Finite Difference Methods for Fully Nonlinear Second Order PDEs

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A Finite Difference Framework: 1-D Case

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A Finite Difference Framework: 1-D Case

High Dimension and High Order Extensions

Conclusion
Terminologies:

- **Semilinear PDEs**: Nonlinear in unknown functions but linear in all their derivatives. e.g.
  \[-\Delta u = e^u, \quad u_t - \Delta u + (u^2 - 1)u = 0\]

- **Quasilinear PDEs**: Nonlinear in lower order derivatives of unknown functions but linear in highest order derivatives. e.g.
  \[-\text{div} (|\nabla u|^{p-2}\nabla u) = f, \quad u_t - \text{div} \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + 1}} \right) = 0\]

- **Fully nonlinear PDEs**: Nonlinear in highest order derivatives of unknown functions
First order fully nonlinear PDEs

\[ F(Du, u, x) = 0 \]

Examples:

- Eikonal equation: \( |Du| = f \)
- Hamilton-Jacobi equations: \( u_t + H(Du, u, x, t) = 0 \)
First order fully nonlinear PDEs

\[ F(Du, u, x) = 0 \]

Examples:
- Eikonal equation: \( |Du| = f \)
- Hamilton-Jacobi equations: \( u_t + H(Du, u, x, t) = 0 \)

Second order fully nonlinear PDEs

\[ F(D^2 u, Du, u, x) = 0 \]

Examples:
- Monge-Ampère equation: \( \det(D^2 u) = f \)
- HJ-Bellman equation: \( \inf_{\nu \in \mathcal{V}} (L_{\nu} u - f_{\nu}) = 0 \)
Fully nonlinear PDEs arise in

- Differential Geometry
- Mass Transportation
- Optimal Control
- Semigeostrophic Flow
- Meteorology
- Antenna Design
- Image Processing and Computer Vision
- Gas Dynamics
- Astrophysics
- Grid generation
Example 1: (Minkowski Problem) Let $K > 0$ be a constant, find $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that the Gauss curvature of the graph of $u$ is equal to $K$ at every point $x$ in $\Omega$

$\Omega = [-0.57, 0.57]^2$; \hspace{1em} BC: $u = x^2 + y^2 - 1$; \hspace{1em} $K = 0.1, 1, 2, 2.1$
Example 1: (Minkowski Problem) Let $K > 0$ be a constant, find $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that the Gauss curvature of the graph of $u$ is equal to $K$ at every point $x$ in $\Omega$

\[ \Omega = [-0.57, 0.57]^2; \quad \text{BC: } u = x^2 + y^2 - 1; \quad K = 0.1, 1, 2, 2.1 \]

In the PDE language, the problem is described by

\[
\frac{\det(D^2 u)}{(1 + |Du|^2)^{\frac{n+2}{2}}} = K \quad \text{or} \quad \det(D^2 u) = K(1 + |Du|^2)^{\frac{n+2}{2}}
\]
Example 2: (Monge-Kantorovich Optimal Transport Problem)
Let \( \rho^+, \rho^- \) be two nonnegative (density) functions on \( \mathbb{R}^n \) with equal mass. Let \( \mathcal{A} \) denote the set of all mass preserving maps with respect to \( \rho^+ \) and \( \rho^- \), that is, \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) satisfies

\[
\rho^+(x) = \det(\nabla \Phi(x))\rho^-(\Phi(x)) \quad \forall x \in \text{supp}(\rho^+). \tag{MPc}
\]

Given \( c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \) (cost density function), the MK Optimal Transport Problem seeks \( \Psi \in \mathcal{A} \) such that

\[
I(\Psi) = \inf_{\Phi \in \mathcal{A}} I(\Phi) := \int_{\mathbb{R}^d} c(x, \Phi(x))\rho^+(x)dx
\]
Example 2: (Monge-Kantorovich Optimal Transport Problem)
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Given $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ (cost density function), the MK Optimal Transport Problem seeks $\Psi \in \mathcal{A}$ such that

$$I(\Psi) = \inf_{\Phi \in \mathcal{A}} I(\Phi) := \int_{\mathbb{R}^n} c(x, \Phi(x))\rho^+(x)dx$$

If $\Psi = \nabla u$ (i.e., $u$ is a potential of $\Psi$), then (MPc) implies

$$\rho^+(x) = \det(D^2 u(x))\rho^-(\nabla u(x)) \quad \forall x \in \text{supp}(\rho^+)$$
Example 3: (Stochastic Optimal Control) Suppose a stochastic process $x(\tau)$ is governed by the stochastic differential equation

$$d x(\tau) = f (\tau, x(\tau), u(\tau)) + \sigma (\tau, x(\tau), u(\tau)) \, dW(\tau), \quad \tau \in (t, T]$$

$$x(t) = x \in \Omega \subset \mathbb{R}^n,$$

$W$: Wiener process \quad $u$: control vector

and let

$$J(t, x, u) = E_t x \left[ \int_t^T L (\tau, x(\tau), u(\tau)) \, d\tau + g (x(T)) \right].$$

Stochastic optimal control problem involves minimizing $J(t, x, u)$ over all $u \in U$ for each $(t, x) \in (0, T] \times \Omega.$
Bellman Principle

Suppose $u^* \in U$ such that

$$u^* \in \arg\min_{u \in U} J(t, x, u),$$

and define the value function

$$v(t, x) = J(t, x, u^*).$$

Then, $v$ is the minimal cost achieved starting from the initial value $x(t) = x$, and $u^*$ is the optimal control that attains the minimum.
Bellman Principle (Continued)

Let $\Omega \subset \mathbb{R}^n$, $T > 0$, and $U \subset \mathbb{R}^m$. The Bellman Principle says $v$ is the solution of

$$v_t = F(D^2 v, \nabla v, v, x, t) \quad \text{in} \ (0, T] \times \Omega,$$

for

$$F(D^2 v, \nabla v, v, x, t) = \inf_{u \in U} (L_u v - h^u),$$

$$L_u v = \sum_{i=1}^{n} \sum_{j=1}^{n} a^u_{i,j}(t, x)v_{x_i x_j} + \sum_{i=1}^{n} b^u_i(t, x)v_{x_i} + c^u(t, x)v$$

with

$$A^u := \frac{1}{2} \sigma \sigma^T \quad \quad b^u := f(t, x, u)$$

$$c^u := 0 \quad \quad h^u := L(t, x, u)$$
Bellman Principle (Continued)

After \( v \) is found, to find the control \( u^* \) and the corresponding stochastic process \( x \),

- Solve \( u^* = \arg\min_{u \in U} [L_u v - f_u] \)
- Plug \( u^* \) into the SDE and solve for \( x \)

Thus, solving the Stochastic Optimal Control Problem can be recast as solving a fully nonlinear 2nd order (parabolic) Bellman equation
**Definition:** $F[u] := F(D^2u, \nabla u, u, x)$ is said to be **elliptic** if

$$F(A, p, r, x) \leq F(B, p, r, x) \quad \forall A, B \in \text{SL}(n), \ A - B > 0$$

**Remark:**
When $F(A, p, r, x)$ is differentiable in $A$, then $F$ is elliptic at $\hat{u}$ if

$$\frac{\partial F[\hat{u}]}{\partial A} > 0,$$

OR if the linearization of $F$ at $\hat{u}$ is elliptic as a linear operator.
(1) Classical solution theory (before 1985): [Chapter 17, Gilbarg & Trudinger], [Chapter 16, Lieberman]

(2) Viscosity solution theory (after 1985). Solution concept was due to Crandall and Lions (1983). Two breakthroughs:
   ▶ R. Jensen (1986): Uniqueness by maximum principle
   ▶ H. Ishii (1987): Existence by Perron’s method

(3) A complete regularity theory was established in ’80s and ’90s, some key milestones are
   ▶ Krylov-Safonov $C^\alpha$ estimates for linear elliptic PDEs of non-divergence form (’79) (which replaces De Giorgi-Nash $C^\alpha$ estimates for linear elliptic PDEs of divergence form)
   ▶ Evans-Krylov $C^{2,\alpha}$ estimates (’82)
   ▶ Caraffelli’s $W^{2,p}$ estimates for the Monge-Ampère equation and general elliptic PDEs
**Definitions:** Assume $F$ is elliptic in a function class $\mathcal{A} \subset B(\overline{\Omega})$ (set of bounded functions),

(i) $u \in \mathcal{A}$ is called a **viscosity subsolution** of $F[u] = 0$ if $
\forall \varphi \in C^2$, when $u^* - \varphi$ has a local maximum at $x_0$ then

$$F^*(D^2\varphi(x_0), D\varphi(x_0), u^*(x_0), x_0) \leq 0$$

(ii) $u \in \mathcal{A}$ is called a **viscosity supersolution** of $F[u] = 0$ if $
\forall \varphi \in C^2$, when $u_* - \varphi$ has a local minimum at $x_0$ then

$$F^*(D^2\varphi(x_0), D\varphi(x_0), u_*(x_0), x_0) \geq 0$$

(iii) $u \in \mathcal{A}$ is called a **viscosity solution** of $F[u] = 0$ if $u$ is both a sub- and supersolution of $F[u] = 0$

where $u^*(x) := \lim \sup_{x' \to x} u(x')$ and $u_*(x) := \lim \inf_{x' \to x} u(x')$ are the upper and lower semi-continuous envelopes of $u$
Geometrically,

\[ u(x_0) = \varphi(x_0) \]

\[ F(D^2\varphi(x_0), \nabla \varphi(x_0), \varphi(x_0), x_0) \leq 0 \]

\[ F(D^2\varphi(x_0), \nabla \varphi(x_0), \varphi(x_0), x_0) \geq 0 \]

**Figure:** A geometric interpretation of viscosity solutions
Geometrically,

\[
\varphi
\]

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\]

**Figure:** A geometric interpretation of viscosity solutions

**Remark:** The concept of viscosity solution is non-variational. It is based on a “differentiation by parts” approach, instead of the usual integration by parts approach used to define weak solutions for linear, semilinear, and quasilinear PDEs.
$64K$ Question:

How to compute viscosity solutions?
Available Numerical Methodologies

- **Finite Difference Methods** based on directly approximating derivatives by difference quotients

- **Galerkin-type Methods** based on variational principles and approximating infinite dimension spaces by finite dimension spaces (finite element, finite volume, spectral Galerkin, discontinuous Galerkin)

- **Everything else** such as *Collocation Method, Meshless Method, Lattice Boltzmann method, radial basis function methods, etc.*
Numerical Challenges

In contrast with the success of PDE analysis, little progress on developing numerical methods was made until very recently

- None of above methods work (directly) for general fully nonlinear 2nd order PDEs
- Situation is even worse for Galerkin-type methods because there is no variational principle to start with! Recall that the concept of viscosity solutions is non-variational!!
- Conditional uniqueness is hard to deal with numerically
- Tons of other numerical issues such convergence, efficiency, fast solvers, implementations, etc.
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- Situation is even worse for Galerkin-type methods because there is no variational principle to start with! Recall that the concept of viscosity solutions is non-variational!!
- Conditional uniqueness is hard to deal with numerically
- Tons of other numerical issues such convergence, efficiency, fast solvers, implementations, etc.

Remark: On the other hand, to certain degree, all above methods work for linear, semilinear, and quasilinear PDEs. At least they can be formulated without too much difficulties
Consider Monge-Ampère problem

\[
det(D^2u) = f \quad \text{in } \Omega = (0, 1)^2
\]

\[
u = g \quad \text{on } \partial \Omega
\]

\(f\) and \(g\) are chosen such that \(u = e^{(x_1^2 + x_2^2)/2} \in C^\infty(\Omega)\) is the exact solution. We discretize the Monge-Ampère equation using the standard nine-point finite difference method

\[
(D_{xx}^2 u_{ij}) (D_{yy}^2 u_{ij}) - (D_{xy}^2 u_{ij}) = f_{i,j},
\]
Figure: 16 computed solutions of the Monge-Ampère equation using a nine-point stencil on a $4 \times 4$ grid. $2^{(N-2)^2}$ solutions on an $N \times N$ grid.
Nevertheless, there are some recent attempts for constructing Galerkin-type methods for Monge-Ampère type equations.

- Dean and Glowinski ('03-'06) proposed some least squares and augmented Lagrange methods for Monge-Ampère type equations.
- Böhmer ('08) proposed and analyzed an $L^2$ projection method using $C^1$ finite elements.
- Brenner-Gudi-Neilan-Sung ('09-'10) constructed and analyzed some $C^0$ discontinuous Galerkin $L^2$-projection methods for Monge-Ampère type equations.
- F.-Neilan ('06-'10) has developed an indirect approach based on combining the newly developed vanishing moment method (VMM) and Galerkin type methods.
- F.-Lewis ('10-'11) extended the vanishing moment method (VMM) to Hamilton-Jacobi-Bellman type equations.
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Goals:

- To construct some (2nd order) finite difference methods (FDMs) which guarantee to converge to viscosity solutions of the underlying fully nonlinear 2nd order PDE problems

Remark:
To simplify the presentation, I shall describe the ideas and methods using 1-D equations. High dimension (and high order) generalizations will be given at the end.
Goals:

- To construct some (2nd order) finite difference methods (FDMs) which guarantee to converge to viscosity solutions of the underlying fully nonlinear 2nd order PDE problems

- To develop a blueprint/receipt for designing FDMs and a new FD convergence framework/theory which is simpler than Barles-Souganidis’ (’91) framework and more suitable for FDMs and local discontinuous Galerkin (LDG) methods

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To simplify the presentation, I shall describe the ideas and methods using 1-D equations. High dimension (and high order) generalizations will be given at the end
Notation:

Let $\Omega = [a, b]$ and $\Omega_h := \{x_j\}_{j=1}^J$ be a (uniform) mesh on $\Omega$ with mesh size $h = h_x$. Define the forward and backward difference operators by

$$\delta^+_x v(x) := \frac{v(x + h) - v(x)}{h}, \quad \delta^-_x v(x) := \frac{v(x) - v(x - h)}{h}$$

for a continuous function $v$ defined in $\Omega$ and

$$\delta^+_x V_j := \frac{V_{j+1} - V_j}{h}, \quad \delta^-_x V_j := \frac{V_j - V_{j-1}}{h},$$

for a grid function $V$ on $\Omega_h$

Remark:

$\delta^+_x$ and $\delta^-_x$ will be our building blocks in the sense that we shall approximate 1st and 2nd order derivatives using combinations and compositions of them
Existing Finite Difference Methods

- Barles-Souganidis (’91) proposed an abstract framework for approximating $F[u] = 0$ by $S[u^\rho] := S(u^\rho, x, \rho) = 0$ (FD or otherwise). They proved that $u^\rho$ converges to $u$ locally uniformly if $S$ is monotone, stable, and consistent. An $O(h^\alpha)$ convergence rate in $C^0$-norm was established recently by Caffarelli-Souganidis (’08).

- Oberman (’08) constructed a wide-stencil FD scheme for the Monge-Ampére equation which is the only FD scheme known to satisfy Barles-Souganidis’ criterion so far.
Existing Finite Difference Methods

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- Oberman (’08) constructed a wide-stencil FD scheme for the Monge-Ampére equation which is the only FD scheme known to satisfy Barles-Souganidis’ criterion so far.

- Krylov (’02-05) proposed several monotone FD schemes for Bellman type equations $F[u] = 0$.

- Kao-Trudinger (’90–’93) developed several monotone/positive finite difference methods for $F[u] = 0$. 
Barles-Souganidis (’91) proposed an abstract framework for approximating $F[u] = 0$ by $S[u^\rho] := S(u^\rho, x, \rho) = 0$ (FD or otherwise). They proved that $u^\rho$ converges to $u$ locally uniformly if $S$ is monotone, stable, and consistent.

Remark:

- Barles-Souganidis’ framework does not tell how to construct a convergent scheme, no hints are given either.
- Their monotonicity and consistency are difficult to verify for a given scheme.
- We feel that imposing monotonicity on $S[u^\rho]$ (as a function of $u^\rho$) is too restrictive because differential operator $F[u]$ does not have that property.
Ideas used for 1st order Hamilton-Jacobi equations

Consider fully nonlinear 1st order HJ equation

\[ H(u_x, x) = 0 \]

A general FD scheme on the mesh \( \mathcal{T}_h := \{x_j\}_{j=1}^J \) has the form

\[ \hat{H}(\delta^- U_j, \delta^+ U_j, x_j) = 0 \]

\( \hat{H} \) is called a numerical Hamiltonian

**Consistency:** \( \hat{H}(p, p, x_j) = H(p, x_j) \)

**Monotonicity:** \( \hat{H}(\uparrow, \downarrow, x_j) \)

**Remark/Observation:**
- \( \hat{H} \) is a function of both \( \delta^- U_j \) and \( \delta^+ U_j \). This is crucial because \( u_x \) may be discontinuous at \( x_j \).
- Both **monotonicity and consistency** are not hard to verify in practice. The uniform convergence for such schemes was proved by Crandall-Lions ('84)
Finite difference approximations of $u_{xx}$ in $F(u_{xx}, x) = 0$

- Since $u_{xx}$ may be discontinuous at $x_j$, it should be approximated from both sides of $x_j$
Finite difference approximations of $u_{xx}$ in $F(u_{xx}, x) = 0$

- Since $u_{xx}$ may be discontinuous at $x_j$, it should be approximated from both sides of $x_j$
- There are 3 possible FD approximations of $u_{xx}(x_j)$:
  
  $\delta_x^- \delta_x^- u(x_j), \quad \delta_x^+ \delta_x^- u(x_j), \quad \delta_x^+ \delta_x^+ u(x_j)$

- It is trivial to check

  $\delta_x^- \delta_x^- u(x) = \delta_x^2 u(x_j - h) = \delta_x^2 u(x_{j-1})$

  $\delta_x^+ \delta_x^- u(x) = \delta_x^2 u(x_j)$

  $\delta_x^+ \delta_x^+ u(x) = \delta_x^2 u(x_j + h) = \delta_x^2 u(x_{j+1})$

  where

  $\delta_x^2 u(x_j) := \frac{u(x_j - h) - 2u(x_j) + u(x_j + h)}{h^2} = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2}$
A general finite difference framework for $F[u] = 0$

Inspired by the above observations, we propose the following form of FD methods for $F[u] = 0$:

$$F_h[U, x_j] := \hat{F}(\delta_x^2 U_{j-1}, \delta_x^2 U_j, \delta_x^2 U_{j+1}, x_j) = 0$$

**Definition:** $\hat{F}$ is called a numerical operator

**Questions:**

- What is $\hat{F}$?
- Is there a guideline for constructing $\hat{F}$? (Recall that such a guideline is missing in Barles-Souganidis’ framework)
Criterions for “good” numerical operator $\hat{F}$

- **Consistency**: $\hat{F}(p, p, p, x_j) = F(p, x_j)$
- **Monotonicity**: $\hat{F}(\uparrow, \downarrow, \uparrow, x_j)$
- **Solvability and Stability**: $\exists h_0 > 0$ and $C_0 > 0$, which is independent of $h$, such that $F_h[U, x_j] = 0$ has a (unique) solution $U$ and $\|U\|_{\ell^\infty(T_h)} < C_0$ for $h < h_0$
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**Remark:**

- If $F$ is *not* continuous, $\hat{F}$ may not be continuous either. In this case the consistency definition should be changed to

\[
\lim_{p_k \to p, k=1,2,3} \liminf_{\xi \to x} \hat{F}(p_1, p_2, p_3, \xi) \geq F_*(p, x) \quad \text{and} \quad \limsup_{p_k \to p, k=1,2,3} \lim_{\xi \to x} \hat{F}(p_1, p_2, p_3, x) \leq F^*(p, x)
\]

- Above consistency and monotonicity are different from Barles-Souganidis’ definitions, but the solvability and stability are same
For consistent, monotone and stable FD schemes, we have the following theorem, which may be regarded as an analogue of Crandall-Lions’ convergence theorem for fully nonlinear 1st order equations.

**Theorem (F., Kao, Lewis, ’11)**

Let $u_h$ denote the piecewise constant extension of $U$ on $\{[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]\}$. Suppose the numerical operator $\hat{F}$ is consistent, monotone and stable, then $u_h$ converges to the unique viscosity solution of the Dirichlet problem for $F(u_{xx}, x) = 0$.

**Idea of Proof:** Follow the PDE analysis and modify Barles-Souganidis’ proof.
Guidelines for constructing $\hat{F}$

Once again, we look for hints/ideas from FD methods for fully nonlinear 1st order Hamilton-Jacobi equations (and hyperbolic conservation laws)!

**Theorem (E. Tadmor (’97))**

*Every “convergent” monotone finite difference scheme for HJ equations (and hyperbolic conservation laws) must contain a numerical diffusion (or viscosity) term.*
Once again, we look for hints/ideas from FD methods for fully nonlinear 1st order Hamilton-Jacobi equations (and hyperbolic conservation laws)!

**Theorem (E. Tadmor (’97))**

*Every “convergent” monotone finite difference scheme for HJ equations (and hyperbolic conservation laws) must contain a numerical diffusion (or viscosity) term.***

In other words, every “convergent” monotone finite difference scheme for HJ equations implicitly approximates the differential equation

\[-\alpha_h \Delta u + H(\nabla u, x) = 0\]

for sufficiently large and nonlinear \(\alpha > 0\),
Lax-Friedrichs scheme:

\[ \hat{H}^{LF}(p_1, p_2) := H\left(\frac{p_1 + p_2}{2}\right) - \alpha^{LF}(p_2 - p_1) \]

Godunov scheme:

\[ \hat{H}^G(p_1, p_2) := \text{ext}_{p \in I(p_1, p_2)} H(p) = H^G(p_1, p_2) - \alpha^G(p_2 - p_1) \]

where \( I(p_1, p_2) := [p_1 \wedge p_2, p_1 \vee p_2] \) and

\[ \text{ext}_{p \in I(p_1, p_2)} := \begin{cases} 
\min_{p \in I(p_1, p_2)} & \text{if } p_1 \leq p_2 \\
\max_{p \in I(p_1, p_2)} & \text{if } p_1 > p_2 
\end{cases} \]
**Question:** What “quantity" plays the role of “numerical diffusion (viscosity) term" as above for fully nonlinear 2rd order PDEs?
Question: What “quantity" plays the role of “numerical diffusion (viscosity) term” as above for fully nonlinear 2rd order PDEs?

Answer/Conjecture: Numerical moment! We conjecture that a “good" scheme for fully nonlinear 2rd order PDEs should contain some kind numerical moment term. In other words, such a “good" scheme implicitly approximates the regularized PDE

$$\alpha h^2 \Delta^2 u + F(D^2 u, \nabla u, u, x) = 0$$

for sufficiently large (and nonlinear) $\alpha > 0$. This is perfectly consistent with the vanishing moment method (at the PDE level) introduced and studied by F. and Neilan ('07-'10):

$$\epsilon \Delta^2 u^\epsilon + F(D^2 u^\epsilon, \nabla u^\epsilon, u^\epsilon, x) = 0$$

with $\epsilon = \alpha h^2$
Examples of “good” numerical operators/schemes

Lax-Friedrichs-like schemes (FKL ’11):

\[
\hat{F}_1(p_1, p_2, p_3) := F\left(\frac{p_1 + p_2 + p_3}{3}\right) + \alpha(p_1 - 2p_2 + p_3)
\]

\[
\hat{F}_2(p_1, p_2, p_3, x_j) := F(p_2) + \alpha(p_1 - 2p_2 + p_3)
\]

\[
\hat{F}_3(p_1, p_2, p_3) := F\left(\frac{p_1 + p_3}{2}\right) + \alpha(p_1 - 2p_2 + p_3)
\]
Examples of “good” numerical operators/schemes

Godunov-like schemes (FKL ’11):

\[ \hat{F}_4(p_1, p_2, p_3) := \text{ext}_{p \in I(p_1, p_2, p_3)} F(p) \]

where \( I(p_1, p_2, p_3) := [p_1 \land p_2 \land p_3, p_1 \lor p_2 \lor p_3] \) and

\[ \text{ext}_{p \in I(p_1, p_2, p_3)} := \begin{cases} 
\min_{p \in I(p_1, p_2, p_3)} & \text{if } p_2 \geq \max\{p_1, p_3\}, \\
\max_{p \in I(p_1, p_2, p_3)} & \text{if } p_2 \leq \min\{p_1, p_3\}, \\
\min_{p_1 \leq p \leq p_2} & \text{if } p_1 < p_2 < p_3, \\
\min_{p_3 \leq p \leq p_2} & \text{if } p_3 < p_2 < p_1. 
\end{cases} \]
Godunov-like schemes (FKL '11) (continued):

\[ \hat{F}_5(p_1, p_2, p_3) := \text{extr} \ F(p) \]

where \( I(p_1, p_2, p_3) := [p_1 \land p_2 \land p_3, p_1 \lor p_2 \lor p_3] \) and

\[
\text{extr}_{p \in I(p_1, p_2, p_3)} := \begin{cases} 
\min_{p \in I(p_1, p_2, p_3)} & \text{if } p_2 \geq \max\{p_1, p_3\}, \\
\max_{p \in I(p_1, p_2, p_3)} & \text{if } p_2 \leq \min\{p_1, p_3\}, \\
\max_{p_2 \leq p \leq p_3} & \text{if } p_1 < p_2 < p_3, \\
\max_{p_2 \leq p \leq p_1} & \text{if } p_3 < p_2 < p_1.
\end{cases}
\]
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High dimension extensions

The main issue is how to discretize $u_{x_kx_\ell} u(x)$. There are 4 possible ways:

\[
\delta_{-x_k} \delta_{-x_\ell} u(x), \quad \delta_{+x_k} \delta_{+x_\ell} u(x), \quad \delta_{-x_k} \delta_{+x_\ell} u(x), \quad \delta_{+x_k} \delta_{-x_\ell} u(x)
\]
High dimension and high order extensions

High dimension extensions

The main issue is how to discretize \( u_{x_k x_{\ell}} u(x) \). There are 4 possible ways:

\[
\delta_{x_k}^- \delta_{x_{\ell}}^- u(x), \quad \delta_{x_k}^+ \delta_{x_{\ell}}^+ u(x), \quad \delta_{x_k}^- \delta_{x_{\ell}}^+ u(x), \quad \delta_{x_k}^+ \delta_{x_{\ell}}^- u(x)
\]

we are saved by the following identity:

\[
\delta_{x_k}^- \delta_{x_{\ell}}^- U_{ij} + \delta_{x_k}^+ \delta_{x_{\ell}}^+ U_{ij} - \delta_{x_k}^- \delta_{x_{\ell}}^+ U_{ij} - \delta_{x_k}^+ \delta_{x_{\ell}}^- U_{ij}
\]

\[
= h_{x_k}^2 h_{x_{\ell}}^2 \delta_{x_k}^2 \delta_{x_{\ell}}^2 U_{ij} = h_{x_k}^2 h_{x_{\ell}}^2 \delta_{x_k}^2 \delta_{x_{\ell}}^2 U_{ij}
\]

because it is an \( O(h_{x_k}^2 + h_{x_{\ell}}^2) \) approximation to \( u_{x_k x_{k} x_{\ell} x_{\ell}} \), which is a prototypical 4th order mixed derivative term in \( \Delta^2 u \)

To ensure consistency, monotonicity and stability, hence, convergence, the multi-D numerical operator \( \hat{F} \) must depend on all four approximate derivatives \( \delta_{x_k}^- \delta_{x_{\ell}}^- U_{ij}, \quad \delta_{x_k}^+ \delta_{x_{\ell}}^+ U_{ij}, \quad \delta_{x_k}^- \delta_{x_{\ell}}^+ U_{ij}, \quad \delta_{x_k}^+ \delta_{x_{\ell}}^- U_{ij} \), \( \hat{F} \) is decreasing (\( \downarrow \)) in first two arguments and increasing (\( \uparrow \)) in last two arguments.
High dimension and high order extensions

High order extensions

▶ “Bad" news: We tried but failed (as expected) to construct higher than 2rd order *monotone* FD schemes.
High dimension and high order extensions

High order extensions

- **“Bad” news:** We tried but failed (as expected) to construct higher than 2nd order *monotone* FD schemes.
- **“Good” news:** Inspired by a recent work by *Yan-Osher* (JCP, ’11) for fully nonlinear 1st order Hamilton-Jacobi equations, we are able to construct high order LDG (local discontinuous Galerkin) methods.
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Concluding Remarks

- We propose a new and “easy to verify” framework for developing “good” FD (and LDG) methods for fully nonlinear 2nd elliptic PDEs
- We also give a guideline for designing consistent, monotone and stable FD schemes. **Key concept** is numerical moment, which may be regarded as an analogue of numerical viscosity from 1st order PDEs
- We see a strong interplay between PDE analysis and numerical analysis in this work
- Many open problems and challenging theoretical and practical issues need to be addressed: rate of convergence in $C^0$-norm, boundary layers, parabolic PDEs, nonlinear solvers, etc.
References

- arxiv.org/abs/1109.1183 (small book on VMM)
References

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Thanks for Your Attention!