow \]

The regime we HAVE TO consider is $N \to \infty$, $\delta \to 0$ with $N\delta^3 = \alpha > 0$
Choice of the scaling

- To recover the fermionic equation we need to keep track of the "exclusion mechanism" even for large $N \mapsto$ this is the "origin" of the "fermionic" part of the collision kernel (... $- \alpha f_{1}...$)

- The "exclusion mechanism" is implemented by the admissibility requirement and that can be seen as a constraint on the volume occupied by the $N$ particles constituting the system $\mapsto \chi_{\delta}(V_{N}) = 1 \Rightarrow$ occupied volume $\sim N\delta^{3}$

\[
\downarrow
\]
The regime we HAVE TO consider is $N \to \infty, \delta \to 0$ with $N\delta^{3} = \alpha > 0$

**Remark 1:** if $N\delta^{3} \to 0$ we go back to the classical homogeneous Boltzmann eqn
Choice of the scaling

- To recover the fermionic equation we need to keep track of the "exclusion mechanism" even for large $N \rightarrow \text{this is the "origin" of the "fermionic" part of the collision kernel (...} - \alpha f f_1...$

- The "exclusion mechanism" is implemented by the admissibility requirement and that can be seen as a constraint on the volume occupied by the $N$ particles constituting the system $\leftarrow \bar{\chi}_\delta(V_N) = 1 \Rightarrow$ occupied volume $\sim N\delta^3$

The regime we HAVE TO consider is $\text{N} \rightarrow \infty, \delta \rightarrow 0 \text{ with } N\delta^3 = \alpha > 0$

**Remark 1:** if $N\delta^3 \rightarrow 0$ we go back to the classical homogeneous Boltzmann eqn

**NOTE:** $\text{supp } W_0^N \subseteq A_\delta^N \Rightarrow \nexists f_{\text{in}} \text{ s.t. } W_0^N = f_{\text{in}}^\otimes N$

$\leftarrow$ We will assume asymptotic factorization at $t = 0$: $f_k^N(0) \rightarrow f_{\text{in}}^\otimes k$
Assumptions on the Initial Data and the cross-section $B$

1. There exists a one particle probability distribution $f$ in $(v)$ such that:
   
   $f_{in} \in C^0(\mathbb{R}^3)$ and $f_{in}(v) \rightarrow e^{v_2}$ for some $\epsilon > 0$

   $\sup_{V_k} f_{N_k}(V_k \mapsto 0) \rightarrow 0$ for all $k$ and for any compact set $K \subseteq A_k$, where $A_k = \{ v_k \in \mathbb{R}^3 : v_k \neq v_m \mapsto \text{for any } v_k \mapsto v_m \mapsto \cdots \mapsto k \mapsto \cdots \}$(1)

2. For all $k$ and for any compact set $A$, where $A_k := \{ v_k \in \mathbb{R}^3 : v_k \neq v_m \mapsto \text{for any } v_k \mapsto v_m \mapsto \cdots \mapsto k \mapsto \cdots \}$(2)

3. For all $k = 1 \mapsto 2 \mapsto \cdots$, there exists a constant $z_1 > 0$, not depending on $N$, such that:
   
   $f_{N_k}(V_k \mapsto 0) \leq (z_1)^k e^{v_2}$

Note: (1)-(2) $\Rightarrow$ uniform convergence outside the diagonals.
Assumptions on the Initial Data and the cross-section $\mathcal{B}$

1. $\text{supp } W_0^N \subseteq A_0^N$,

2. there exists a one particle probability distribution $f_{in}(v)$ such that:

   \begin{align*}
   &\cdot f_{in} \in C^0(\mathbb{R}^3) \quad \text{and} \quad f_{in}(v) \leq \frac{e^{-\beta v^2}}{\alpha}, \quad \text{for some } \beta > 0 \\
   &\cdot \sup_{\mathcal{V}_k \in K} \left| f_k^N (V_k, 0) - f_{in}^{\otimes k} (V_k) \right| \to 0, \quad \text{as } N \to \infty, \ \delta \to 0, \ \text{and } N\delta^3 = \alpha,
   \end{align*}

   for all $k$ and for any compact set $K \subset A^k$, where

   \begin{equation}
   A^k := \{ V_k \in \mathbb{R}^{3k} : \ v_{\ell} \neq v_m, \quad \text{for any } \ell \neq m, \ \ell, m = 1, 2, \ldots, k \} \tag{2}
   \end{equation}

3. for all $k = 1, 2, \ldots$, there exist a constant $z_1 > 0$, not depending on $N$, such that:

   \begin{equation}
   f_k^N (V_k, 0) \leq (z_1)^k e^{-\beta V_k^2} \tag{3}
   \end{equation}

Note: (1)-(2) $\iff$ uniform convergence outside the diagonals.

---

i) $\sup_{\omega \in S^2} \sup_{v \in \mathbb{R}^3} B(v; \omega) < C_1 < +\infty$, for some $C_1 > 0$, not depending on $N$

ii) there exists $M > 0$, not depending on $N$, s.t. $B(v; \omega) = 0$, if $|v| > M$, $\forall \ \omega \in S^2$

iii) $B(v; \omega)$ is continuous with respect to $v$
Theorem 1 (Colangeli, P., Pulvirenti '13)

Let the initial data and the kernel $B_{i,j}$ satisfy assumptions 1.2.3. and $i > j$ respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V$, $k \in A_k$,

$$\lim_{N \to +\infty} N^{3} = f_N(t) = f(t)$$

$$\delta t < t_0$$

where $f(t)$ is the unique $L_1$-solution of the $U-U$ equation with initial datum $f$.

**STRATEGY** (Lanford '75), Duhamel (series) expansions and direct control on them for $t < t_0$ ($L_1$ estimates), term by term convergence (pointwise in $A_k$).

**WHY short time?**

NO (uniform) a priori estimates on $|f_N(t)|_1$

**HINT to go beyond $t_0$?**

MAXIMUM PRINCIPLE for the $U-U$ eqn.
Main Result 1. - Short time

Theorem 1  [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel $B_{i,j}^\omega$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V_k \in A^k$

$$\lim_{N \to +\infty, \delta \to 0, N \delta^3 = \alpha} f_k^N(t) = (f(t))^\times_k, \quad \forall \ t < t_0$$

where $f(t)$ is the unique $L^\infty$-solution of the $U-U$ equation with initial datum $f_{\text{in}}$. 
**Main Result 1. - Short time**

**Theorem 1** [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel $B_{i,j}^ω$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V_k \in A^k$

\[
\lim_{N \to +\infty, \delta \to 0} \lim_{N\delta^3 = \alpha} f_k^N(t) = (f(t))^\otimes k, \quad \forall t < t_0
\]

where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{in}$.

**STRATEGY** (\(\sim\) Lanford '75)

\(\leftarrow\) Duhamel (series) expansions and direct control on them for $t < t_0$ ($L^\infty$ estimates)

\(\leftarrow\) term by term convergence (pointwise in $A^k$)
Main Result 1. - Short time

Theorem 1  [Colangeli, P., Pulvirenti ’13]

Let the initial data and the kernel $B_{i,j}^{\omega}$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V_k \in A^k$

$$\lim_{N \to +\infty, \delta \to 0} f_k^N(t) = (f(t))^{\otimes_k}, \quad \forall t < t_0$$

where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{in}$.

**STRATEGY** ($\rightsquigarrow$ Lanford ’75)

$\rightarrow$ Duhamel (series) expansions and direct control on them for $t < t_0$ (L$^\infty$ estimates)

$\rightarrow$ term by term convergence (pointwise in $A^k$)

**Q1.** **WHY short time?** NO (uniform) a priori estimates on $\|f_k^N(t)\|_\infty$
Theorem 1  [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel $B_{i,j}^\omega$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V_k \in \mathcal{A}^k$

$$\lim_{N \to +\infty, \delta \to 0} f_k^N(t) = (f(t))^{\otimes_k}, \quad \forall t < t_0$$

where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{in}$.

**STRATEGY** (\Leftrightarrow Lanford '75)

$\leftarrow$ Duhamel (series) expansions and direct control on them for $t < t_0$ (L$^\infty$ estimates)

$\leftarrow$ term by term convergence (pointwise in $\mathcal{A}^k$)

Q1. **WHY short time?** NO (uniform) a priori estimates on $\|f_k^N(t)\|_\infty$

Q2. **HINT to go beyond** $t_0$?
Main Result 1. - Short time

**Theorem 1**  [Colangeli, P., Pulvirenti ’13]

Let the initial data and the kernel $B_{i,j}$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, there exists $t_0 > 0$ such that, for a.e. $V_k \in \mathcal{A}^k$

$$
\lim_{N \to +\infty, \delta \to 0, N\delta^3 = \alpha} f_k^N(t) = (f(t))^{\otimes k}, \quad \forall \ t < t_0
$$

where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{\text{in}}$.

**STRATEGY** (*≈* Lanford ’75)

$\leftrightarrow$ Duhamel (series) expansions and direct control on them for $t < t_0$ (L$^\infty$ estimates)

$\leftrightarrow$ term by term convergence (pointwise in $\mathcal{A}^k$)

Q1. **WHY short time?** NO (uniform) a priori estimates on $\|f_k^N(t)\|_\infty$

Q2. **HINT to go beyond $t_0$?** MAXIMUM PRINCIPLE for the U-U eqn.
Main Result 2. - Global in time PoC

Theorem 2. [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel \( B^{i,j} \) satisfy assumptions 1, 3, and iii), respectively. Then, for any \( t > 0 \),

\[
\lim_{N \to +\infty} N^{-\frac{3}{2}} f_N^k(t) = \mathcal{U} f(t) \quad (\text{iii})
\]

where \( f(t) \) is the unique \( L_1 \)-solution of the \( U-U \) equation with initial datum \( f \) in \( R^3 \).

STRATEGY concerning \( Q1 \):

- define a suitable norm \( \| \cdot \| \) s.t.

\[
\| f_N^k(t) \| \leq \mathcal{C} \quad (\text{iii})
\]

concerning \( Q2 \):

- theorem max principle provides a global \( L_1 \) control on the \( U-U \) hierarchy, term by term convergence holds pointwise, \( L_1 \) makes the three topologies above to "match" well, convergence at \( t > 0 \) is weaker than that we require at \( t = 0 \):

PILING UP
Main Result 2. - Global in time PoC

**Theorem 2**  [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel $B_{i,j}^\omega$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, for any $t > 0$

$$\lim_{N \to +\infty, \delta \to 0} \left\| f_N^k(t) - (f(t))^\otimes k \right\|_{L^1(\Lambda_k)} = 0, \quad \forall \text{ compact sets } \Lambda_k \subset \mathbb{R}^{3k}$$

where $f(t)$ is the unique $L^\infty$-solution of the $U-U$ equation with initial datum $f_{in}$.
Main Result 2. - Global in time PoC

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STRATEGY
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where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{\text{in}}$.

STRATEGY

← concerning Q1.: define a suitable norm $\| \cdot \|_\delta$ s.t. $\| f_k^N(t) \|_{\delta,k} \leq (c)^k \forall \ t$
Theorem 2 [Colangeli, P., Pulvirenti ’13]

Let the initial data and the kernel $B_{i,j}$ satisfy assumptions 1. – 3. and i) – iii), respectively. Then, for any $t > 0$

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where $f(t)$ is the unique $L^\infty$-solution of the U-U equation with initial datum $f_{in}$.

STRATEGY

$\leftarrow$ concerning Q1.: define a suitable norm $\| \cdot \|_{\delta}$ s.t. $\| f_N^k(t) \|_{\delta,k} \leq (c)^k \forall t$

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Main Result 2. - Global in time PoC

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**STRATEGY**

$\leftarrow$ concerning **Q1.**: define a suitable norm $\| \cdot \|_\delta$ s.t. $\left\| f_k^N(t) \right\|_{\delta, k} \leq (c)^k \forall t$

$\leftarrow$ concerning **Q2.**: the maximum principle provides a global $L^\infty$-control on the $U-U$ hierarchy

$\leftarrow$ term by term convergence holds pointwise
Main Result 2. - Global in time PoC

Theorem 2 \[\text{[Colangeli, P., Pulvirenti '13]}\]

Let the initial data and the kernel \(B_{i,j}^{\omega}\) satisfy assumptions 1. – 3. and i) – iii), respectively. Then, for any \(t > 0\)

\[
\lim_{N \to +\infty, \delta \to 0} \left\| f_k^N(t) - (f(t)) \otimes_k \right\|_{L^1(\Lambda_k)} = 0, \quad \forall \text{ compact sets } \Lambda_k \subset \mathbb{R}^{3k}
\]

where \(f(t)\) is the \underline{unique} \(L^\infty\)-solution of the \(U-U\) equation with initial datum \(f_{\text{in}}\).

STRATEGY

\(\leftrightarrow\) concerning \(Q1.\): define a suitable norm \(\left\| \cdot \right\|_\delta\) s.t. \(\left\| f_k^N(t) \right\|_{\delta,k} \leq (c)^k \forall t\)

\(\leftrightarrow\) concerning \(Q2.\): the maximum principle provides a global \(L^\infty\)-control on the \(U-U\) hierarchy

\(\leftrightarrow\) term by term convergence holds \underline{pointwise}

\(\leftrightarrow\) \(L^1_{\text{loc}}\) makes the three topologies above to "match" well
Main Result 2. - Global in time PoC

**Theorem 2**  [Colangeli, P., Pulvirenti '13]

Let the initial data and the kernel $B_{i,j}^\omega$ satisfy assumptions 1. - 3. and i) – iii), respectively. Then, for any $t > 0$

$$\lim_{N \to +\infty, \delta \to 0, N\delta^3 = \alpha} \left\| f_k^N(t) - (f(t))^{\otimes k} \right\|_{L^1(\Lambda_k)} = 0, \quad \forall \text{ compact sets } \Lambda_k \subset \mathbb{R}^{3k}$$

where $f(t)$ is the unique $L^\infty$-solution of the $U-U$ equation with initial datum $f_{in}$.

**STRATEGY**

$\leftrightarrow$ concerning Q1.: define a suitable norm $\| \cdot \|_\delta$ s.t. $\| f_k^N(t) \|_{\delta,k} \leq (c)^k \forall t$

$\leftrightarrow$ concerning Q2.: the maximum principle provides a global $L^\infty$-control on the $U-U$ hierarchy

$\leftrightarrow$ term by term convergence holds pointwise

$\leftrightarrow L^1_{loc}$ makes the three topologies above to ”match” well

$\leftrightarrow$ convergence at $t > 0$ is WEAKER than that we require at $t = 0$: PILING UP $\sum$
Idea of the Proof of Thm 1. - HIERARCHIES

BBGKY hierarchy of \((N)\) eqns for the marginals \(\{f_N^k(t)\}\)

\[ \frac{\partial}{\partial t} f_N^k = \frac{1}{N!} \sum_{L^N_k \in L_N^1} \left( f_{N+1}^k + f_{N+2}^s \right) \]

Defining the flow \(S_N^k(t)\): \(e^{L_N^k t N} \)

\[ ||S_k(t)|| \approx e^{k^2 C_B t N} \]

U-U hierarchy of \((1)\) eqns (formal limit of (4))

\[ \frac{\partial}{\partial t} f_1^k = \sum_{s=1}^3 C_k^s, k+1 \approx O(k) \]

\(\approx\) quadratic term (classical Boltzmann opt.)

\[ C_k^s, k+2 \approx O(k^2 ) \]

\(\approx\) cubic term (fermionic part in the U-U opt.)

\[ C_k^s, k+3 = O(k^2 ) \]

\(\approx\) quartic term = 0 by symmetry

\[ \frac{\partial}{\partial t} f_1^k = \sum_{s=1}^3 C_k^s, k+1 \]


Idea of the Proof of Thm 1. - HIERARCHIES

- **BBGKY hierarchy** of \((N)\) eqns for the marginals \(\{f^N_k(t)\}_k\)

\[
\partial_t f^N_k = \underbrace{\frac{1}{N} L^N_k f^N_k}_{O(\frac{k^2}{N}) \text{ in } L^\infty} + \underbrace{\frac{1}{N} \sum_{s=1}^2 L^N_{k,k+s} f^N_{k+s}}_{O(\frac{k^2}{N}) \text{ in } L^\infty} + \underbrace{\sum_{s=1}^3 C^N_{k,k+s} f^N_{k+s},}_{O(1) \text{ in } L^\infty} \quad k = 1, 2, \ldots, N
\]
Idea of the Proof of Thm 1. - HIERARCHIES

- **BBGKY hierarchy** of \((N)\) eqns for the marginals \(\{f_k^N(t)\}\)

\[
\partial_t f_k^N = \underbrace{\frac{1}{N} L_k^N f_k^N}_{O\left(\frac{k^2}{N}\right) \text{ "in } \mathcal{L}^\infty\text{"}} + \underbrace{\frac{1}{N} \sum_{s=1}^{2} L_{k,k+s}^N f_{k+s}^N}_{O\left(\frac{k^2}{N}\right) \text{ "in } \mathcal{L}^\infty\text{"}} + \underbrace{\sum_{s=1}^{3} C_{k,k+s}^N f_{k+s}^N}_{O(1) \text{ "in } \mathcal{L}^\infty\text{"}}, \quad k = 1, 2, \ldots, N \tag{4}
\]

\(\downarrow\) Defining the flow \(S_k^N(t) := e^{\frac{L_k^N t}{N}}\), we have \(\|S_k(t)\|_\infty \leq e^{\frac{k^2 c_B t}{N}}\)
Idea of the Proof of Thm 1. - HIERARCHIES

- **BBGKY hierarchy** of \((N)\) eqns for the marginals \(\{f_k^N(t)\}_k\)

\[
\partial_t f_k^N = \frac{1}{N} \left( \underbrace{L_k^N}_{O \left( \frac{k^2}{N} \right) \text{ "in } L^\infty \text{"}} \right) f_k^N + \frac{1}{N} \sum_{s=1}^{2} L_{k,k+s}^N f_{k+s}^N + \sum_{s=1}^{3} C_{k,k+s}^N f_{k+s}^N, \quad k = 1, 2, \ldots, N \tag{4}
\]

\(\mapsto\) Defining the flow \(S_k^N(t) := e^{\frac{L_k^N t}{N}}\), we have \(\|S_k(t)\|_\infty \leq e^{\frac{k^2 C_B t}{N}}\)

- **U-U hierarchy** of \((\infty)\) eqns (formal limit of (4))

\[
\partial_t f_k^\infty = \sum_{s=1}^{3} C_{k,k+s} f_{k+s}^\infty, \quad k = 1, 2, \ldots \tag{5}
\]

\(\mapsto\) \(C_{k,k+1} = O(k) \sim \text{quadratic term (classical Boltzmann opt.)},\)

\(\mapsto\) \(C_{k,k+2} = O(k \alpha) \sim \text{cubic term (fermionic part in the U-U opt.)},\)

\(\mapsto\) \(C_{k,k+3} = O(k \alpha^2) \sim \text{quartic term} = 0 \sim \text{by symmetry} \Rightarrow \partial_t f_k^\infty = \sum_{s=1}^{2} C_{k,k+s} f_{k+s}^\infty\)
Idea of the Proof of Thm 1. - DUHAMEL EXP $f^N_k(t)$
• Fix $M_1 > 0$. BBGKY hierarchy + (iterated) Duhamel formula

$$f_k^N(t) = \left[ \sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O^N} \int dt_n S_k^N(t - t_1) O_{k,k+i(1)}^N S_{k+i(1)}^N(t_1 - t_2) \ldots \\
\ldots O_{k+i(n-1),k+i(n)}^N S_{k+i(n)}^N(t_n) f_{k+i(n)}^N(0) \right] + R_{M_1}^N(t), \quad (6)$$

where

\begin{itemize}
  \item $\int dt_n := \int_0^t dt_1 \ldots \int_0^{t_n-1} dt_n$
  \item $\sum_{j(1)\ldots j(n)} := \sum_{(j(1),\ldots,j(n))\in\{1,2,3\}^n}$ and $i(m) := \sum_{\ell=1}^m j(\ell), \quad \forall \ m = 1, \ldots, n$
  \item $\sum_{O^N} := \sum_{O_{k,k+i(1)}^N \ldots O_{k+i(n-1),k+i(n)}^N}^N$, with $O_{k+\ell,k+m}^N$ appearing in the BBGKY hierarchy
\end{itemize}
Idea of the Proof of Thm 1. - DUHAMEL EXP $f_k^N(t)$

- Fix $M_1 > 0$. BBGKY hierarchy + (iterated) Duhamel formula

$$f_k^N(t) = \left[ \sum_{n=0}^{M_1} \sum_{j(1) \ldots j(n)} \sum_{O^N} \int dt_n S_k^N(t - t_1) O_{k,k+i(1)}^N S_{k+i(1)}^N(t_1 - t_2) \ldots$$

$$\ldots O_{k+i(n-1),k+i(n)}^N S_{k+i(n)}^N(t_n) f_{k+i(n)}^N(0) \right] + R_{M_1}^N(t), \quad (6)$$

where

- $\int dt_n := \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n,$

- $\sum_{j(1) \ldots j(n)} := \sum_{(j(1), \ldots, j(n)) \in \{1,2,3\}^n}$

- $\sum_{O^N} := \sum_{O_{k,k+i(1)}^N \ldots O_{k+i(n-1),k+i(n)}^N}$, with $O_{k+k,m}$ appearing in the BBGKY hierarchy

$\iff$ Hyp. on $\{f_k^N(0)\}_k$ ($\Rightarrow ||f_k^N(0)||_{\infty} \leq (z_1)^k$) + Hyp. on $B$ ($\Rightarrow$ good estimates for $O^N$):

$$||R_{M_1}^N(t)||_{\infty} \leq (c_1)^k (\lambda)^{M_1}, \text{ with } \lambda < 1 \iff t < t_1^*,$$

where $t_1^* \sim \frac{1}{C_{B,\alpha}(z_1)}.$
Idea of the Proof of Thm 1. - Term by Term convergence

• Hyp. on the initial data + Hyp. on each term of the (finite) sum in (6) converges pointwise to:

\[ X_j^{(1)} \ldots X_j^{(n)} \]

\[ O_k^{(1)} + i^{(1)} \]

\[ O_k^{(2)} + i^{(2)} \ldots O_k^{(n_1)} + i^{(n)} \]

in \( \mathcal{A} \)

where

\[ \mathcal{A} = X_{j^{(1)}, \ldots, j^{(n)}}^{2} \{1, 2\}^{n} \]

and

\[ i^{(m)} = m \]

\[ \mathcal{A} = X_{O_k^{(1)} + i^{(1)}, \ldots, O_k^{(n_1)} + i^{(n)}}^{\mathcal{A}} \mathcal{A} \]

with \( O_k^{(1)}, \ldots, O_k^{(n_1)} \) appearing in the U-U hierarchy

Why only pointwise?

Because we have to deal with:

\[ \int v_1 dv_2 \int d!_1 \]

\[ \int v_1, v_2, ! \]

\[ |v_1, v_2, !| = 3 \]

\[ dw g(v_1 \mapsto v_2 \mapsto v_3 \mapsto !) \]

for some \( g \in C_0^0 \). The above convergence is proven by the Lebesgue Dominated Convergence thm and we have NO control on "what happens" wrt to \( v_1 \) (beyond pointwise convergence) "quantum effect".
Idea of the Proof of Thm 1. - Term by Term convergence

• Hyp. on the initial data + Hyp. on $B \Rightarrow$ each term of the (finite) sum in (6) converges pointwise to:

$$\sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))},$$

(7)

where

$$\cdot \sum_{j(1)\ldots j(n)} := \sum_{(j(1),\ldots,j(n))\in\{1,2\}^n}$$

$$\cdot \sum_{O} := \sum_{O_{k,k+i(1)}\cdots O_{k+i(n-1),k+i(n)}}$$

and $i(m) := \sum_{\ell=1}^{m} j(\ell), \forall \ m = 1, \ldots, n$

with $O_{k+\ell,k+m}$ appearing in the U-U hierarchy.
Idea of the Proof of Thm 1. - Term by Term convergence

• Hyp. on the initial data + Hyp. on $B$ ⇒ each term of the (finite) sum in (6) converges pointwise to:

$$
\sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))},
$$

where

- $\sum_{j(1)\ldots j(n)} := \sum_{(j(1),\ldots,j(n))\in\{1,2\}^n}$
- $\sum_{O} := \sum_{O_{k,k+i(1)}\cdots O_{k+i(n-1),k+i(n)}}$

Why only "pointwise"?

Because we have to deal with:

$$
\int dv_2 \int d\omega \left( \frac{1}{\delta^3} \int_{\Delta_{v_1,v_2,v_3}:|\Delta_{v_1,v_2,v}|=\delta^3} dw \, g(v_1, v_2, w, \omega) \right) \rightarrow \int dv_2 \int d\omega \, g(v_1, v_2, v_3, \omega)
$$

for some $g \in C^0$. The above convergence is proven by the Lebesgue Dominated Convergence thm and we have NO control on "what happens" wrt to $v_1$ (beyond pointwise convergence) \(\rightsquigarrow "quantum\ \text{effect}"\).
Idea of the Proof of Thm 1. - DUHAMEL EXP $f_k^\infty(t)$
Therefore, the (finite) sum in (6) converges pointwise to:

\[
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes (k+i(n))}.
\]
Therefore, the (finite) sum in (6) converges pointwise to:

\[
\sum_{n=0}^{M_1} \sum_{j(1) \ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes (k+i(n))}.
\]  

(8)

Consider the $k$-particle function defined as:

\[
\sum_{n=0}^{M_1} \sum_{j(1) \ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes (k+i(n))} + R_{M_1}(t),
\]  

(9)

where $R_{M_1}(t) := \sum_{n=0}^{+\infty} - \sum_{n=0}^{M_1}$.
Therefore, the (finite) sum in (6) converges pointwise to:

\[ \sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \ldots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes (k+i(n))}. \]  

(8)

Consider the $k$-particle function defined as:

\[ \sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \ldots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes (k+i(n))} + R_{M_1}(t), \]  

(9)

where $R_{M_1}(t) := \sum_{n=0}^{+\infty} - \sum_{n=0}^{M_1}$. 

$\leftrightarrow$ Hyp. on $f_{in}$ ($\Rightarrow ||f_{in}||_\infty \leq \frac{1}{\alpha}$) + Hyp. on $B$ ($\Rightarrow$ good estimates for $O$)

\[ ||R_{M_1}(t)||_\infty \leq (\tilde{c}_1)^k (\tilde{\lambda})^n, \text{ with } \tilde{\lambda} < 1 \text{ IFF } t < t^*_2, \text{ where } t^*_2 \sim \frac{1}{C_{B,\alpha} (\frac{1}{\alpha})} \]
Idea of the Proof of Thm 1. - DUHAMEL EXP \( f_k^\infty(t) \)

- Therefore, the (finite) sum in (6) converges pointwise to:

\[
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{\mathcal{O}} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))}.
\]  

(8)

Consider the \( k \)-particle function defined as:

\[
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{\mathcal{O}} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))} + R_{M_1}(t),
\]  

(9)

where \( R_{M_1}(t) := \sum_{n=0}^{+\infty} - \sum_{n=0}^{M_1} \).

\( \leftrightarrow \) Hyp. on \( f_{in} (\Rightarrow ||f_{in}||_\infty \leq \frac{1}{\alpha}) \) + Hyp. on \( B (\Rightarrow \text{good estimates for } O) \)

\[
||R_{M_1}(t)||_\infty \leq (\tilde{c}_1)^k (\tilde{\lambda})^n, \text{ with } \tilde{\lambda} < 1 \text{ IFF } t < t_2^*, \text{ where } t_2^* \sim \frac{1}{C_{B,\alpha} \left(\frac{1}{\alpha}\right)}
\]

Remark: Defining \( t_0 := \min\{t_1^*, t_2^*\} \Rightarrow \text{for } t \in [0, t_0) \text{ BOTH exps are bounded!} \)
Therefore, the (finite) sum in (6) converges pointwise to:

\[
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))}.
\] (8)

Consider the \(k\)-particle function defined as:

\[
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in}^{\otimes(k+i(n))} + R_{M_1}(t),
\] (9)

where \(R_{M_1}(t) := \sum_{n=0}^{\infty} - \sum_{n=0}^{M_1} \).

\(\Rightarrow\) Hyp. on \(f_{in}\) (\(\Rightarrow\) \(\|f_{in}\|_{\infty} \leq \frac{1}{\alpha}\)) + Hyp. on \(B\) (\(\Rightarrow\) good estimates for \(O\))

\[\|R_{M_1}(t)\|_{\infty} \leq (\tilde{c}_1)^k (\tilde{\lambda})^n, \text{ with } \tilde{\lambda} < 1 \text{ IFF } t < t_2^*, \text{ where } t_2^* \sim \frac{1}{C_{B,\alpha} \left(\frac{1}{\alpha}\right)}\]

**Remark:** Defining \(t_0 := \min\{t_1^*, t_2^*\} \Rightarrow \text{for } t \in [0, t_0) \text{ BOTH exps are bounded!} \)

\(\Rightarrow\) By a direct check one gets that (9) solves the U-U hierarchy with in. datum \(\{f_{in}^{\otimes k}\}_k\)
Therefore, the (finite) sum in (6) converges pointwise to:

$$
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in} \otimes (k+i(n))
$$

Consider the $k$-particle function defined as:

$$
\sum_{n=0}^{M_1} \sum_{j(1)\ldots j(n)} \sum_{O} \frac{t^n}{n!} O_{k,k+i(1)} O_{k+i(1),k+i(2)} \cdots O_{k+i(n-1),k+i(n)} f_{in} \otimes (k+i(n)) + R_{M_1}(t), \quad (9)
$$

where $R_{M_1}(t) := \sum_{n=0}^{\infty} - \sum_{n=0}^{M_1}$. 

$\hookrightarrow$ Hyp. on $f_{in}$ ($\Rightarrow ||f_{in}||_{\infty} \leq \frac{1}{\alpha}$) + Hyp. on $B$ ($\Rightarrow$ good estimates for $O$)

$$
||R_{M_1}(t)||_{\infty} \leq (\tilde{c}_1)^k (\tilde{\lambda})^n, \text{ with } \tilde{\lambda} < 1 \text{ IFF } t < t_2^*, \quad \text{where } t_2^* \sim \frac{1}{C_{B,\alpha} (\frac{1}{\alpha})}
$$

Remark: Defining $t_0 := \min\{t_1^*, t_2^*\}$ $\Rightarrow$ for $t \in [0, t_0)$ BOTH exps are bounded!

$\hookrightarrow$ By a direct check one gets that (9) solves the U-U hierarchy with in. datum $\{f_{in} \otimes k\}_k$

$\hookrightarrow$ By the uniqueness of $L^\infty$-solns in the class $\{f_k : ||f_k||_{\infty} \leq (c)^k\}$: $(f(t)) \otimes k = (9)$
Idea of the Proof of Thm 1. - Short time Convergence

\[
E_1^k(t) = P_{M_1} n=0 E_n(t) f(k + i(n)) + R_{M_1}(t) \hookrightarrow \text{with } ||R_{M_1}(t)||_1 \leq c_1 k M_1,\]

\[
E_2^k(t) = P_{M_1} n=0 E_n(t) f(k + i(n)) \text{ in } R_{M_1}(t) \hookrightarrow \text{with } ||R_{M_1}(t)||_1 \leq e c_1 k M_1,\]

If \( t < t_0 \) and \( e < 1 \), then the bound on the remainders are uniform in \( N \) and \( E_n(t) f(k + i(n))! E_n(t) f(k + i(n)) \text{ in pointwise as } N!1 \hookrightarrow 0 \hookrightarrow N \) and

\[
R_{M_1}(0) \leq c_1 k M_1,\]

\[
||f(k + i(n))||_1 \leq e c_1 k M_1,\]

Concerning \( E_2 \), we know that

\[
||f(t)||_1 \leq e c_1 k M_1,\]

\[
||f(t)||_1 \leq e c_1 k M_1 \hookrightarrow 8 t|z| \{z, \}
\]

Maximum Principle for \( U \).
Idea of the Proof of Thm 1. - Short time Convergence

\[ \begin{align*}
E1. \quad f_k^N(t) &= \sum_{n=0}^{M_1} E_n^N(t) f_{k+i(n)}^N(0) + R_{M_1}^N(t), \quad \text{with} \quad \|R_{M_1}^N(t)\|_\infty \leq (c_1)^k (\lambda)^{M_1}, \\
E2. \quad (f(t))^{\otimes k} &= \sum_{n=0}^{M_1} E_n(t) f_{in}^{\otimes (k+i(n))} + R_{M_1}(t), \quad \text{with} \quad \|R_{M_1}(t)\|_\infty \leq (\tilde{c}_1)^k (\tilde{\lambda})^{M_1}
\end{align*} \]

- \( t < t_0 \) \Rightarrow \lambda < 1 \text{ and } \tilde{\lambda} < 1

- the bound on the remainders are UNIFORM in \( N \) and \( \delta \)

- \( E_n^N(t) f_{k+i(n)}^N(0) \rightarrow E_n(t) f_{in}^{\otimes (k+i(n))} \) pointwise as \( N \rightarrow \infty, \delta \rightarrow 0, N\delta^3 = \alpha \)
Idea of the Proof of Thm 1. - Short time Convergence

\[ f_k^N(t) = \sum_{n=0}^{N} e_n^N(t) f_{k+i(n)}^N(0) + R_{M_1}^N(t), \quad \text{with} \quad \|R_{M_1}^N(t)\|_\infty \leq (c_1)^k (\lambda)^{M_1}, \]

\[ (f(t))^{\otimes k} = \sum_{n=0}^{N} e_n(t) f_{in}^{\otimes (k+i(n))} + R_{M_1}(t), \quad \text{with} \quad \|R_{M_1}(t)\|_\infty \leq (\tilde{c}_1)^k (\tilde{\lambda})^{M_1} \]

\[ t < t_0 \Rightarrow \lambda < 1 \text{ and } \tilde{\lambda} < 1 \]

\[ \text{the bound on the remainders are UNIFORM in } N \text{ and } \delta \]

\[ e_n^N(t) f_{k+i(n)}^N(0) \rightarrow e_n(t) f_{in}^{\otimes (k+i(n))} \text{ pointwise as } N \rightarrow \infty, \delta \rightarrow 0, N\delta^3 = \alpha \]

\[ f_k^N(t) \rightarrow (f(t))^{\otimes k} \text{ pointwise for } t < t_0 \]

\[ \|f_k^N(0)\|_\infty \leq (z_1)^k \text{ and } \|f_{in}^{\otimes k}\|_\infty \leq (\frac{1}{\alpha})^k \]
Idea of the Proof of Thm 1. - Short time Convergence

\[ E1. \quad f_k^N(t) = \sum_{n=0}^{M_1} \mathcal{E}_n^N(t) f_{k+i(n)}(0) + R_{M_1}^N(t), \quad \text{with} \quad \|R_{M_1}^N(t)\|_\infty \leq (c_1)^k (\lambda)^{M_1}, \]

\[ E2. \quad (f(t))^\otimes k = \sum_{n=0}^{M_1} \mathcal{E}_n(t) f_{in}^{(k+i(n))} + R_M(t), \quad \text{with} \quad \|R_M(t)\|_\infty \leq (\tilde{c}_1)^k (\tilde{\lambda})^{M_1}. \]

\[ \begin{align*}
& \Rightarrow \quad t < t_0 \quad \Rightarrow \quad \lambda < 1 \quad \text{and} \quad \tilde{\lambda} < 1 \\
& \Rightarrow \quad \text{the bound on the remainders are UNIFORM in } N \text{ and } \delta \\
& \Rightarrow \quad \mathcal{E}_n^N(t) f_{k+i(n)}(0) \longrightarrow \mathcal{E}_n(t) f_{in}^{(k+i(n))} \quad \text{pointwise as } N \to \infty, \delta \to 0, N\delta^3 = \alpha \\
\end{align*} \]

\[ f_k^N(t) \to (f(t))^\otimes k \quad \text{pointwise for } \quad \begin{cases} t < t_0 \\ \|f_k^N(0)\|_\infty \leq (z_1)^k \quad \text{and} \quad \|f_{in}^{\otimes k}\|_\infty \leq (\frac{1}{\alpha})^k \end{cases} \]

**RMK:** the short time constraint comes ONLY from the control of \( E1. \): \( \|f_k^N(t)\|_\infty \leq ?? \)
Idea of the Proof of Thm 1. - Short time Convergence

\[ E1. \quad f^N_k(t) = \sum_{n=0}^{M_1} \mathcal{E}^n(t) f^N_{k+i(n)}(0) + R^N_{M_1}(t), \quad \text{with} \quad \|R^N_{M_1}(t)\|_{\infty} \leq (c_1)^k (\lambda)^{M_1}, \]

\[ E2. \quad (f(t))^{\otimes k} = \sum_{n=0}^{M_1} \mathcal{E}^n(t) f^{\otimes (k+i(n))}_{in} + R_{M_1}(t), \quad \text{with} \quad \|R_{M_1}(t)\|_{\infty} \leq (\tilde{c}_1)^k (\tilde{\lambda})^{M_1} \]

\[ \begin{align*}
&\quad \quad \begin{cases}
\text{if } t < t_0 \Rightarrow \lambda < 1 \text{ and } \tilde{\lambda} < 1 \\
\text{the bound on the remainders are UNIFORM in } N \text{ and } \delta \\
\text{pointwise as } N \to \infty, \delta \to 0, N\delta^3 = \alpha
\end{cases}
\end{align*} \]

\[ f^N_k(t) \to (f(t))^{\otimes k} \quad \text{pointwise for } \begin{cases} t < t_0 \end{cases} \]

\[ \|f^N_k(0)\|_{\infty} \leq (z_1)^k \quad \text{and} \quad \|f^{\otimes k}_{in}\|_{\infty} \leq \left(\frac{1}{\alpha}\right)^k \]

RMK: the short time constraint comes ONLY from the control of \( E1. \): \[ \|f^N_k(t)\|_{\infty} \leq ?? \]

\[ \begin{align*}
&\quad \quad \begin{cases}
\text{concerning } E2., \text{ we know that } \|f_{in}\|_{\infty} \leq \frac{1}{\alpha} \Rightarrow \|f(t)\|_{\infty} \leq \frac{1}{\alpha}, \quad \forall t
\end{cases}
\end{align*} \]

\[ \text{MAXIMUM PRINCIPLE for } U - U \]
Idea of the Proof of Thm 2. - The norm $\| \cdot \|_{\delta}$

For any $k = 1 \to 2 \to \ldots \to N$, we define the following norm:

$$\| g_k \|, \quad k := \sup_{1 \leq 1 \leq k_{13}} \ldots \sup_{1 \leq 1 \leq k_{13}k} Z_1 \ldots Z_k | g_k(V_k) | \to$$

for any $g_k : \mathbb{R}^{3k} \to \mathbb{R}$.

It is easy to check that:

• For any $k$ a compact set $\cdot \mapsto R^{3k}$ there exists a positive constant $C_k < +1$ such that:

$$k g_k L_1(\cdot) \leq C_k g_k, \quad k \to$$

for any $g_k : \mathbb{R}^{3k} \to \mathbb{R}$.

Moreover:

• All $L_1$-estimates we have for the operators in the BBGKY hierarchy hold as well wrt $\| \cdot \|$(not completely trivial......)

• $\sup W_N(t) \mapsto A_N$ for any $t \geq 0$.

$| | f_N(t) | |, \quad k < \mapsto e^{2/c_k} \forall t \geq 0$ and $k < 1$.

The uniform boundedness argument WORKS as well and now it CAN be iterated! (in a clever way......)
Idea of the Proof of Thm 2. - The norm $|| \cdot ||_\delta$

For any $k = 1, 2, \ldots, N$, we define the following norm:

$$|| g_k ||_{\delta,k} := \sup_{\Delta_1 \ldots \Delta_k} \frac{1}{\delta^{3k}} \int_{\Delta_1} dv_1 \ldots \int_{\Delta_k} dv_k |g_k(V_k)|, \quad \text{for any } g_k : \mathbb{R}^{3k} \to \mathbb{R} \quad (10)$$
Idea of the Proof of Thm 2. - The norm $|| \cdot ||_\delta$

For any $k = 1, 2, \ldots, N$, we define the following norm:

$$||g_k||_{\delta, k} := \sup_{\Delta_1 \ldots \Delta_k} \frac{1}{\delta^{3k}} \int_{\Delta_1} dv_1 \ldots \int_{\Delta_k} dv_k |g_k(V_k)|,$$

for any $g_k : \mathbb{R}^{3k} \to \mathbb{R}$ (10)

It is easy to check that:

- For any $k \geq 1$ and any compact set $\Lambda_k \subset \mathbb{R}^{3k}$ there exists a positive constant $C_{\Lambda_k} < +\infty$ such that:

$$||g_k||_{L^1(\Lambda_k)} \leq C_{\Lambda_k} ||g_k||_{\delta, k}, \quad \forall \ g_k : \mathbb{R}^{3k} \to \mathbb{R}$$

(11)
Idea of the Proof of Thm 2. - The norm $|| \cdot ||_\delta$

For any $k = 1, 2, \ldots, N$, we define the following norm:

$$
||g_k||_{\delta,k} := \sup_{\Delta_1 \ldots \Delta_k} \frac{1}{\delta^{3k}} \int_{\Delta_1} dv_1 \ldots \int_{\Delta_k} dv_k |g_k(V_k)|, \quad \text{for any } g_k : \mathbb{R}^{3k} \to \mathbb{R} \quad (10)
$$

It is easy to check that:

- For any $k \geq 1$ and any compact set $\Lambda_k \subset \mathbb{R}^{3k}$ there exists a positive constant $C_{\Lambda_k} < +\infty$ such that:

$$
||g_k||_{L^1(\Lambda_k)} \leq C_{\Lambda_k} ||g_k||_{\delta,k}, \quad \forall \ g_k : \mathbb{R}^{3k} \to \mathbb{R} \quad (11)
$$

Moreover:

- ALL $L^\infty$-estimates we have for the operators in the BBGKY hierarchy hold as well wrt $|| \cdot ||_\delta$ (*not completely trivial......*)
For any $k = 1, 2, \ldots, N$, we define the following norm:

$$\|g_k\|_{\delta,k} := \sup_{\Delta_1 \ldots \Delta_k} \frac{1}{\delta^{3k}} \int_{\Delta_1} \cdots \int_{\Delta_k} dv_1 \cdots dv_k \ |g_k(V_k)|,$$

for any $g_k : \mathbb{R}^{3k} \rightarrow \mathbb{R}$ \hspace{1cm} (10)

It is easy to check that:

- For any $k \geq 1$ and any compact set $\Lambda_k \subset \mathbb{R}^{3k}$ there exists a positive constant $C_{\Lambda_k} < +\infty$ such that:

$$\|g_k\|_{L^1(\Lambda_k)} \leq C_{\Lambda_k} \|g_k\|_{\delta,k}, \quad \forall \ g_k : \mathbb{R}^{3k} \rightarrow \mathbb{R}$$

(11)

Moreover:

- **ALL $L^\infty$-estimates we have for the operators in the BBGKY hierarchy** hold as well wrt $\| \cdot \|_{\delta}$ *(not completely trivial......)*

- supp $W^N(t) \subseteq A^N_\delta$ for any $t \geq 0$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $\|f^N_k(t)\|_{\delta,k} < \left( \frac{e^2}{\alpha} \right)^k$, \quad $\forall \ t \geq 0$ and $k \geq 1$
For any $k = 1, 2, \ldots, N$, we define the following norm:

$$||g_k||_{\delta,k} := \sup_{\Delta_1 \cdots \Delta_k} \frac{1}{\delta^{3k}} \int_{\Delta_1} dv_1 \cdots \int_{\Delta_k} dv_k |g_k(V_k)|, \quad \text{for any } g_k : \mathbb{R}^{3k} \to \mathbb{R} \quad (10)$$

It is easy to check that:

- For any $k \geq 1$ and any compact set $\Lambda_k \subset \mathbb{R}^{3k}$ there exists a positive constant $C_{\Lambda_k} < +\infty$ such that:

$$\|g_k\|_{L^1(\Lambda_k)} \leq C_{\Lambda_k} \|g_k\|_{\delta,k}, \quad \forall \ g_k : \mathbb{R}^{3k} \to \mathbb{R} \quad (11)$$

Moreover:

- **ALL $L^\infty$ estimates** we have for the operators in the BBGKY hierarchy hold as well wrt $\| \cdot \|_\delta$ (*not completely trivial.....*)

- $\text{supp } W^N(t) \subseteq A^N_\delta$ for any $t \geq 0 \Rightarrow \|f^N_k(t)\|_{\delta,k} < \left( \frac{e^2}{\alpha} \right)^k, \quad \forall \ t \geq 0 \text{ and } k \geq 1$

$\Rightarrow$ **The uniform boundedness argument WORKS** as well and now it CAN be iterated! (*in a clever way...*)
For any $t > 0$, we partition the interval $[0 \rightarrow t]$ into intervals of fixed amplitude $\tau < t_0(\epsilon) \leq 1 \epsilon 2^{\epsilon} \epsilon$. Clearly, $s = t \tau > t_0(\epsilon)$.

Iterating $s$ times the argument discussed in Thm 1, we get:

$$f^N_k(t) = M_1 \sum_{n_1=0} M_2 \sum_{n_2=0} \cdots M_s \sum_{n_s=0} E^N_{n_1}(\tau) E^N_{n_2}(\tau) \cdots E^N_{n_s}(\tau) f^N_k(0) + R^N_{M_1}(V^k \rightarrow t) + R^N_{M_2}(V^k \rightarrow t) + \cdots + R^N_{M_s}(V^k \rightarrow t)$$

where $I((n_1 \rightarrow \cdots \rightarrow n_s)) := P_s \sum_{i=1}^s i(n_i)$.

Moreover, by choosing $M_2 \rightarrow \cdots \rightarrow M_s$ sufficiently large (precise condition), for any $n_1 \rightarrow \cdots \rightarrow s$ we have:

$$|R^N_{M_\eta}(t)| \leq k \frac{(c_\eta)}{k(\epsilon)} M_\eta \eta \rightarrow 1$$

and $\tau < t_0(\epsilon)$.
Idea of the Proof of Thm 2. - PILING UP ∑ for $f^N_k(t)$

For any $t > 0$, we partition the interval $[0, t]$ in $s$ intervals of fixed amplitude $\tau < t_0(\alpha) \sim \frac{1}{c_B,\alpha \left(\frac{e^2}{\alpha}\right)}$. Clearly, $s = \frac{t}{\tau} > \frac{t}{t_0(\alpha)} := s_0$. Iterating $s$ times the argument discussed in Thm 1, we get:

$$f^N_k(t) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} E^{N}_{n_1}(\tau) E^{N}_{n_2}(\tau) \cdots E^{N}_{n_s}(\tau) f^N_{k+l(n_1, \ldots, n_s)}(0) +$$

$$+ R^{N}_{M_s}(V_k, t) + R^{N}_{M_{s-1}}(V_k, t) + \cdots + R^{N}_{M_1}(V_k, t), \quad (12)$$

where $I(n_1, \ldots, n_s) := \sum_{\ell=1}^{s} i(n_\ell)$. 

Moreover, by choosing $M_2 \hookrightarrow \cdots \hookrightarrow M_s$ sufficiently large (precise condition), for any $\ell = 1 \hookrightarrow \cdots \hookrightarrow s$ we have:

$$||R^{N}_{M_\ell}(t)||, k \leq (c_\ell) k \left(\frac{\tau}{\Theta}\right) M_\ell \hookrightarrow$$

for some $c_\ell > 0$.
For any $t > 0$, we partition the interval $[0, t]$ in $s$ intervals of fixed amplitude $\tau < t_0(\alpha) \sim \frac{1}{c_{B, \alpha} \left( \frac{e^2}{\alpha} \right)}$. Clearly, $s = \frac{t}{\tau} > \frac{t}{t_0(\alpha)} := s_0$. Iterating $s$ times the argument discussed in Thm 1, we get:

\[
f_k^N(t) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}^N(\tau) \mathcal{E}_{n_2}^N(\tau) \cdots \mathcal{E}_{n_s}^N(\tau)f_{k+l(n_1, \ldots, n_s)}^N(0) + \]

\[
+ \quad R_{M_s}^N(V_k, t) + R_{M_{s-1}}^N(V_k, t) + \cdots + R_{M_1}^N(V_k, t), \tag{12}
\]

where $I(n_1, \ldots, n_s) := \sum_{\ell=1}^{s} i(n_\ell)$. Moreover, by choosing $M_2, \ldots, M_s$ sufficiently large (precise condition), for any $\ell = 1, \ldots, s$ we have:

\[
||R_{M_\ell}^N(t)||_{\delta, k} \leq (c_\ell)^k (\lambda_\alpha)^{M_\ell \ell}, \quad \text{for some } c_\ell > 0 \tag{13}
\]

and

\[
\tau < t_0(\alpha) \Rightarrow \lambda_\alpha < 1
\]
Idea of the Proof of Thm 2. - PILING UP $\sum$ for $f_k^N(t)$

For any $t > 0$, we partition the interval $[0, t]$ in $s$ intervals of fixed amplitude $\tau < t_0(\alpha) \sim \frac{1}{c_B,\alpha \left(\frac{\varepsilon^2}{\alpha}\right)}$. Clearly, $s = \frac{t}{\tau} > \frac{t}{t_0(\alpha)} := s_0$. Iterating $s$ times the argument discussed in Thm 1, we get:

$$f_k^N(t) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}^N(\tau) \mathcal{E}_{n_2}^N(\tau) \cdots \mathcal{E}_{n_s}^N(\tau) f_{k+l(n_1,\ldots,n_s)}^N(0) +$$

$$+ R_{M_1}^N(V_k, t) + R_{M_{s-1}}^N(V_k, t) + \cdots + R_{M_1}^N(V_k, t), \quad (12)$$

where $l(n_1,\ldots,n_s) := \sum_{\ell=1}^s i(n_\ell)$. Moreover, by choosing $M_2,\ldots, M_s$ sufficiently large (precise condition), for any $\ell = 1,\ldots,s$ we have:

$$||R_{M_\ell}^N(t)||_{\delta,k} \leq (c_\ell)^k (\lambda_\alpha)^{M_\ell / \ell}, \quad \text{for some } c_\ell > 0 \quad (13)$$

and

$$\tau < t_0(\alpha) \Rightarrow \lambda_\alpha < 1$$

HINTS:
We need to "reach" $t = 0 + M_2,\ldots, M_s$ large is crucial to make the amplitude $\tau$ FIXED
Idea of the Proof of Thm 2. - PILING UP $\sum$ for $(f(t))^\otimes k$

Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

$$M_1 X_{n_1} = 0 \quad M_2 X_{n_2} = 0 \quad \cdots \quad M_s X_{n_s} = 0 \quad E_{n_1}(\tau) \quad E_{n_2}(\tau) \quad \cdots \quad E_{n_s}(\tau) f(\tau)$$

in (14)

Consider the $k$-particle function:

$$M_1 X_{n_1} = 0 \quad M_2 X_{n_2} = 0 \quad \cdots \quad M_s X_{n_s} = 0 \quad E_{n_1}(\tau) \quad E_{n_2}(\tau) \quad \cdots \quad E_{n_s}(\tau) f(\tau)$$

in $+ R M_s(t) + \cdots + R M_1(t) \hookrightarrow (15)$

where $R M_1(t) := P^1_{n_1=1} M_1 + 1 E_{n_1}(\tau) f(\tau) \hookrightarrow (k+i(n_1, \ldots, n_s))$ and the other remainders are computed recursively.

Choosing $M_2 \hookrightarrow \cdots \hookrightarrow M_s$ as before, for any $` = 1 \hookrightarrow \cdots \hookrightarrow s$ we have:

$$||R M_` (t)||_1 \leq (c_0`)^{k(`)} M` \hookrightarrow \text{ for some } c_0` > 0 \quad (16)$$

By a direct check one gets that (15) solves the U-U hierarchy with in. datum $\{f(\tau)\}$

By the uniqueness of $L_1$-solns in the class $\{f_k:\ ||f_k||_1 \leq (c_k)\}$, $(f(t))$.
Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

\[
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{\text{in}}^{\otimes (k+I(n_1,\ldots,n_s))} \]

(14)
Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

$$
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{i^n}^{\otimes (k+I(n_1, \ldots, n_s))}
$$

(14)

Consider the $k$-particle function:

$$
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{i^n}^{\otimes (k+I(n_1, \ldots, n_s))} + R_{M_s}(t) + \cdots + R_{M_1}(t),
$$

(15)

where $R_{M_1}(t) := \sum_{n_1=M_1+1}^{\infty} \mathcal{E}_{n_1}(\tau) (f(t - \tau))^{\otimes (k+I(n_1))}$ and the other remainders are computed recursively.
Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

$$\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{in}^{\otimes (k+i(n_1,\ldots,n_s))}$$  \hspace{1cm} (14)

Consider the $k$-particle function:

$$\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{in}^{\otimes (k+i(n_1,\ldots,n_s))} + R_{M_s}(t) + \cdots + R_{M_1}(t), \hspace{1cm} (15)$$

where $R_{M_1}(t) := \sum_{n_1=M_1+1}^{\infty} \mathcal{E}_{n_1}(\tau) (f(t-\tau))^{\otimes (k+i(n_1))}$ and the other remainders are computed recursively. Choosing $M_2, \ldots, M_s$ as before, for any $\ell = 1, \ldots, s$ we have:

$$\|R_{M_\ell}(t)\|_\infty \leq (c'_\ell)^k (\lambda_\alpha)^{M_\ell / \ell}, \hspace{1cm} \text{for some } c'_\ell > 0 \hspace{1cm} (16)$$
Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

\[
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) \cdot f_{in}^{\otimes (k+I(n_1,\ldots,n_s))}
\]

(14)

Consider the \( k \)-particle function:

\[
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) \cdot f_{in}^{\otimes (k+I(n_1,\ldots,n_s))} + R_{M_s}(t) + \cdots + R_{M_1}(t),
\]

(15)

where \( R_{M_1}(t) := \sum_{n_1=M_1+1}^{\infty} \mathcal{E}_{n_1}(\tau) \cdot (f(t - \tau))^{\otimes (k+I(n_1))} \) and the other remainders are computed recursively. Choosing \( M_2,\ldots,M_s \) as before, for any \( \ell = 1,\ldots,s \) we have:

\[
\|R_{M_\ell}(t)\|_\infty \leq (c'_\ell)^k \cdot (\lambda_\alpha)^{\frac{M_\ell}{\ell}}, \quad \text{for some } c'_\ell > 0
\]

(16)

\( \rightarrow \) By a direct check one gets that (15) solves the U-U hierarchy with in. datum \( \{f_{in}^{\otimes_k}\}_k \)
Since term by term convergence holds (see Thm 1), the (finite) sums in (12) converge pointwise to:

$$
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{in}^{\otimes (k+I(n_1,\ldots,n_s))}
$$

Consider the $k$-particle function:

$$
\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{in}^{\otimes (k+I(n_1,\ldots,n_s))} + R_{M_s}(t) + \cdots + R_{M_1}(t), \tag{15}
$$

where $R_{M_1}(t) := \sum_{n_1=M_1+1}^{\infty} \mathcal{E}_{n_1}(\tau) (f(t - \tau))^{\otimes (k+I(n_1))}$ and the other remainders are computed recursively. Choosing $M_2, \ldots, M_s$ as before, for any $\ell = 1, \ldots, s$ we have:

$$
\|R_{M_\ell}(t)\|_{\infty} \leq (c'_\ell)^k (\lambda_\alpha)^{M_\ell / \ell}, \quad \text{for some } c'_\ell > 0 \tag{16}
$$

$\rightarrow$ By a direct check one gets that (15) solves the U-U hierarchy with in. datum $\{f_{in}^{\otimes k}\}_k$

$\rightarrow$ By the uniqueness of $L^\infty$-solns in the class $\{f_k : \|f_k\|_{\infty} \leq (c)^k\}$: $$(f(t))^{\otimes k} = (15)$$
Idea of the Proof of Thm 2. - Global in time Convergence

\[ f(N_k(t)) = P_{M_1} n_1 = 0 P_{M_2} n_2 = 0 \cdots P_{M_s} n_s = 0 E_{N_1}(\tau) E_{N_2}(\tau) \cdots E_{N_s}(\tau) f(\infty) + I(n_1, \ldots, n_s)(0) + P_{s=1} R_{M_s}(t), \]

where, for any \( s \) \( \triangleleft \cdots \triangleleft \), there exists \( c_s \triangleleft c_0 > 0 \) such that

\[ \| R_{M_s}(t) \| \leq k \| \tau \| M_s \triangleleft \| R_{M_s}(t) \| \leq (c_0 \| \tau \| M_s \triangleleft 1 I \]
Idea of the Proof of Thm 2. - Global in time Convergence

**E1.** \( f_k^N(t) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}^N(\tau) \mathcal{E}_{n_2}^N(\tau) \cdots \mathcal{E}_{n_s}^N(\tau) f_{k+I(n_1,\ldots,n_s)}(0) + \sum_{\ell=1}^{s} R_{M_{\ell}}^N(t) \)

**E2.** \( (f(t))^{\otimes k} = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} \mathcal{E}_{n_1}(\tau) \mathcal{E}_{n_2}(\tau) \cdots \mathcal{E}_{n_s}(\tau) f_{(k+I(n_1,\ldots,n_s))}^{\otimes k} + \sum_{\ell=1}^{s} R_{M_{\ell}}(t) \)

where, for any \( \ell = 1,\ldots,s \), there exist \( c_\ell, c'_\ell > 0 \) s. t.

\[ ||R_{M_{\ell}}^N(t)||_{\delta,k} \leq (c_\ell)^k (\lambda_\alpha) \frac{M_\ell}{\ell}, \quad ||R_{M_{\ell}}(t)||_{\infty} \leq (c'_\ell)^k (\lambda_\alpha) \frac{M_\ell}{\ell} \]

\( \lambda_\alpha < 1 \)

\( \triangleright \) the bound on the remainders are UNIFORM in \( N \) and \( \delta \)

\( \triangleright \) pointwise as \( N \to \infty, \delta \to 0, N\delta^3 = \alpha \) AND both strings are unif. bounded (see Thm 1)
Idea of the Proof of Thm 2. - Global in time Convergence

\[ E_1. \quad f_k^N(t) = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} E_{n_1}^N(\tau) E_{n_2}^N(\tau) \cdots E_{n_s}^N(\tau) f_{k+l(n_1,\ldots,n_s)}(0) + \]
\[ + \sum_{\ell=1}^{s} R_{M_{\ell}}^N(t), \]

\[ E_2. \quad (f(t))^{\otimes k} = \sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} \cdots \sum_{n_s=0}^{M_s} E_{n_1}(\tau) E_{n_2}(\tau) \cdots E_{n_s}(\tau) f_{\otimes (k+l(n_1,\ldots,n_s))} + \]
\[ + \sum_{\ell=1}^{s} R_{M_{\ell}}(t) \]

where, for any \( \ell = 1, \ldots, s \), there exist \( c_{\ell}, c'_{\ell} > 0 \) s. t.

\[ ||R_{M_{\ell}}^N(t)||_{\delta,k} \leq (c_{\ell})^k (\lambda_{\alpha})^{\frac{M_{\ell}}{\ell}}, \quad ||R_{M_{\ell}}(t)||_{\infty} \leq (c'_{\ell})^k (\lambda_{\alpha})^{\frac{M_{\ell}}{\ell}} \]

\[ \blacksquare \lambda_{\alpha} < 1 \]

\[ \blacksquare \text{the bound on the remainders are UNIFORM in } N \text{ and } \delta \]

\[ \blacksquare E_{n_1}(\tau) E_{n_2}(\tau) \cdots E_{n_s}(\tau) f_{k+l(n_1,\ldots,n_s)}^N(0) \rightarrow E_{n_1}(\tau) E_{n_2}(\tau) \cdots E_{n_s}(\tau) f_{\otimes (k+l(n_1,\ldots,n_s))} \]

pointwise as \( N \rightarrow \infty, \delta \rightarrow 0, N\delta^3 = \alpha \) AND both strings are unif. bounded (see Thm 1)

\[ \downarrow \]

\[ \lim_{N \rightarrow +\infty, \delta \rightarrow 0} \left| f_k^N(t) - (f(t))^{\otimes k} \right|_{L^1(\Lambda_k)} = 0, \quad \forall \text{ compact sets } \Lambda_k \subset \mathbb{R}^{3k} \]
An example of factorizing Initial Data (IDEA)

Consider the $N$-particle symmetric probability measure:

$$W_N(V_N) = \frac{1}{N!} \prod_{i=1}^{N} \phi(i)(v_i) \hookrightarrow (17)$$

where $P_N$ is the group of permutations on \{1\hookrightarrow...\hookrightarrow N\} and $1\hookrightarrow...\hookrightarrow N$ is a fixed sequence of different cells of volume $3$.

We have:

$$I_{supp W_N} \subseteq A_N$$

If $N_k(V_k) = \frac{1}{N_1}(...\prod_{i=1}^{k+1} P_{i_1...i_k}$:

$$N_r \neq i_s \Rightarrow i_1(v_1)\cdots i_k(v_k) \rightarrow^{3k, \text{ for any}} k \geq 1$$

In particular,

$$f_{N_1}(v) = \frac{1}{N} \sum_{i=1}^{N} \phi(i)(v) \rightarrow^{\text{empirical measure}} (18)$$

Choose the sequence of cells $1\hookrightarrow...\hookrightarrow N$ in such a way that, for any $k \in C_0 \mathbb{R}_3$,

$$Z dv f_{N_1}(v) \rightarrow Z dv f_{in}(v) \hookrightarrow$$

as $N \rightarrow 0 \rightarrow N$,

where $f_{in}(v) \in C_0 \mathbb{R}_3$, $supp f_{in} = A$ with $|A| > N_3 = \phi > 0$ and $||f_{in}||_1 < 1$.

Then, for any $k$, $f_{N_k}$ is WEAKLY converging to $f_{\infty k}$.

Good BUT too weak convergence....
Consider the $N$-particle symmetric probability measure:

$$W_N(V_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}_N} \frac{1}{\delta^{3N}} \prod_{i=1}^{N} \chi_{\Delta_{\pi(i)}}(v_i),$$

(17)

where $\mathcal{P}_N$ is the group of permutations on $\{1, \ldots, N\}$ and $\Delta_1 \ldots \Delta_N$ is a fixed sequence of different cells of volume $\delta^3$. 
Consider the $N$-particle symmetric probability measure:

$$
W_N(V_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}_N} \frac{1}{\delta^{3N}} \prod_{i=1}^{N} \chi_{\Delta_{\pi(i)}}(v_i),
$$

(17)

where $\mathcal{P}_N$ is the group of permutations on $\{1, \ldots, N\}$ and $\Delta_1 \ldots \Delta_N$ is a fixed sequence of different cells of volume $\delta^3$. We have:

- $\text{supp } W_N \subseteq A^N_\delta$

- $f_k^N(V_k) = \frac{1}{N(N-1)\ldots(N-k+1)} \sum_{i_1 \ldots i_k: \ i_r \neq i_s} \frac{\chi_{\Delta_{i_1}}(v_1)\ldots\chi_{\Delta_{i_k}}(v_k)}{\delta^{3k}}, \text{ for any } k \geq 1$

In particular,

$$
f_1^N(v) = \frac{1}{N} \sum_{i=1}^{N} \frac{\chi_{\Delta_i}(v)}{\delta^3} \sim \text{empirical measure}
$$

(18)
Consider the \( N \)-particle symmetric probability measure:

\[
W_N(V_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}_N} \frac{1}{\delta^{3N}} \prod_{i=1}^{N} \chi_{\Delta_{\pi(i)}}(v_i),
\]

where \( \mathcal{P}_N \) is the group of permutations on \( \{1, \ldots, N\} \) and \( \Delta_1 \ldots \Delta_N \) is a fixed sequence of different cells of volume \( \delta^3 \). We have:

\[
\bullet \quad \text{supp } W_N \subseteq A_\delta^N
\]

\[
\bullet \quad f_k^N(V_k) = \frac{1}{N(N-1)\ldots(N-k+1)} \sum_{i_1 \ldots i_k: \text{ } i_r \neq i_s} \frac{\chi_{\Delta_{i_1}}(v_1) \ldots \chi_{\Delta_{i_k}}(v_k)}{\delta^{3k}}, \quad \text{for any } k \geq 1
\]

In particular,

\[
f_1^N(v) = \frac{1}{N} \sum_{i=1}^{N} \frac{\chi_{\Delta_i}(v)}{\delta^3} \sim \text{empirical measure}
\]

Chose the sequence of cells \( \Delta_1 \ldots \Delta_N \) in such a way that, for any \( \varphi \in C_b^0(\mathbb{R}^3) \),

\[
\int dv \ f_1^N(v) \varphi(v) \to \int dv \ f_{in}(v) \varphi(v), \quad \text{as } N \to \infty, \delta \to 0, N\delta^3 = \alpha > 0,
\]

where \( f_{in}(v) \in C_c^0(\mathbb{R}^{3k}) \), \( \text{supp } f_{in} = A \) with \( |A| > N\delta^3 = \alpha \) and \( \|f_{in}\|_\infty < \frac{1}{\alpha} \).
An example of factorizing Initial Data (IDEA)

Consider the $N$-particle symmetric probability measure:

$$W_N(V_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}_N} \frac{1}{\delta^{3N}} \prod_{i=1}^{N} \chi_{\Delta_{\pi(i)}}(v_i),$$  \hspace{1cm} (17)

where $\mathcal{P}_N$ is the group of permutations on $\{1, \ldots, N\}$ and $\Delta_1 \ldots \Delta_N$ is a fixed sequence of different cells of volume $\delta^3$. We have:

- $\text{supp } W_N \subseteq A_\delta^N$

- $f_k^N(V_k) = \frac{1}{N(N-1)\ldots(N-k+1)} \sum_{i_1 \ldots i_k: \text{for any } k \geq 1} \frac{\chi_{\Delta_{i_1}}(v_1) \ldots \chi_{\Delta_{i_k}}(v_k)}{\delta^{3k}}$

In particular,

$$f_1^N(v) = \frac{1}{N} \sum_{i=1}^{N} \frac{\chi_{\Delta_i}(v)}{\delta^3} \sim \text{empirical measure}$$ \hspace{1cm} (18)

Chose the sequence of cells $\Delta_1 \ldots \Delta_N$ in such a way that, for any $\varphi \in C_0^0(\mathbb{R}^3)$,

$$\int dv \; f_1^N(v) \varphi(v) \to \int dv \; f_{\text{in}}(v) \varphi(v), \quad \text{as} \quad N \to \infty, \delta \to 0, N\delta^3 = \alpha > 0,$$ \hspace{1cm} (19)

where $f_{\text{in}}(v) \in C_0^0(\mathbb{R}^{3k})$, supp $f_{\text{in}} = A$ and $|A| > N\delta^3 = \alpha$ and $\|f_{\text{in}}\|_\infty < \frac{1}{\alpha}$. Then, for any $k$, $f_k^N$ is WEAKLY converging to $f_{\text{in}}^k$. Good BUT too weak convergence....
An example of factorizing Initial Data (IDEA)
An example of factorizing Initial Data (IDEA)

Uniform Distribution on (suitable) cells is a good idea but local smoothing is needed!

Trick: Introduce an intermediate scale $O(p)$ on which correlations vanish in the limit since they live on the smaller scale $O(p')$.

"New" partition of $R^3$ made by cells $e$ of side $p$ ($\ll \text{volume } 3^2$). Each "large" cell $e$ contains a lot of "small" cells of volume $3^2$.

We consider a symmetric $N$-particle distribution "saying that" particles are uniformly distributed on each "larger" cell $e$ with the addition of the exclusion constraint on the smaller cells (contained in $e$).

Particles whose velocities belong to different cells $e$ are independent.

For any $k_1$ and $v_1 \hookrightarrow \ldots \hookrightarrow v_k$, the resource exists $N_0 > 1$ ($0 << 1$) such that, for $N > N_0$ ($< 0$):

$v_1 \hookrightarrow \ldots \hookrightarrow v_k$ with $e_i \neq e_{i'}$.

$f_N(v_1) \mapsto X_i (N \hookrightarrow f_{\text{in}})$ $e_i (v_1)$ uniformly "adjusting" $i (N \hookrightarrow f_{\text{in}})$.
An example of factorizing Initial Data (IDEA)

UNIFORM DISTRIBUTION on (suitable) cells is a good idea **BUT** LOCAL SMOOTHING is needed!

**TRICK:** Introduce an intermediate scale $O(\sqrt{\delta})$ on which correlations vanish in the limit (since they live on the smaller scale $O(\delta)$).

"New" partition of $\mathbb{R}^3$ made by cells $\tilde{\Delta}$ of side $\sqrt{\delta}$ ($\sim$ volume $\delta^{3/2}$). Each "large" cell $\tilde{\Delta}$ contains A LOT of "small" cells $\Delta$ of volume $\delta^3$. 
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"New" partition of \( \mathbb{R}^3 \) made by cells \( \widetilde{\Delta} \) of side \( \sqrt{\delta} \) \( (\sim \) volume \( \delta^{3/2} \)). Each ”large” cell \( \widetilde{\Delta} \) contains A LOT of ”small” cells \( \Delta \) of volume \( \delta^3 \).

We consider a symmetric \( N \)-particle distribution ”saying that” particles are uniformly distributed on each ”larger” cell \( \widetilde{\Delta} \) with the addition of the exclusion constraint on the smaller cells \( \Delta \) (contained in \( \widetilde{\Delta} \)).
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  uniformly distributed on each "larger" cell $\tilde{\Delta}$ with the addition of the exclusion
  constraint on the smaller cells $\Delta$ (contained in $\tilde{\Delta}$).

→ Particles whose velocities belong to **DIFFERENT CELLS $\tilde{\Delta}$** are **INDEPENDENT**
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Particles whose velocities belong to DIFFERENT CELLS $\tilde{\Delta}$ are INDEPENDENT

For any $k \geq 1$ and $v_1, \ldots, v_k \in \mathcal{A}^k$, there surely exists $N_0 >> 1$ ($\delta_0 << 1$) such that, for $N > N_0$ ($\delta < \delta_0$): $v_1 \in \Delta_{i_1}, \ldots, v_k \in \Delta_{i_k}$ with $\Delta_{i_\ell} \neq \Delta_{i_m}$. 
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For any $k \geq 1$ and $v_1, \ldots, v_k \in \mathcal{A}^k$, there surely exists $N_0 >> 1$ ($\delta_0 << 1$) such that, for $N > N_0$ ($\delta < \delta_0$): $v_1 \in \tilde{\Delta}_{i_1}, \ldots, v_k \in \tilde{\Delta}_{i_k}$ with $\tilde{\Delta}_{i_\ell} \neq \tilde{\Delta}_{i_m}$.

\[ f^N_1(v_1) \approx \sum_i \gamma_i(N, f_{in}) \chi_{\tilde{\Delta}_i}(v_1) \rightarrow f_{in}(v_1) \text{ uniformly } \sim \text{ "adjusting" } \gamma_i(N, f_{in})! \]
Comments and Perspectives

• Actually, we can prove a more general result, recovering "statistical solutions" of the U-U eqn (Hewitt-Savage Theorem + Statistical Mechanics Tools to deal with non-factorized initial data.....)

• What about BOSONS (and Bose-Einstein Condensation)??

• Numerical Simulations ?
• Actually, we can prove a more general result, recovering "statistical solutions" of the U-U eqn (Hewitt-Savage Theorem + Statistical Mechanics Tools to deal with non-factorized initial data.....)

• What about BOSONS (and Bose-Einstein Condensation)??

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Thank you for your attention!