Perturbation theory for the defocusing nonlinear Schrödinger equation

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What is a dark soliton?
Outline

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- Dark solitons as solutions of the defocusing NLS
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The old theory
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- References
What is a Dark Soliton?

Dark solitons are manifested as localized dips in intensity that decay off of a continuous wave background. They are termed black if the intensity dip goes to zero and grey otherwise.

In terms of mathematics: Localized solutions of PDEs with non-zero boundary conditions and non-zero phase shift.
Do we like dark solitons?

Reasons to dislike:
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- Fundamental excitations of the universal defocusing NLS equation
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- Fundamental excitations of the universal defocusing NLS equation
- Observed in many applications: liquids, discrete mechanical systems, thin magnetic films, optical media, BECs, etc
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- Observed in many applications: liquids, discrete mechanical systems, thin magnetic films, optical media, BECs, etc

Conclusion: We don’t like dark solitons!
Solutions of the defocusing NLS

Normalized 1D NLS: \( i \frac{\partial u}{\partial z} - \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0 \)

Dark soliton solution

\[ u(z, t) = \sqrt{n_0} \exp(-i\mu z) (B \tanh \zeta + iA) \]

\[ \zeta \equiv \sqrt{\mu}B (t - \sqrt{\mu}A z) \]
Solutions of the defocusing NLS

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- \( A \) is the soliton velocity, \( B \) is the soliton depth
Solutions of the defocusing NLS

Normalized 1D NLS: $i \frac{\partial u}{\partial z} - \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0$

Dark soliton solution

$$u(z, t) = \sqrt{n_0} \exp(-i\mu z) \left( B \tanh \zeta + iA \right)$$

- $\zeta \equiv \sqrt{\mu} B \left( t - \sqrt{\mu} A z \right)$
- $A$ is the soliton velocity, $B$ is the soliton depth
- $A^2 + B^2 = 1$, so hereafter $A = \sin \phi$, $B = \cos \phi$
Solutions of the defocusing NLS

Normalized 1D NLS: \[ i \frac{\partial u}{\partial z} - \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0 \]

Dark soliton solution \[ u(z, t) = \sqrt{n_0} \exp(-i\mu z) (B \tanh \zeta + iA) \]

- \[ \zeta \equiv \sqrt{\mu} B (t - \sqrt{\mu} A z) \]
- \[ A \] is the soliton velocity, \( B \) is the soliton depth
- \[ A^2 + B^2 = 1 \], so hereafter \( A = \sin \phi \), \( B = \cos \phi \)
- Black solitons: \( B = 1 \), gray solitons \( B < 1 \)

Density notch with phase jumps:
\[ \Delta \phi = 2 \left[ \tan^{-1} \left( \frac{A}{B} \right) - \frac{\pi}{2} \right] \]
Integrals of motion

The equation

\[ iu_z - \frac{1}{2} u_{tt} + |u|^2 u = 0 \]

also has the following conserved quantities:

Energy: \( E = \int_{-\infty}^{+\infty} (u^2_\infty - |u|^2) \, dt \)

Momentum: \( I = \int_{-\infty}^{+\infty} \frac{i}{2} (uu_t^* - u^* u_t) \, dt \)

Center of mass: \( R = \int_{-\infty}^{+\infty} t (u^2_\infty - |u|^2) \, dt \)

Hamiltonian: \( H = \int_{-\infty}^{+\infty} \frac{1}{2} \left( |u_t|^2 + (u^2_\infty - |u|^2)^2 \right) \, dt \)
What happens under perturbation

Suppose now that we add a perturbation, namely

\[ iu_z - \frac{1}{2}u_{tt} + |u|^2 u = \epsilon F[u] \]

where \( \epsilon \ll 1 \). With

\[ u_s(z, t) = (A + iB \tanh [B(t - Az - t_0)]) e^{i\sigma_0} \]

we need to study the evolution of the soliton’s parameters under \( F[u] \).
The steps through an example

Consider the two photon absorption problem

\[ i u_z + \frac{1}{2} u_{tt} - |u|^2 u = -iK|u|^2 u \]

**Step 1:** Distinguish soliton (dip) from background

\[ u(z, t) = u_\infty(z)v(z, t) \]

so that

\[ i \frac{\partial u_\infty}{\partial z} - |u_\infty|^2 u_\infty = -iK|u_\infty|^2 u_\infty \]
The steps through an example

Consider the two photon absorption problem

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**Step 2:** Define new independent variables \( d\zeta = |u_\infty(z)|dz \) and \( \xi = |u_\infty(z)|t \) so that

\[ iv_\zeta + \frac{1}{2}v_{\xi\xi} - (|v|^2 - 1) v = -iK (|v|^2 - 1) v \]

where \( v(\zeta, \xi) = \cos \phi \tan T - i \sin \phi \), \( T = \cos \phi \left( \xi - \int_{-\infty}^{+\infty} \sin \phi \, d\zeta \right) \)
The steps through an example

Consider the two photon absorption problem

\[ iu_z + \frac{1}{2}u_{tt} - |u|^2 u = -iK|u|^2 u \]

Step 3: Solve the modified conserved quantities

\[ \frac{dH}{d\zeta} = -\epsilon \int_{-\infty}^{+\infty} \left( \tilde{F}[v]\tilde{v}^*_\zeta - \tilde{F}^*[v]v_\zeta \right) d\xi \]

where \( \tilde{F}[v] = -iK(|v|^2 - 1)v \) so that

\[ \frac{d\phi}{d\zeta} = \frac{\epsilon}{2\cos^2\phi \sin\phi} \text{Re} \left\{ \int_{-\infty}^{+\infty} \tilde{F}[v]v^*_\zeta \ d\xi \right\} \]
Finally, the background evolves according to

\[ u_\infty(z) = \frac{u_\infty(0)}{\sqrt{1 + 2 Ku_\infty^2(0)z}} e^{i\theta(z)}, \quad \theta(z) = \int_0^z u_\infty^2(z') \, dz' \]

and the soliton according to

\[ \frac{d\phi}{dz} = \frac{1}{3} Ku_\infty^2(z) \sin(2\phi) \]

\[ t_0(z) = \int_0^z \sin(z') \, dz' \]
Analysis vs simulations

Taken from Kivshar and Yang, PRE pp167, v49, 1994

FIG. 4. Contour plots demonstrating the effect of TPA on a dark soliton for $K = 0.05$ and $\phi(0) = 0.2\pi$ (left column) and $\phi(0) = 0.1\pi$ (right column) based on (a) adiabatic approximation given by Eqs. (41) and (46) and (b) numerical simulations of Eq. (40).
FIG. 4. Contour plots demonstrating the effect of TPA on a dark soliton for $K = 0.05$ and $\phi(0) = 0.2\pi$ (left column) and $\phi(0) = 0.1\pi$ (right column) based on (a) adiabatic approximation given by Eqs. (41) and (46) and (b) numerical simulations of Eq. (40).
Analysis vs simulations

Something is missing!
The true picture
Basic idea

- Go beyond adiabatic approximation
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- The complete soliton consists of: **core**, **background and shelf**
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- The complete soliton consists of: **core, background and shelf**
- The shelf is the **discrepancy** between the approximate soliton solution and the background
- To bridge the “inner” soliton and “outer” background we use a moving boundary layer
Basic idea

- Go beyond adiabatic approximation
- The complete soliton consists of: core, background and shelf
- The shelf is the discrepancy between the approximate soliton solution and the background
- To bridge the “inner” soliton and “outer” background we use a moving boundary layer
- The shelf dynamics will be derived from the perturbed conservation laws
Consider the perturbed NLS in the form

$$i\psi_z - \frac{1}{2}\psi_{tt} + |\psi|^2\psi = \epsilon F[\psi]$$

and remove the background as usual with

$$\psi = u(z, t) \exp(i \int_0^z u_\infty \, dz')$$

$$iu_z - \frac{1}{2}u_{tt} + (|u|^2 - u_\infty^2) u = \epsilon F[u]$$

The additional steps are as follows:

- Break the problem into two regions
The Complete Theory Outline

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The additional steps are as follows:

- Break the problem into **two regions**
- **Outer region** consists of the background; **inner region** consists of soliton and shelf
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The additional steps are as follows:

- Break the problem into two regions
- **Outer region** consists of the background; **inner region** consists of soliton and shelf
- Break \( u \) into **phase and magnitude** \( u = q \exp(i\phi) \)
Consider the perturbed NLS in the form

\[ i\psi_z - \frac{1}{2} \psi_{tt} + |\psi|^2 \psi = \epsilon F[\psi] \]

and remove the background as usual with \( \psi = u(z, t) \exp(i \int_0^Z u_\infty \, dz') \)

\[ iu_z - \frac{1}{2} u_{tt} + (|u|^2 - u_\infty^2) u = \epsilon F[u] \]

The additional steps are as follows:

- Break the problem into two regions
- **Outer region** consists of the **background**; **inner region** consists of soliton and shelf
- Break \( u \) into **phase and magnitude** \( u = q \exp(i \phi) \)
- Expand in **series of epsilon**

\[ q = q_0 + \epsilon q_1 + O(\epsilon^2), \quad \phi = \phi_0 + \epsilon \phi_1 + O(\epsilon^2) \]
The Complete Theory Outline

Consider the perturbed NLS in the form

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and remove the background as usual with \( \psi = u(z, t) \exp(i \int_0^z u_\infty \, dz') \)

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- **Outer region** consists of the background; **inner region** consists of soliton and shelf
- Break \( u \) into phase and magnitude \( u = q \exp(i\phi) \)
- Expand in series of epsilon

\[ q = q_0 + \epsilon q_1 + O(\epsilon^2), \quad \phi = \phi_0 + \epsilon \phi_1 + O(\epsilon^2) \]

- **Note:** The expansions are valid only in the inner region!
The expansion

At first order

\[ q_0 e^{i\phi_0} = (A + iB \tanh [B (t - Az - t_0)]) e^{i\sigma_0} \]

and \( A^2 + B^2 = u_\infty^2 \). Introduce a new scale \( Z = \epsilon z \), then:

- At \( O(\epsilon) \) the shape of the shelf is described by \( q_1^{\pm} \) and \( \phi_{1t}^{\pm} \) where \( q_1(Z, t) \to q_1^{\pm}(Z) \) and \( \phi_{1t}(Z, t) \to \phi_{1t}^{\pm}(Z) \) as \( t \to \pm\infty \)
The expansion

At first order

\[ q_0 e^{i\phi_0} = (A + iB \tanh [B(T - Az - t_0)]) e^{i\sigma_0} \]

and \(A^2 + B^2 = u_{\infty}^2\). Introduce a new scale \(Z = \epsilon z\), then:

- At \(O(\epsilon)\) the shape of the shelf is described by \(q_1^{\pm}\) and \(\phi_{1t}^{\pm}\) where \(q_1(Z, t) \to q_1^{\pm}(Z)\) and \(\phi_{1t}(Z, t) \to \phi_{1t}^{\pm}(Z)\) as \(t \to \pm\infty\).

- Use the modified conserved quantities

\[
\frac{dH}{dz} = \epsilon \left( E \frac{d}{dZ} u_{\infty}^2 + 2 \text{Re} \int_{-\infty}^{+\infty} F[u] u_z^* \, dt \right)
\]

\[
\frac{dE_D}{dz} = 2\epsilon \text{Im} \int_{-\infty}^{+\infty} F[u_{\infty}] u_{\infty} - F[u] u^* \, dt
\]

\[
\frac{dl}{dz} = 2\epsilon \text{Re} \int_{-\infty}^{+\infty} F[u] u_t^* \, dt
\]

\[
\frac{dR}{dz} = -l + 2\epsilon \text{Im} \int_{-\infty}^{+\infty} t \left( F[u_{\infty}] u_{\infty} - F[u] u^* \right) \, dt
\]
The evolution of the parameters

Then moving along the frame of reference $T = t - \int_0^Z A(\epsilon s) \, ds - t_0$

\[
\frac{d}{dZ} u_\infty = \text{Im} [F[u_\infty]]
\]

\[
2B \frac{d}{dZ} A = \left( \text{Re} \int_{-\infty}^{+\infty} F[u_0] u_0^* T dT \right)
\]

\[
u_\infty \frac{d}{dZ} \sigma_0 = \left( B_Z - \text{Im} \int_{-\infty}^{+\infty} F[u_\infty] u_\infty - F[u_0] u_0^* dT + \text{Re}[F[u_\infty]] \right)
\]

\[
q_1^+ = \frac{1}{2} \left( \sigma_0 Z + \Delta \phi_0 Z \right) / (u_\infty - A)
\]

\[
q_1^- = \frac{1}{2} \left( \sigma_0 Z - \Delta \phi_0 Z \right) / (u_\infty + A)
\]

\[
\phi_1^+ T = -2q_1^+, \quad \phi_1^- T = 2q_1^-
\]

\[
B_Z = (u_\infty u_\infty Z - AA_Z) / B
\]

\[
\Delta \phi_0 Z = (2AB_Z - 2BA_Z) / u_\infty^2 \quad \Delta \phi_0 = 2 \tan^{-1}(A / B)
\]
Two photon absorption

As before we take $F[u] = -i \gamma |u|^2 u$. Then:

$$\frac{d}{dZ}u_\infty = -\gamma u_\infty^3$$

$$\frac{d}{dZ}A = -\gamma \left( \frac{2}{3} A^2 + \frac{1}{3} u_\infty^2 \right) A$$

$$\frac{d}{dZ}\Delta \phi_0 = -\frac{4}{3} \gamma AB, \quad \Delta \phi_0 = 2 \tan^{-1}(A/B)$$

$$\sigma_{0Z} = \gamma \frac{B}{u_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 \right)$$

$$q_1^\pm = \gamma \frac{(u_\infty \pm A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 \pm \frac{4}{3} Au_\infty \right)$$

$$\phi_{1T}^+ = -2\gamma \frac{(u_\infty + A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 + \frac{4}{3} Au_\infty \right)$$

$$\phi_{1T}^- = 2\gamma \frac{(u_\infty - A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 - \frac{4}{3} Au_\infty \right)$$
Theory vs simulation

The diagram shows two functions as a function of distance $z$:
- $u_\infty(z)$
- $A(z)$

The blue line represents numerics, while the red dashed line represents asymptotics.
Are we missing anything?
The Power-Energy Saturation (PES) equation

To analyze dark solitons in mode-locked (ML) lasers we use a model expressed in the following dimensionless form,

\[ i\psi_z - \frac{1}{2}\psi_{tt} + |\psi|^2\psi = \frac{ig}{1 + E/E_0}\psi + \frac{i\tau}{1 + E/E_0}\psi_{tt} - \frac{il}{1 + P/P_0}\psi \]

where the complex electric field envelope \( \psi(z, t) \) is subject to the boundary conditions \( |\psi(z, t)| \rightarrow |\psi_\infty| \) as \( |t| \rightarrow \infty \). Here, 

- \( E(z) = \int_{-\infty}^{+\infty} (|\psi_\infty|^2 - |\psi|^2) \, dt \) is the dark-pulse energy,
- \( P(z, t) = |\psi_\infty|^2 - |\psi|^2 \) is the instantaneous power, while \( E_0 \) and \( P_0 \) are related to the saturation energy and power, respectively. Furthermore, \( g, \tau, \) and \( l \) are all positive, real constants, with the corresponding terms representing saturable gain, spectral filtering, and saturable loss.
Interesting dynamics

Integrate the PES with initial profile with a $\pi$-phase jump. The gain parameter $g$ is varied, while $E_0 = P_0 = 1$ and $\tau = l = 0.1$.

Locking onto stable dark solitons is only achieved when the gain term is sufficiently strong, i.e. the parameter $g$ is large enough to counter balance the losses.
Interesting interactions

Evolve the PES with initial condition $\psi(0, t) = \tilde{\psi}_\infty \tanh(\tilde{\psi}_\infty t + t_0)$ for $t < 0$ and $\psi(0, t) = -\tilde{\psi}_\infty \tanh(\tilde{\psi}_\infty t - t_0)$ for $t > 0$, where, $\tilde{\psi}_\infty$ can be approximated by $\psi_\infty / 2$; the dark-pulse energy is now twice that of the single soliton.

The major difference with the bright case, where the interaction is logarithmically slow, is that pulses repel sooner here due to the shelf interactions.
Perturbation theory to the PES model

To determine the evolution of the background wave in the framework of the PES, we assume that $\psi(z) = \psi_\infty(z) \exp[i\theta(z)]$. Separating real and imaginary parts, yields the following equation for the background amplitude $\psi_\infty(z)$:

$$\frac{d\psi_\infty}{dz} = \frac{g}{1 + 2|\psi_\infty|/E_0} \psi_\infty - i\psi_\infty.$$

Here, an approximate solution in the form $|\psi(z, t)| = |\psi_\infty \tanh(\psi_\infty t)|$ is assumed, which gives $E = 2|\psi_\infty|$.

A stable equilibrium (attractor) exists and can be found setting $d\psi_\infty/dz = 0$, namely,

$$|\psi_\infty| = \frac{E_0}{2} \left(\frac{g}{l} - 1\right)$$

This is the resulting background amplitude of the dark soliton and agrees with direct numerical simulation. Thus, dark solitons tend to an equilibrium (mode-lock) with constant energy and background.
Perturbation theory to the PES model

To put the problem in the correct notation we set:

\[ \epsilon F[u] = i \left( \frac{g}{1 + E/E_{\text{sat}}} u + \frac{\tau}{1 + E/E_{\text{sat}}} u_{tt} - \frac{l}{1 + P/P_{\text{sat}}} u \right) \]

where it is assumed that the small parameter \( \epsilon \) is implicitly contained in the right hand side of the PES. Hence,

\[
\frac{d}{dZ} E^{(0)} = 2\epsilon \text{Im} \left\{ \int_{-\infty}^{\infty} (F[u_\infty] u_\infty - F[u_0] u_0^*) \, dt \right\},
\]

\[
\frac{d}{dZ} \sigma_0 = \left( B_Z - \frac{1}{2} E_Z^{(0)} \right) / u_\infty,
\]

where \( E^{(0)} \) is the first order approximation for the energy, i.e.,

\[ E = E^{(0)} + \epsilon E^{(1)} + O(\epsilon^2), \]

and we have also used the fact that \( \text{Re}\{F[u_\infty]\} = 0 \) for this perturbation. Notice that the energy \( E^{(0)} \) at \( O(1) \), has contributions from both the core of the soliton and the shelf.
Perturbation theory to the PES model

We find that the evolution of the above subset of soliton parameters is described by the following closed system of equations,

\[
\frac{d}{dz} u_\infty = \frac{g}{1 + E^{(0)}/E_{\text{sat}}} u_\infty - l u_\infty ,
\]

\[
\frac{d}{dz} A = \frac{g}{1 + E^{(0)}/E_{\text{sat}}} A - \frac{lP_{\text{sat}}}{B \sqrt{B^2 + P_{\text{sat}}}} \tanh^{-1} \left( \frac{B}{\sqrt{B^2 + P_{\text{sat}}}} \right) A ,
\]

\[
\frac{d}{dz} E^{(0)} = \frac{4g}{1 + E^{(0)}/E_{\text{sat}}} B + \frac{2\tau/3}{1 + E^{(0)}/E_{\text{sat}}} B^3
\]

\[- 4l \frac{u_\infty^2 + P_{\text{sat}}}{\sqrt{B^2 + P_{\text{sat}}}} \tanh^{-1} \left( \frac{B}{\sqrt{B^2 + P_{\text{sat}}}} \right) .
\]

Note that the above equations are expressed in terms of \( z \), since the small parameter \( \epsilon \) is implicitly contained in the perturbation.
Comments

- There is a **discrepancy** between the soliton energy and the total energy and indicates that a **shelf will develop around the soliton**.

- The **shelf height** can be calculated as part of the perturbation analysis and, for the considered form of the perturbation, it turns out that it has a small size.

- Indeed, using typical parameter values, it can be seen that the **shelf height is** $O(10^{-3})$, which is too small to be observed in a plot of the soliton.
Perturbation theory to the PES model

Solid (blue) lines and dashed (red) lines correspond to the numerical results and the asymptotic analytical predictions, respectively. The vertical line indicates the spatial distance at which the shelves begin interacting. Here, the parameter values are $g = 0.5$, $\tau = 0.1$, $l = 0.1$ and $E_{\text{sat}} = P_{\text{sat}} = 1$. 
Gray Solitons

It can be shown that any deviation from a purely stationary, black soliton state (i.e., any grey soliton) will eventually degenerate into a continuous wave with renormalized energy $E = 0$.

Here, parameter values are $g = 0.3$, $\tau = 0.05$, $l = 0.1$ and $E_{\text{sat}} = P_{\text{sat}} = 1$. 

The NLS to KdV Connection

Consider the perturbed NLS

\[
    i \frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = F[\psi],
\]

where \( F[\psi] \) is a general functional perturbation.

**Step 1:** Write the solution in terms of a time dependent background function and a function \( u(t, x) \) such that \( \psi(t, x) = u_\infty(t) u(t, x) \) where the background and the function \( u(t, x) \) satisfy the system

\[
    i \frac{\partial u_\infty}{\partial t} + |u_\infty|^2 u_\infty = F[u_\infty]
\]

\[
    i \frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u_\infty|^2 (|u|^2 - 1) u = \frac{F[u_\infty u] - F[u_\infty] u}{u_\infty}
\]
The NLS to KdV Connection

Consider the perturbed NLS

\[ i \frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = F[\psi], \]

where \( F[\psi] \) is a general functional perturbation.

Step 2: Change the independent variables

\[ d\tau = |u_\infty|^2 dt \quad \text{and} \quad d\xi = |u_\infty| dx, \]

such that the above equation may be obtained from

\[ i \frac{\partial u}{\partial \tau} - \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} + (|u|^2 - 1) u = \frac{F[u_\infty u] - F[u_\infty]}{u_\infty |u_\infty|^2} \]
The NLS to KdV Connection

Consider the perturbed NLS

\[ i \frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = F[\psi], \]

where \( F[\psi] \) is a general functional perturbation.

**Step 3**: Employ the so-called Madelung transformation \( u(\tau, \xi) = \rho \exp(i\phi) \) (\( \rho \) and \( \phi \) denote the amplitude and phase of \( u \) respectively) to reduce the NLS to the hydrodynamic equations.

**Step 4**: Define new scales such that

\[ T = \varepsilon^3 \tau, \quad X = \varepsilon (\xi - C\tau), \]

where \( C \) is a constant to be determined.
The NLS to KdV Connection

Consider the perturbed NLS

$$i\frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = F[\psi],$$

where $F[\psi]$ is a general functional perturbation.

**Step 5:** Expand amplitude and phase in powers of $\varepsilon$ as follows:

$$\rho = \rho_0 + \varepsilon^2 \rho_2 + \varepsilon^4 \rho_4 + \varepsilon^6 \rho_6 + \ldots,$$

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + \varepsilon^5 \phi_5 + \varepsilon^7 \phi_7 + \ldots,$$

**Finally:** Match different orders of $\varepsilon$. 
Examples

Consider the pNLS equation:

\[ i \frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = i \nu \frac{\partial^2 \psi}{\partial x^2}, \]

Write the solution as \( \psi(t, x) = u_\infty(t) u(t, x) \), where the background function satisfies \( u_\infty(t) = u_0 \exp(-i|u_0|^2 t) \) (where \( u_0 \) is a complex constant).

Make the change

\[ \tau = u_0^2 t, \quad \xi = u_0 x, \]

Write \( u = \rho \exp(i\phi) \) and separate real and imaginary parts.
Examples

Expand and match different orders of $\epsilon$

$\mathcal{O}(\epsilon^0)$: $\rho_0^2 = 1$,

$\mathcal{O}(\epsilon^2)$: $2\rho_2 + C \frac{\partial \phi_1}{\partial X} = 0$,

$\mathcal{O}(\epsilon^3)$: $C \frac{\partial \rho_2}{\partial X} + \frac{1}{2} \frac{\partial^2 \phi_1}{\partial X^2} = 0$,

$\mathcal{O}(\epsilon^4)$: $2\rho_4 + 3\rho_2^2 + \frac{\partial \phi_3}{\partial X} - \frac{1}{2} \frac{\partial^2 \rho_2}{\partial X^2} - \frac{\partial \phi_1}{\partial T} = \nu \frac{\partial \rho_2}{\partial X}$,

$\mathcal{O}(\epsilon^5)$: $-\frac{\partial \rho_4}{\partial X} + 3\rho_2 \frac{\partial \rho_2}{\partial X} - \frac{1}{2} \frac{\partial^2 \phi_3}{\partial X^2} + \frac{\partial \rho_2}{\partial T} = -4\nu \rho_2^2 + \nu \frac{\partial^2 \rho_2}{\partial X^2}$.

The compatibility conditions at $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ yields $C = 1$ and from the last two orders we obtain the perturbed KdV equation:

$$\frac{\partial \rho_2}{\partial T} + 3\rho_2 \frac{\partial \rho_2}{\partial X} - \frac{1}{8} \frac{\partial^3 \rho_2}{\partial X^3} = -2\nu \rho_2^2 + \nu \frac{\partial^2 \rho_2}{\partial X^2}.$$
Examples

Similarly from

\[ i \frac{\partial \psi}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = i \delta |\psi|^2 \psi. \]

we get

\[ \frac{\partial \rho_2}{\partial T} + 3 \rho_2 \frac{\partial \rho_2}{\partial X} - \frac{1}{8} \frac{\partial^3 \rho_2}{\partial X^3} = \delta \rho_2. \]
References


