Non-dispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in $\mathbb{R}^3$

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joint work with C. Ortoleva
Energy critical NLS in $\mathbb{R}^3$

We consider the initial value problem

$$i\psi_t = -\Delta \psi - |\psi|^4 \psi, \quad x \in \mathbb{R}^3, \quad \text{(NLS)}$$

$$\psi|_{t=0} = \psi_0 \in \dot{H}^1(\mathbb{R}^3).$$

This Cauchy problem is locally well posed in $\dot{H}^1$: for any $\psi_0 \in \dot{H}^1(\mathbb{R}^3)$ there exists $0 < T \leq +\infty$ and a unique solution $\psi \in C([0, T), \dot{H}^1) \cap L^{10}(I \times \mathbb{R}^3)$ for any compact interval $I \subset [0, T)$, and either $T = +\infty$ or $T < +\infty$ and then $\|\psi(t)\|_{L^{10}((0, T) \times \mathbb{R}^3)} = \infty$. 
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Fundamental properties

- conservation of the energy:
  \[ E(\psi(t)) \equiv \int \left( |\nabla \psi(x, t)|^2 - \frac{1}{3} |\psi(x, t)|^6 \right) dx = E(\psi_0) \]

- conservation of the mass: if \( \psi_0 \in L^2 \),
  \[ \int |\psi(x, t)|^2 \, dx = \int |\psi_0(x)|^2 \, dx \]

- virial identity: if \( \langle x \rangle \psi_0 \in L^2 \),
  \[ \frac{d^2}{dt^2} \int |x|^2 |\psi(x, t)|^2 dx = 8 \int (|\nabla \psi(x, t)|^2 - |\psi(x, t)|^6) \, dx \]

- scaling:
  if \( \psi(x, t) \) is a solution of (NLS), then for all \( \lambda > 0 \), \( \lambda^{1/2} \psi(\lambda x, \lambda^2 t) \)
  is also a solution
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Global existence/ finite time blow up results

- global existence and scattering for small $\dot{H}^1$ data
- existence of finite time blow up solutions by virial identity
- stationary solutions:

$$W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}, \quad \Delta W + W^5 = 0$$

- dynamics below the critical level: $E(\psi_0) < E(W)$ (radial solutions, Kenig, Merle):
  (i) if $\|\nabla \psi_0\|_{L^2} < \|\nabla W\|_{L^2}$ then the solution is global and scatters to zero as a free wave in both directions;
  (ii) if $\psi_0 \in L^2$ and $\|\nabla \psi_0\|_{L^2} > \|\nabla W\|_{L^2}$, then the solution blows up in finite time in both direction.

- classification of the dynamics on the critical level (Duyckaerts and Merle): in addition to the finite time blow up and scattering to zero (and $W$ itself), one has the existence of solutions that converge as $t \to \infty$ to a rescaled ground state.
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Question

Q : Behavior of the solutions above the critical level?
Energy critical wave equation

\[ u_{tt} - \Delta u - u^5 = 0, \quad x \in \mathbb{R}^3 \quad (NLW) \]

\[ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3), \quad u_t|_{t=0} = u_1 \in L^2(\mathbb{R}^3). \]

Locally well posed

Duyckaerts, Kenig, Merle: classification of radial solutions
Classification of radial solutions of (NLW)

Let $u$ be a radial solution of (NLW) and $T > 0$ its maximal positive time of definition. Then one of the following holds.

- **Type I blow up**: $T < \infty$ and $\lim_{t \to T} \|(u(t), u_t(t))\|_{H^1 \times L^2} = \infty$

- **Type II blow up**: $T < \infty$ and there exists $v_0 \in \dot{H}^1$, $v_1 \in L^2$, an integer $J \in \mathbb{N}^*$ and for all $j \in \{1, \ldots, J\}$, a sign $\varepsilon_j$ and a positive function $\lambda_j(t)$ defined for $t$ close to $T$ such that, as $t \to T$,
  
  $$(T - t)^{-1} \ll \lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t),$$

  
  $$u(t) = \sum_{j=1}^{J} \varepsilon_j \lambda_j^{1/2}(t)W(\lambda_j(t)x) + v_0 + o_{\dot{H}^1}(1), \quad u_t(t) = v_1 + o_{L^2}(1).$$

- **Global solution**: $T = +\infty$ and there exists a solution $v_L$ of the linear wave equation, an integer $J \in \mathbb{N}$ and for all $j \in \{1, \ldots, J\}$, a sign $\varepsilon_j$ and a positive function $\lambda_j(t)$ defined for large $t$ such that, as $t \to +\infty$,
  
  $$t^{-1} \ll \lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t),$$

  
  $$u(t) = \sum_{j=1}^{J} \varepsilon_j \lambda_j^{1/2}(t)W(\lambda_j(t)x) + v_L(t) + o_{\dot{H}^1}(1), \quad u_t(t) = \partial_t v_L(t) + o_{L^2}(1).$$
Constructive results for NLW

- **Krieger, Schlag, Tataru**: for any \( \nu > 0 \), existence of type II blow up solutions with \( J = 1 \), \( \lambda(t) = t^{-1-\nu} \) and \( (v_0, v_1) \) arbitrarily small in \( \dot{H}^1 \times L^2 \)
- more recently, **Donninger, Krieger, Schlag**: similar result with \( \lambda(t) = t^{-1-\nu} \exp(\varepsilon_0 \sin(\ln(t))), \nu > 3, \varepsilon_0 \) small
- **Donninger, Krieger**: global solutions with \( J = 1 \), \( \lambda(t) = t^\nu \), \( \nu \) small.
Main result

Theorem

There exists $\beta_0 > 0$ such that for any $\nu$, $\alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \leq \beta_0$ and any $\delta > 0$ there exist $T > 0$ and a radial solution $\psi \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ to (NLS) of the form:

$$\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t),$$

(1)

where $\lambda(t) = t^{\nu}$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ verifies:

$$\|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} \leq \delta,$$

$$\|\zeta(t)\|_{L^\infty} \leq C t^{-\frac{1+\nu}{2}},$$

(2)

for all $t \geq T$.

Furthermore, there exists $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, such that, as $t \to +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \to 0$ in $\dot{H}^1 \cap \dot{H}^2$. 
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Remarks

- Theorem 1 remains valid, in fact, with $\dot{H}^2$ replaced by $\dot{H}^k$ for any $k \geq 2$ (with $\beta_0$ depending on $k$).
- The restriction on $\nu$ and $\alpha_0$ that appears in Theorem 1 seems to be technical. One might expect the same result to be true for any $\nu > -1/2$ and any $\alpha_0 \in \mathbb{R}$.
- With the similar techniques one can prove the existence of radial finite time blow up solutions of the form
  \[ \psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t), \lambda(t) = (T - t)^{-1/2 - \nu}, \]
  \[ \alpha(t) = \alpha_0 \ln(T - t), \] where $\zeta(t)$ is arbitrary small in $\dot{H}^1 \cap \dot{H}^2$ and $\nu > 1$, $\alpha_0 \in \mathbb{R}$ can be chosen arbitrarily (in the spirit of the Krieger-Tataru-Schlag result for NLW)
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Proof

The rest of the talk is devoted to the proof of this result.

- **Step 1**: construct an approximate solution to (NLS) that has the form required by the theorem and solves the equation up to an error of order $O_{H^2}(t^{-\kappa})$ with some $\kappa > 2$ independent of $\nu$. This part of the construction is closely related to the works of Donninger, Krieger, Schlag, Tataru.

- **Step 2**: convert the approximate solution into an exact solution by solving the equation for the remainder with zero initial data at infinity. Main technical tool: energy type estimates for the linearized evolution.
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Construction of an approximate solution

Proposition

For any \( \nu \) and \( \alpha_0 \) sufficiently small and any \( 0 < \delta \leq 1 \) there exists a radial approximate solution \( \psi^{ap}(x, t) \) to (NLS) such that the following holds for \( t \geq T \) with some \( T = T(\nu, \alpha_0, \delta) > 0 \).

(i) \( \psi^{ap} \) has the form:
\[
\psi^{ap}(x, t) = e^{i\alpha(t)} \lambda^{1/2}(t)(W(\lambda(t)x) + \chi^{ap}(\lambda(t)x, t)),
\]
where \( \chi^{ap}(y, t), \) \( y = \lambda(t)x, \) verifies
\[
\| \chi^{ap}(t) \|_{\dot{H}^k} \leq C\delta^{\nu+k-1/2}t^{-\nu(k-1)}, \quad k = 1, 2, \quad (3)
\]
\[
\| \chi^{ap}(t) \|_{L^\infty} \leq Ct^{-(1+2\nu)/2}, \quad (4)
\]
\[
\| |y|^{-k}\nabla^l \chi^{ap}(t) \|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}, \quad k + l = 2, \quad (5)
\]

Furthermore, there exists \( \zeta^* \in \dot{H}^s \), for any \( s > \frac{1}{2} - \nu \), such that, as \( t \to +\infty \),
\[
e^{i\alpha(t)} \lambda^{1/2}(t)\chi^{ap}(\lambda(t)\cdot, t) - e^{it\Delta} \zeta^* \to 0 \text{ in } \dot{H}^1 \cap \dot{H}^2.
\]

(ii) The corresponding error \( R = -i\psi_t^{ap} - \Delta\psi^{ap} - |\psi^{ap}|^4\psi^{ap} \) satisfies
\[
\| R(t) \|_{\dot{H}^k} \leq t^{-(2+\frac{1}{8})(1+2\nu)+\nu(k+1)}, \quad k = 0, 1, 2. \quad (6)
\]
Construction of an approximate solution

The construction of $\psi^{ap}$ is achieved by considering separately

- **inner region**: $|x| t^\nu \lesssim 1$
- **self-similar region**: $|x| t^{-1/2} \lesssim 1$
- **remote region**: $|x| t^{-1} \lesssim 1$

In the inner region the solution will be constructed as a perturbation of the profile $e^{i\alpha_0 \ln t^{\nu/2}} W(t^{\nu} x)$. The self-similar and remote regions are the regions where the solution is small and is described essentially by the linear equation $i\psi_t = -\Delta \psi$. In the self-similar region the profile of the solution will be determined uniquely by the matching conditions coming out from the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching procedure.
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Inner region: $0 \leq t^{\nu} |x| \leq t^{1/2+\nu-\epsilon_1}$, $0 < \epsilon_1 < 1/2 + \nu$ to be fixed later.

Write $\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)\hat{\psi}(\rho, t)$, $\rho = \lambda(t)|x|$, then $\hat{\psi}(\rho, t)$ solves

$$it^{-2\nu}\hat{\psi}_t - \alpha_0 t^{-(1+2\nu)}\hat{\psi} + i\nu t^{-(1+2\nu)}\left(\frac{1}{2} + \rho \partial_\rho\right)\hat{\psi} = -\Delta\hat{\psi} - |\hat{\psi}|^4\hat{\psi}. \quad (7)$$

Write $\hat{\psi}(\rho, t) = W(\rho) + \chi(\rho, t)$.

Ansatz for $\chi$:

$$\chi(\rho, t) = \sum_{k=1}^{\infty} t^{-k(1+2\nu)}\chi_k(\rho) \quad (8)$$
Inner region

\[ 0 \leq t^\nu |x| \leq t^{1/2 + \nu - \epsilon_1}, \quad 0 < \epsilon_1 < 1/2 + \nu \] to be fixed later.

Write \( \psi(x, t) = e^{i\alpha(t)} \lambda^{1/2}(t) \hat{\psi}(\rho, t), \quad \rho = \lambda(t)|x| \), then \( \hat{\psi}(\rho, t) \) solves

\[
it^{-2\nu} \hat{\psi}_t - \alpha_0 t^{-(1+2\nu)} \hat{\psi} + \nu t^{-(1+2\nu)} \left( \frac{1}{2} + \rho \partial_\rho \right) \hat{\psi} = -\Delta \hat{\psi} - |\hat{\psi}|^4 \hat{\psi}. \tag{7}
\]

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Ansatz for \( \chi \):

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\]
Inner region: recurrent system for \((\chi_k)\)

Eq. (7) is equivalent to:

\[
\begin{align*}
H\vec{\chi}_1 &= \alpha_0 W \left( \begin{array}{c} 1 \\ -1 \end{array} \right) + i\nu W_1 \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \\
H\vec{\chi}_k &= D_k, \quad k \geq 2,
\end{align*}
\]

(9)

where

\[\vec{\chi}_k = \begin{pmatrix} \chi_k \\ \bar{\chi}_k \end{pmatrix},\]

\[H = -\triangle \sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\]

\[W_1(\rho) = \left( \frac{1}{2} + \rho \partial_\rho \right) W(\rho),\]

\[D_k = D_k(\rho; \chi_j, 1 \leq j \leq k - 1).\]

We subject (9) to zero initial conditions at \(\rho = 0:\)

\[\chi_k(0) = \partial_\rho \chi_k(0) = 0.\]
Inner region: recurrent system for \((\chi_k)\)

Eq. (7) is equivalent to:

\[
\begin{align*}
H\vec{\chi}_1 &= \alpha_0 W\left(\frac{1}{-1}\right) + i\nu W_1\left(\frac{1}{1}\right), \\
H\vec{\chi}_k &= D_k, \quad k \geq 2,
\end{align*}
\]

(9)

where

\[
\vec{\chi}_k = \begin{pmatrix} \chi_k \\ \overline{\chi}_k \end{pmatrix},
\]

\[
H = -\Delta \sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
W_1(\rho) = \left(\frac{1}{2} + \rho\partial_{\rho}\right)W(\rho),
\]

\[
D_k = D_k(\rho; \chi_j, 1 \leq j \leq k - 1).
\]

We subject (9) to zero initial conditions at \(\rho = 0\):

\[
\chi_k(0) = \partial_{\rho}\chi_k(0) = 0.
\]
Inner region : recurrent system for \((\chi_k)\)

**Lemma**

System (9) has a unique solution \((\chi_k)_{k \geq 1}\) verifying:

i) for any \(k \geq 1\), \(\chi_k\) is a \(C^\infty\) function that has an even Taylor expansion at \(\rho = 0\) that starts at order \(2k\);

ii) as \(\rho \to +\infty\), \(\chi_k = O(\rho^{2k-1})\), \(k \geq 1\). More precisely,

\[
\chi_k(\rho) = \sum_{j \leq 2k-1} \alpha_j^{(k)} \rho^j, \quad \rho \to +\infty. \tag{10}
\]

Ansatz (8) is meaningful if \(\rho \ll t^{1/2+\nu}\).

Define:

\[
\psi_{in}^{ap}(x, t) = e^{i\alpha(t)} \lambda^{1/2}(t) (W(\rho) + \sum_{k=1}^{N} t^{-k(1+2\nu)} \chi_k(\rho)).
\]

\(N\) large to be fixed later.
Self-similar region

Self-similar region: \( \frac{1}{2} t^{-\epsilon_1} \leq |x| t^{-1/2} \leq t^{\epsilon_2}, \ 0 < \epsilon_2 < \frac{1}{2} \) to be fixed later.

By (10), as \( \rho \to \infty \), \( y \equiv |x| t^{-1/2} \to 0 \) one has, at least, formally

\[
\psi(x, t) = t^{-1/4} e^{i\alpha(t)} \sum_{j \geq 0} t^{-\frac{1}{4}(2j+1)(1+2\nu)} \sum_{k \geq 0} \alpha_{2k-j-1}^{(k)} y^{2k-j-1},
\]

Ansatz:

\[
\psi(x, t) = t^{-1/4} e^{i\alpha(t)} \sum_{j \geq 0} t^{-\frac{1}{4}(2j+1)(1+2\nu)} A_j(y). \tag{11}
\]
Self-similar region : recurrent system for \((A_j)\)

Substituting (11) into (NLS) one gets

\[
\begin{cases}
(\mathcal{L} + \mu_j)A_j = 0, & j = 0, 1, \\
(\mathcal{L} + \mu_j)A_j = G_j, & j \geq 2,
\end{cases}
\]

(12)

\[
\mathcal{L} = -\Delta + \frac{i}{2} \left( \frac{1}{2} + y \partial_y \right), \quad \mu_j = \alpha_0 + \frac{i}{4}(2j + 1)(1 + 2\nu),
\]

\[
G_j = G_j(y; A_i, i \leq j - 2).
\]

Matching conditions :

\[
A_j(y) = \sum_{k \geq 0} \alpha_{2k-j-1}^{(k)} y^{2k-j-1}, \quad y \to 0.
\]

(13)

**Lemma**

*There exists a unique solution \((A_j)\) of (12) verifying matching conditions (13).*
Self-similar region: recurrent system for \((A_j)\)

Substituting (11) into (NLS) one gets

\[
\begin{cases}
(L + \mu_j)A_j = 0, & j = 0, 1, \\
(L + \mu_j)A_j = G_j, & j \geq 2,
\end{cases}
\]

\[
\begin{align*}
L &= -\Delta + \frac{i}{2} \left( \frac{1}{2} + y \partial_y \right), \\
\mu_j &= \alpha_0 + \frac{i}{4} (2j + 1)(1 + 2\nu), \\
G_j &= G_j(y; A_i, i \leq j - 2).
\end{align*}
\]

Matching conditions:

\[
A_j(y) = \sum_{k \geq 0} \alpha^{(k)}_{2k-j-1} y^{2k-j-1}, \quad y \to 0.
\]

Lemma

There exists a unique solution \((A_j)\) of (12) verifying matching conditions (13).
Self-similar region : recurrent system for \((A_j)\)

Substituting (11) into (NLS) one gets

\[
\begin{aligned}
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(L + \mu_j)A_j &= G_j, \quad j \geq 2,
\end{cases}
\end{aligned}
\]  

(12)

\[
L = -\Delta + \frac{i}{2} \left( \frac{1}{2} + y \partial_y \right), \quad \mu_j = \alpha_0 + \frac{i}{4}(2j + 1)(1 + 2\nu),
\]

\[
G_j = G_j(y; A_i, i \leq j - 2).
\]

Matching conditions :

\[
A_j(y) = \sum_{k \geq 0} \alpha_{2k-j-1}^{(k)} y^{2k-j-1}, \quad y \to 0.
\]  

(13)

Lemma

There exists a unique solution \((A_j)\) of (12) verifying matching conditions (13).
Approximate solution in the self-similar region

Define

$$\psi_{ss}^{ap}(x, t) = e^{i\alpha(t)} t^{-1/4} \sum_{j=0}^{2} t^{-\frac{1}{4}(2j+1)(1+2\nu)} A_j(y).$$

Behavior at infinity: as $y \to \infty$,

$$A_j(y) = d_j f_1(y, \mu_j) + c_j f_2(y, \mu_j), \quad j = 0, 1,$$

$$A_2(y) = d_2 f_1(y, \mu_2) + c_2 f_2(y, \mu_2) + O(y^{-5-5\nu}), \quad (14)$$

where $f_1$, $f_2$ are solutions of $(\mathcal{L} + \mu)f = 0$ with the following behavior at infinity:

$$f_1(y, \mu) = y^{-1/2+2i\mu} (1 + O(y^{-2})), \quad$$

$$f_2(y, \mu) = e^{i\frac{\nu^2}{4}} y^{-5/2-2i\mu} (1 + O(y^{-2})).$$
Remote region

Remote region : $\frac{1}{2} t^{1/2+\epsilon_2} \leq |x|$. 

By previous analysis : for $|x| t^{-1/2} \gg 1$, $|x| t^{-1} \ll 1$,

$$\psi_{ss}^{ap}(x, t) \sim \sum_{j=0, 1, 2} d_j r^{-1-\nu-j(1+2\nu)+2i\alpha_0} + e^{i \frac{|x|^2}{4t}} \frac{1}{t^{3/2}} \sum_{j=0, 1, 2} c_j \left(\frac{x}{t}\right)^{-2+\nu+j(1+2\nu)-2i\alpha_0}$$

(the first sum in the r.h.s is the contribution of $f_1(y, \mu_j)$ and the second one comes from $f_2(y, \mu_j)$, $j = 0, 1, 2$).
Approximate solution in the remote region

Define:

\[ \psi_{\text{rem}}^{ap}(x, t) = e^{i\alpha(t)} t^{-1/4} \sum_{j=0}^{2} t^{-\frac{1}{4}(2j+1)(1+2\nu)} d_j f_1(y, \mu_j) + v(x, t), \quad (15) \]

\[ v(x, t) = e^{i\frac{|x|^2}{4t}} \left( \hat{\zeta} \left( \frac{x}{t} \right) - \frac{i}{t} \Delta \hat{\zeta} \left( \frac{x}{t} \right) \right), \]

\[ \hat{\zeta}(\xi) = \Theta(\frac{\xi}{\delta}) \sum_{j=0,1,2} c_j |\xi|^{-2+\nu+j(1+2\nu)-2i\alpha_0}, \]

\[ \Theta \in C_0^\infty(R^3) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}. \]

Remark: the second term in the r.h.s. of (15) verifies

\[ iv_t + \Delta v = i t^{-9/2} e^{i\frac{|x|^2}{4t}} (\Delta^2 \hat{\zeta}) \left( \frac{x}{t} \right). \]
Approximate solution in the remote region

Define:

\[ \psi_{rem}^{ap}(x, t) = e^{i\alpha(t)} t^{-1/4} \sum_{j=0}^{2} t^{-\frac{1}{4}(2j+1)(1+2\nu)} d_j f_1(y, \mu_j) + \nu(x, t), \quad (15) \]

\[ \nu(x, t) = \frac{e^{i\frac{|x|^2}{4t}}}{t^{3/2}} \left( \hat{\zeta} \left( \frac{x}{t} \right) - \frac{i}{t} \Delta \hat{\zeta} \left( \frac{x}{t} \right) \right), \]

\[ \hat{\zeta}(\xi) = \Theta\left( \frac{\xi}{\delta} \right) \sum_{j=0,1,2} c_j |\xi|^{-2+\nu+j(1+2\nu)-2i\alpha_0}, \]

\[ \Theta \in C_0^\infty(\mathbb{R}^3) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}. \]

Remark: the second term in the r.h.s. of (15) verifies

\[ iv_t + \Delta \nu = it^{-9/2} e^{i\frac{|x|^2}{4t}} (\Delta^2 \hat{\zeta}) \left( \frac{x}{t} \right). \]
Approximate solution in the remote region

Define :

$$\psi_{rem}^{ap}(x, t) = e^{i\alpha(t)} t^{-1/4} \sum_{j=0}^{2} t^{-\frac{1}{4}(2j+1)(1+2\nu)} d_{j} f_{1}(y, \mu_{j}) + v(x, t),$$

(15)

$$v(x, t) = \frac{e^{i \frac{|x|^{2}}{4t}}}{t^{3/2}} \left( \hat{\zeta} \left( \frac{x}{t} \right) - \frac{i}{t} \Delta \hat{\zeta} \left( \frac{x}{t} \right) \right),$$

$$\hat{\zeta}(\xi) = \Theta(\frac{\xi}{\delta}) \sum_{j=0,1,2} c_{j} |\xi|^{-2+\nu+j(1+2\nu)-2i\alpha_{0}},$$

$$\Theta \in C_{0}^{\infty}(R^{3}) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| \geq 2 \end{cases}.$$  

Remark : the second term in the r.h. s. of (15) verifies

$$iv_{t} + \Delta v = it^{-9/2} e^{i \frac{|x|^{2}}{4t}} (\Delta^{2} \hat{\zeta}) \left( \frac{x}{t} \right).$$
Define

\[
\psi^{ap}(x, t) = \Theta(t^{-1/2+\epsilon_1}x)\psi_{in}^{ap}(x, t) + (1 - \Theta(t^{-1/2+\epsilon_1}x))\Theta(t^{-1/2-\epsilon_2}x)\psi_{ss}^{ap}(x, t) \\
+ (1 - \Theta(t^{-1/2-\epsilon_2}x))\psi_{out}^{ap}(x, t), \quad x \in \mathbb{R}^3.
\]

Fix \( \frac{3}{8} < \epsilon_2 < \frac{1}{2} \), \( \epsilon_1 = \frac{1+2\nu}{27} \), \( N = 27 \).

Then \( \psi^{ap}(x, t) \) verifies the proposition.
Constrution of an exact solution

Write

\[ \psi(x, t) = \psi^{ap}(x, t) + g(x, t), \]
\[ g(x, t) = e^{i\alpha(t)\lambda^{1/2}(t)f(y, \tau)}, \quad y = \lambda(t)x, \quad \tau = \int_0^t \lambda^2(t)dt = \frac{t^{1+2\nu}}{1 + 2\nu}. \]

Then one gets the following equation for the remainder \( f \)

\[ if_\tau = \mathcal{H}(\tau) \vec{f} + \mathcal{V}(\tau) \bar{f} + \mathcal{F}(f) + r, \quad \vec{f} = \left( \begin{array}{c} f \\ \bar{f} \end{array} \right), \]

\[ \mathcal{H}(\tau) = H + i\nu_1 \tau^{-1} \left( \frac{1}{2} + y \cdot \nabla \right), \]
\[ H = -\Delta \sigma_3 - 3W^4 \sigma_3 - 2W^4 \sigma_3 \sigma_1, \quad \nu_1 = \frac{\nu}{2\nu + 1}, \]
\[ \|\mathcal{V}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(\|\alpha_0\| + \|\nu\|)\tau^{-1}, \quad i = 1, 2, \]
\[ \mathcal{F}(f) = O(f^2), \]
\[ \|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}} \]

(\( r(\tau) \) is the error generated by \( \psi^{ap} \)).
Construction of an exact solution

Write
\[ \psi(x, t) = \psi^{ap}(x, t) + g(x, t), \]
\[ g(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)f(y, \tau), \quad y = \lambda(t)x, \quad \tau = \int_0^t \lambda^2(t)\,dt = \frac{t^{1+2\nu}}{1 + 2\nu}. \]

Then one gets the following equation for the remainder \( f \)
\[ if_{\tau} = \mathcal{H}(\tau)\vec{f} + \mathcal{V}(\tau)\vec{f} + \mathcal{F}(f) + r, \quad \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \]
\[ \mathcal{H}(\tau) = H + i\nu_1\tau^{-1}\left(\frac{1}{2} + y \cdot \nabla\right), \]
\[ H = -\triangle \sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \nu_1 = \frac{\nu}{2\nu + 1}, \]
\[ \|\mathcal{V}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)^{-1}, \quad i = 1, 2, \]
\[ \mathcal{F}(f) = O(f^2), \]
\[ \|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2 - \frac{1}{8}} \]

(\( r(\tau) \) is the error generated by \( \psi^{ap} \)).
Construction of an exact solution

Write

$$\psi(x, t) = \psi_{ap}(x, t) + g(x, t),$$

$$g(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)f(y, \tau), \quad y = \lambda(t)x, \quad \tau = \int_0^t \lambda^2(t)dt = \frac{t^{1+2\nu}}{1 + 2\nu}.\$$

Then one gets the following equation for the remainder $f$

$$i\bar{f}_\tau = \mathcal{H}(\tau)\bar{f} + \mathcal{V}(\tau)\bar{f} + \mathcal{F}(f) + r, \quad \bar{f} = \begin{pmatrix} \bar{f} \\ \bar{f} \end{pmatrix},$$

$$\mathcal{H}(\tau) = H + i\nu_1\tau^{-1}\left(\frac{1}{2} + y \cdot \nabla\right),$$

$$H = -\triangle \sigma_3 - 3W^4 \sigma_3 - 2W^4 \sigma_3 \sigma_1, \quad \nu_1 = \frac{\nu}{2\nu + 1},$$

$$\|\mathcal{V}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)\tau^{-1}, \quad i = 1, 2,$$

$$\mathcal{F}(f) = O(f^2),$$

$$\|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}}.$$

($r(\tau)$ is the error generated by $\psi_{ap}$).
Construction of an exact solution

Write

\[ \psi(x, t) = \psi^{ap}(x, t) + g(x, t), \]

\[ g(x, t) = e^{i\alpha(t)\lambda^{1/2}(t)f(y, \tau)}, \quad y = \lambda(t)x, \quad \tau = \int_0^t \lambda^2(t)dt = \frac{t^{1+2\nu}}{1+2\nu}. \]

Then one gets the following equation for the remainder \( f \)

\[ if_{\tau} = \mathcal{H}(\tau)\vec{f} + \mathcal{V}(\tau)\vec{f} + \mathcal{F}(f) + r, \quad \vec{f} = \left( \begin{array}{c} f \\ \bar{f} \end{array} \right), \]

\[ \mathcal{H}(\tau) = H + i\nu_1\tau^{-1}\left(\frac{1}{2} + y \cdot \nabla\right), \]

\[ H = -\Delta \sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \nu_1 = \frac{\nu}{2\nu+1}, \]

\[ \|\mathcal{V}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)\tau^{-1}, \quad i = 1, 2, \]

\[ \mathcal{F}(f) = O(f^2), \]

\[ \|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}} \]

(\( r(\tau) \) is the error generated by \( \psi^{ap} \)).
We consider $H$ as an operator on $L^2_{rad}(\mathbb{R}^3; \mathbb{C}^2)$ with domain $D(H) = H^2_{rad}(\mathbb{R}^3; \mathbb{C}^3)$.

- $\sigma_3 H \sigma_3 = H^*$, $\sigma_1 H \sigma_1 = -H$
- $\sigma_{ess}(H) = \mathbb{R}$.
- the discrete spectrum of $H$ consists of two simple purely imaginary eigenvalues $i\lambda_0$, $-i\lambda_0$, $\lambda_0 > 0$, the corresponding eigenfunctions $\zeta_+$, $\zeta_-$ are in $S(\mathbb{R}^3)$ and can be chosen in such a way that $\zeta_- = \sigma_1 \zeta_+ = \bar{\zeta}_+$.
- $HW\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = HW_1\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = 0$ ($H$ has a resonance at zero).
Energy estimates for $\mathcal{H}(\tau)$

Let $P$ be the spectral projection of $H$ onto the essential spectrum:

$$P = I - P_+ - P_-,$$
$$P_\pm = \frac{\langle \cdot, \sigma_3 \zeta_\mp \rangle}{\langle \zeta_\pm, \sigma_3 \zeta_\mp \rangle} \zeta_\pm,$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^2)$.

**Lemma**

(Duyckaerts, Merle)

$$\left\| e^{-i\tau H} Pf \right\|_{H^1(\mathbb{R}^3)} \leq C \langle \tau \rangle \left\| f \right\|_{H^1(\mathbb{R}^3)},$$

for any $f \in \dot{H}^1(\mathbb{R}^3)$ radial.

By a careful spectral analysis of the operator $H$ around zero energy, we convert the Duyckaerts-Merle lemma into the following result. Consider the projection of the linearized equation $i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f}$, $\mathcal{H}(\tau) = H + i\nu_1\tau^{-1}(\frac{1}{2} + y \cdot \nabla)$, onto the essential spectrum of $H$:

$$i\vec{f}_\tau = PH(\tau)P\vec{f},$$

Let $U(\tau, s)$ be the propagator associated to Eq. (16).
Energy estimates for $\mathcal{H}(\tau)$

Proposition

There exists a constant $C > 0$ such that

$$\| U(\tau, s)f \|_{H^k} \leq C \left( \frac{s}{\tau} \right)^{C_{\nu_1}} \| f \|_{H^k}$$

for any $s \geq \tau > 0$ and any $f \in H^k_{rad}, k = 1, 2$. 