Affine LIBOR models with multiple curves: theory, examples and calibration

Zorana Grbac

TU Berlin

Based on joint work with A. Papapantoleon, J. Schoenmakers and D. Skovmand

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Interest rate derivatives – pre-crisis

- **Forward rate agreement (FRA)** for the future time interval \([T, T + \delta]\): payoff at time \(T + \delta\) is
  \[
  \delta \left( \frac{1}{\delta} \left( \frac{1}{B_T(T+\delta)} - 1 \right) - K \right)
  \]

- **Floating-for-fixed interest rate swap** with maturity \(T_n\), the payment dates \(T_1 < \cdots < T_n\) and the fixed rate \(S\), and starting at \(T_0 \geq 0\): cash flows at each payment date are
  \[
  \delta_k - 1 \left( L_{T_k-1}(T_k-1, T_k) - S \right)
  \]

- **Caplet** with strike \(K\) and maturity \(T_k\), settled in arrears: payoff at settlement date \(T_k+1\) is
  \[
  \delta_k \left( L_{T_k}(T_k, T_{k+1}) - K \right)^+ \]

- **Swaption** with swap rate \(S\) and exercise date \(T\) – option to enter an interest rate swap: payoff at exercise date is \((P^{Sw}(T; T_1, T_n, S))^+\), i.e. positive part of the value of the underlying swap at \(T\)

⇒ The underlying for each of these derivatives is a so-called LIBOR rate.
The fair forward rate is the rate that makes the FRA value equal to zero at time $t$ and is given by

$$
\frac{1}{\delta} \left( \frac{B_t(T)}{B_t(T+\delta)} - 1 \right) =: L_t(T, T+\delta),
$$

where $B.(T)$ denotes the zero coupon bond price with maturity $T$.

$L_t(T, T+\delta)$ is called the forward LIBOR rate and the process $L.(T, T+\delta)$ is a martingale under the forward measure $\mathbb{P}^{T+\delta}$ which uses the bond price $B.(T+\delta)$ as a numeraire:

$$
L_t(T, T+\delta) = \mathbb{E}^{\mathbb{P}^{T+\delta}} [L_T(T, T+\delta) | \mathcal{F}_t]
$$

The value of the interest rate swap at time $t$ is now simply:

$$
1 - B_t(T_n) - S \sum_{k=1}^{n} \delta_k - 1 B_t(T_k)
$$

A caplet can be transformed into a put option on a zero coupon bond.

A swaption can be seen as a put option on a coupon bearing bond.
The fair forward rate is the rate that makes the FRA value equal to zero at time $t$ and is given by

$$\frac{1}{\delta} \left( \frac{B_t(T)}{B_t(T + \delta)} - 1 \right) =: L_t(T, T + \delta),$$

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$L_t(T, T + \delta)$ is called the forward LIBOR rate and the process $L.(T, T + \delta)$ is a martingale under the forward measure $\mathbb{P}_{T+\delta}$ which uses the bond price $B.(T + \delta)$ as a numeraire:

$$L_t(T, T + \delta) = \mathbb{E}^{\mathbb{P}_{T+\delta}}[L_T(T, T + \delta) | \mathcal{F}_t]$$

The value of the interest rate swap at time $t$ is now simply:

$$1 - B_t(T_n) - S \sum_{k=1}^{n} \delta_{k-1} B_t(T_k)$$

A caplet can be transformed into a put option on a zero coupon bond

A swaption can be seen as a put option on a coupon bearing bond

Here LIBOR is seen as a risk-free rate!
What is LIBOR?

LIBOR stands for London InterBank Offered Rate. It is produced for 10 currencies with 15 maturities quoted for each, ranging from overnight to 12 Months producing 150 rates each business day. LIBOR is computed as a trimmed average of the interbank borrowing rates assembled from the LIBOR contributing banks.

More precisely, every contributing bank has to submit an answer to the following question: "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?"

<table>
<thead>
<tr>
<th>Bank</th>
<th>Rate</th>
</tr>
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<tbody>
<tr>
<td>Barclays Bank plc</td>
<td>2.15</td>
</tr>
<tr>
<td>Bank of Tokyo-Mitsubishi-UFJ Ltd</td>
<td>2.15</td>
</tr>
<tr>
<td>HSBG</td>
<td>2.12</td>
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<tr>
<td>Royal Bank of Scotland Group</td>
<td>2.11</td>
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<tr>
<td>UBS AG</td>
<td>2.105</td>
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<tr>
<td>Abbey National</td>
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<td>Bank of America</td>
<td>2.1</td>
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<tr>
<td>Citibank NA</td>
<td>2.1</td>
</tr>
<tr>
<td>Mizuho Corporate Bank</td>
<td>2.1</td>
</tr>
<tr>
<td>Rabobank</td>
<td>2.1</td>
</tr>
<tr>
<td>Royal Bank of Canada</td>
<td>2.1</td>
</tr>
<tr>
<td>WestLB AG</td>
<td>2.1</td>
</tr>
<tr>
<td>BNP-Paribas</td>
<td>2.05</td>
</tr>
<tr>
<td>Lloyds Banking Group</td>
<td>2</td>
</tr>
<tr>
<td>Deutsche Bank AG</td>
<td>1.95</td>
</tr>
<tr>
<td>JP Morgan Chase</td>
<td>1.95</td>
</tr>
</tbody>
</table>

\[
\text{bbalibor Rate} = \frac{\sum \text{Rates}}{\text{Number of Rates}} = 2.10063
\]
The financial crisis

Euribor - Eoniaswap spreads

- spread 1m
- spread 3m
- spread 6m
- spread 12m
A number of anomalies has appeared in the interest rate markets due to the credit crisis in 2008.

A large gap between the LIBOR (EURIBOR) and the OIS (EONIA) rate; LIBOR cannot be considered risk-free any longer, it reflects credit and liquidity risk of the interbank sector.

Different forward curves for different tenors; large basis spreads corresponding to various tenors.

Counterparty risk and funding issues.

Large number of papers: Kijima et al. (2009), Kenyon (2010), Mercurio (2009, 2010), Bianchetti (2009), Henrard (2009), Moreni and Pallavicini (2010), Fujii et al. (2010), Piterbarg (2010), Morini (2009), Filipović and Trolle (2012), Crépey, G. and Nguyen (2012), Gallitschke, Müller and Seifried (2012), ...
The post-crisis setup

- Tenor structures:
  \[ T^x = \{ 0 < T^x_0 < T^x_1 < \cdots < T^x_N = T_N \}, \ x \in \{1, 3, 6, 12\} \]

- Discount curve: OIS zero coupon bonds
  \[ T \mapsto B(0, T) = B^{OIS}(0, T) \]

- Forward measures \( \mathbb{P}^x_k \) – numeraire \( B(\cdot, T^x_k) \)

*Figure* : Illustration of different tenor structures.
The post-crisis setup II

**Definition**
The time-$t$ OIS forward rate for $[T_{k-1}^x, T_k^x]$ is

\[ F_k^x(t) = \frac{1}{\delta_x} \left( \frac{B(t, T_{k-1}^x)}{B(t, T_k^x)} - 1 \right) \]

**Definition (Mercurio)**
The time-$t$ FRA rate $L_k^x$ is the fixed rate to be exchanged at $T_k^x$ for the LIBOR rate $L_{T_{k-1}^x}(T_{k-1}^x, T_k^x)$ such that the swap has value zero:

\[ L_k^x(t) = \mathbb{E}_k^x[L_{T_{k-1}^x}(T_{k-1}^x, T_k^x)|\mathcal{F}_t] \]

- FRA – OIS spread: $S_k^x(t) := L_k^x(t) - F_k^x(t)$
The post-crisis setup III

Requirements:
- $F^x_k(t) \geq 0$ and $F^x_k$ is a $\mathbb{P}^x_k$-martingale
- $L^x_k(t) \geq 0$ and $L^x_k$ is a $\mathbb{P}^x_k$-martingale
- $S^x_k(t) \geq 0 \iff L^x_k(t) \geq F^x_k(t)$ (market observations)

Options:
1. Model $F^x_k$ and $L^x_k$
2. Model $L^x_k$ and $S^x_k$
3. Model $F^x_k$ and $S^x_k$

The first choice is the most convenient in terms of tractability and calibration, but more difficult to guarantee that the implied spreads will be non-negative (detailed discussion in Mercurio (2010)).
Constructing ordered martingales $\geq 1$

1. Take a random variable $Y_t^u \geq 1$, $\mathcal{F}_T$-measurable and integrable, and set:

$$M_t^u = \mathbb{E}[Y_T^u|\mathcal{F}_t],$$

then $(M_t^u)_{0 \leq t \leq T}$ is a martingale and $M_t^u \geq 1$ for all $t \in [0, T]$. 
Constructing ordered martingales $\geq 1$

1. Take a random variable $Y^u_T \geq 1$, $\mathcal{F}_T$-measurable and integrable, and set:
   \[ M_t^u = \mathbb{E}[Y_T^u | \mathcal{F}_t], \]
   then $(M_t^u)_{0 \leq t \leq T}$ is a martingale and $M_t^u \geq 1$ for all $t \in [0, T]$.

2. Require that $u \mapsto Y_T^u$ is increasing, then
   \[ u \mapsto M_t^u \]
   is a.s. increasing for all $t \in [0, T]$.
Constructing ordered martingales $\geq 1$

1. Take a random variable $Y_T^u \geq 1$, $\mathcal{F}_T$-measurable and integrable, and set:

   $M_t^u = \mathbb{E}[Y_T^u | \mathcal{F}_t],$

   then $(M_t^u)_{0 \leq t \leq T}$ is a martingale and $M_t^u \geq 1$ for all $t \in [0, T]$.

2. Require that $u \mapsto Y_T^u$ is increasing, then $u \mapsto M_t^u$

   is a.s. increasing for all $t \in [0, T]$.

3. Affine LIBOR model (Keller-Ressel, Papapantoleon and Teichmann, 2012):

   - $Y_T^u = e^{\langle u, X_T \rangle}$
   - $X$ positive affine process
   - set

   $L_t(T_k, T_{k+1}) = \frac{1}{\delta_k} \left( \frac{M_t^{u_k}}{M_t^{u_{k+1}}} - 1 \right)$
Affine processes

1. $X = (X_t)_{0 \leq t \leq T_N}$ a time-homogeneous Markov process

2. $X$ takes values in $D = \mathbb{R}_d^{\geq 0}$

3. $X$ is affine, if the moment generating function satisfies:

\[ \mathbb{E}_x[ \exp \langle u, X_t \rangle ] = \exp (\phi_t(u) + \langle \psi_t(u), x \rangle ) \]

4. where $\phi_t(u)$ and $\psi_t(u)$ are functions taking values in $\mathbb{R}$ and $\mathbb{R}^d$ and $x \in D$. 
$\phi_t(u)$ and $\psi_t(u)$ are defined on $[0, T] \times \mathcal{I}_T$, where

$$\mathcal{I}_T := \left\{ u \in \mathbb{R}^d : \mathbb{E}_x \left[ e^{\langle u, X_T \rangle} \right] < \infty, \text{ for all } x \in D \right\},$$

the 'domain of exponential moments'.

2 Technical assumption: $0 \in \mathcal{I}_T^\circ$.

3 The process $X$ is a regular affine process in the spirit of DFS, and a semimartingale.

4 Old and new examples $\Rightarrow$ CIR process, Lévy subordinators, non-Gaussian OU processes, and multivariate extensions.
The multiple-curve affine LIBOR model

- Modeling OIS forward rates: \((u^x_k)^{k\geq 0}\)

\[
1 + \delta_x F^x_k(t) = \frac{B(t, T^x_{k-1})}{B(t, T^x_k)} = \frac{M^u_{k-1}}{M^u_k}
\]
The multiple-curve affine LIBOR model

- Modeling OIS forward rates: \((u^x_k)_{k \geq 0}\)

\[
1 + \delta_x F^x_k(t) = \frac{B(t, T^x_{k-1})}{B(t, T^x_k)} = \frac{M^u_{k-1}}{M^u_k}
\]

- Modeling FRA rates: \((v^x_k)_{k \geq 0}\)

\[
1 + \delta_x L^x_k(t) = \frac{M^v_{k-1}}{M^u_k}
\]
The multiple-curve affine LIBOR model

- Modeling OIS forward rates: \((u_k^x)_{k \geq 0}\)

\[
1 + \delta_x F_k^x(t) = \frac{B(t, T_{k-1}^x)}{B(t, T_k^x)} = \frac{M_{t}^{u_{k-1}^x}}{M_{t}^{u_k^x}}
\]

- Modeling FRA rates: \((v_k^x)_{k \geq 0}\)

\[
1 + \delta_x L_k^x(t) = \frac{M_{t}^{v_{k-1}^x}}{M_{t}^{u_k^x}}
\]

- The \(\mathbb{P}_k^x\)-martingale property is obvious

- Orderings:

\[
\begin{align*}
  u_k^x &\geq u_{k+1}^x &\Rightarrow& & F_k^x \geq 0 \\
  v_k^x &\geq u_k^x &\Rightarrow& & L_k^x \geq F_k^x 
\end{align*}
\]

\{ \text{automatic: initial values} \} \quad (\star)
Consider the time structure \( T \), let \( B(0, T_k), k \in \mathcal{K} \), be the initial term structure of non-negative OIS discount factors and assume that
\[
B(0, T_1) \geq \cdots \geq B(0, T_N).
\]

Then the following hold:

1. If \( \gamma_X > \frac{B(0, T_1)}{B(0, T_N)} \), then there exists a decreasing sequence \( u_1 \geq u_2 \geq \cdots \geq u_N = 0 \) in \( \mathcal{I}_T \cap \mathbb{R}^d_{\geq 0} \), such that
   \[
   M_0^{u_k} = \frac{B(0, T_k)}{B(0, T_N)} \quad \text{for all } k \in \mathcal{K}.
   \]

   In particular, if \( \gamma_X = \infty \), then the multiple curve affine LIBOR model can fit any initial term structure of OIS rates.

2. If \( X \) is one-dimensional, the sequence \( (u_k)_{k \in \mathcal{K}} \) is unique.
now fix an $x \in \mathcal{X}$ and the tenor structure $\mathcal{T}^x$. For each $k \in \mathcal{K}^x$, set

$$u_k^x := u_j,$$

where $j \in \mathcal{K}$ is such that $T_j = T_k^x$.

proceed with fitting the initial term structure of the FRA rates
Proposition

Consider the setting of Proposition 1, fix an $x \in X$ and the corresponding the tenor structure $T^x$. Let $L^x_k(0), k \in K^x$, be the initial term structure of non-negative FRA rates and assume that for every $k \in K^x$

$$L^x_k(0) \geq \frac{1}{\delta_x} \left( \frac{B(0, T^x_{k-1})}{B(0, T^x_k)} - 1 \right).$$

The following hold:

1. If $\gamma_X > (1 + \delta_x L^x_2(0))B(0, T^x_2)/B(0, T^x_N)$, then there exists a sequence $(v^x_k)_{k \in K^x}$ in $\mathcal{I}_T \cap \mathbb{R}^d_{\geq 0}$, such that $v^x_k \geq u^x_k$ and

$$M^v_k = (1 + \delta_x L^x_{k+1}(0))M^u_{k+1}, \quad \text{for all } k \in \overline{K}^x.$$

In particular, if $\gamma_X = \infty$, then the multiple curve affine LIBOR model can fit any initial term structure of FRA rates.

2. If $X$ is one-dimensional, the sequence $(v^x_k)_{k \in \overline{K}^x}$ is unique.
Analytical tractability

**Proposition**

The process $X$ is a **time-inhomogeneous affine process** under the measure $\mathbb{P}_x^k$, for every $x, k$, with

$$
\mathbb{E}_{x,k}^x \left[ e^{\langle w, X_t \rangle} \right] = \exp \left( \phi_{t,x}^k (w) + \langle \psi_{t,x}^k (w), x \rangle \right),
$$

where

$$
\phi_{t,x}^k (w) := \phi_t (\psi_{T_N-t} (u_x^k) + w) - \phi_t (\psi_{T_N-t} (u_x^k))
$$

$$
\psi_{t,x}^k (w) := \psi_t (\psi_{T_N-t} (u_x^k) + w) - \psi_t (\psi_{T_N-t} (u_x^k)).
$$
Caplet pricing

- express the payoff of a caplet as:
  \[
  \delta_x(L_{T_{k-1}^x} (T_{k-1}^x, T_k^x) - K)^+ = (1 + \delta_x L_{T_{k-1}^x} (T_{k-1}^x, T_k^x) - 1 + \delta_x K)^+ \\
  = \left( e^{A_{x_{k}}^x + B_{k}^x \cdot X_{T_{k-1}^x}^x} - K \right)^+
  \]

where \( \overline{K} := 1 + \delta_x K \)

- apply Fourier methods

\[
\mathbb{C}_0(K, T_k^x) = \delta_x B(0, T_k^x) \mathbb{E}_k^x \left[ (L_{T_{k-1}^x} (T_{k-1}^x, T_k^x) - K)^+ \right] \\
= \frac{\overline{K} B(0, T_k^x)}{2\pi} \int_{\mathbb{R}} K^{iv-R} \frac{\Lambda_{A_{k}^x + B_{k}^x \cdot X_{T_{k-1}^x}^x} (R - iv)}{(R - iv)(R - 1 - iv)} dv
\]

where \( \Lambda_{A_{k}^x + B_{k}^x \cdot X_{T_{k-1}^x}^x} \) is the \( \mathbb{P}_k^x \)-moment generating function of \( X_{T_{k-1}^x}^x \).
**Swaption pricing**

**Post-crisis:** no cancelations $\Rightarrow$ not a put option on a coupon bearing bond. We have

\[
S_0^+(K, T^{x}_{pq}) = B(0, T^{x}_{pq}) \mathbb{E}_p^x \left[ \left( \sum_{i=p+1}^{q} e^{\tilde{A}_i + \tilde{B}_i \cdot X_{T^{x}_p}} - \sum_{i=p+1}^{q} K^{x} e^{\tilde{C}_i + \tilde{D}_i \cdot X_{T^{x}_p}} \right)^+ \right]
\]

\[
= B(0, T^{x}_{pq}) \mathbb{E}_p^x \left[ \left( - \sum_{i=1}^{2l} c_i e^{A_i + B_i \cdot X_{T^{x}_p}} \right)^+ \right]
\]

can be treated as a basket option on $Y_i := A_i + B_i \cdot X_{T^{x}_p}$

\[
= \frac{B(0, T^{x}_{pq})}{(2\pi)^{2l}} \int_{\mathbb{R}^{2l}} \hat{g}(iR - \nu)M_Y(R + i\nu) d\nu.
\]

- Efficient numerical approximation?
Swaption pricing II

\[ S_0^+ (K, T^x_{pq}) = B(0, T_N) \mathbb{E}_N \left[ \left( \sum_{j=p+1}^{q} M^{x}_{j-1} - \sum_{j=p+1}^{q} K_x M^{u_x}_{j} \right)^+ \right] \]

\[ = B(0, T_N) \mathbb{E}_N \left[ \left( \sum_{j=p+1}^{q} M^{x}_{j-1} - \sum_{j=p+1}^{q} K_x M^{u_x}_{j} \right) \mathbf{1}_{\{f(X^{T^x}_{T^p}) \geq 0\}} \right] \]

\[ = B(0, T_N) \sum_{j=p+1}^{q} M^{x}_{j-1} \mathbb{E}_j^{x} \mathbf{1}_{\{f(X^{T^x}_{T^p}) \geq 0\}} \]

\[ - K_x \sum_{j=p+1}^{q} B(0, T^x_j) \mathbb{E}_j^{x} \mathbf{1}_{\{f(X^{T^x}_{T^p}) \geq 0\}} \]

- Linear approximation of the exercise boundary (Singleton & Umantsev, 2002)
Swaption pricing III

Swaption 3m EURIBOR 2x2

Swaption 3m EURIBOR 2x8

Swaption 3m EURIBOR 5x5

Swaption 3m EURIBOR 2x2

Swaption 3m EURIBOR 2x8

Swaption 3m EURIBOR 5x5

Swaption 3m EURIBOR 2x2

Swaption 3m EURIBOR 2x8

Swaption 3m EURIBOR 5x5

Swaption 3m EURIBOR 2x2

Swaption 3m EURIBOR 2x8

Swaption 3m EURIBOR 5x5
Connection to LMMs

Assume that $X$ is an affine diffusion.

- **OIS dynamics**

$$\frac{dF_k^x}{F_k^x} = 1 + \delta_x F_k^x \sum_{l=1}^{d} \left( \psi_{T-t}^l (u_{k-1}^x) - \psi_{T-t}^l (u_k^x) \right) (\sigma^l)^T \sqrt{X_{x,k;l}} dW_{x,k}^l$$

- **FRA dynamics**

$$\frac{dL_k^x}{L_k^x} = 1 + \delta_x L_k^x \sum_{l=1}^{d} \left( \psi_{T-t}^l (v_{k-1}^x) - \psi_{T-t}^l (u_k^x) \right) (\sigma^l)^T \sqrt{X_{x,k;l}} dW_{x,k}^l$$

- **Built-in displacement**

- **Structure of volatility**: completely determined by $X$
Recall:

\[ \mathbb{E}_x \left[ \exp \langle u, X_t \rangle \right] = \exp \left( \phi_t(u) + \langle \psi_t(u), x \rangle \right) \]

**Lemma (Riccati equations)**

The functions \( \phi \) and \( \psi \) satisfy the (generalized) Riccati ODEs

\[
\frac{\partial}{\partial t} \phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0, \\
\frac{\partial}{\partial t} \psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u,
\]

for all suitable \( 0 \leq t + s \leq T \) and \( u \in \mathcal{I}_T \).
Lemma (Riccati equations II)

[DFS] showed that

\[ F(u) := \frac{\partial}{\partial t} \bigg|_{t=0+} \phi_t(u) \quad \text{and} \quad R(u) := \frac{\partial}{\partial t} \bigg|_{t=0+} \psi_t(u) \]

exist for all \( u \in \mathcal{I}_T \) and are continuous in \( u \). Moreover, \( F \) and \( R \) satisfy Lévy–Khintchine-type equations:

\[
F(u) = \langle b, u \rangle + \int_{\mathbb{D}} (e^{\langle \xi, u \rangle} - 1) m(d\xi)
\]

and

\[
R_i(u) = \langle \beta_i, u \rangle + \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \int_{\mathbb{D}} (e^{\langle \xi, u \rangle} - 1 - \langle u, h^i(\xi) \rangle) \mu_i(d\xi),
\]

where \((b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}\) are admissible parameters.
Affine processes V

Admissible parameters for $\mathbb{R}_{\geq 0}^2$-valued affine processes:

\[
b = \left( \begin{array}{c} + \\ + \end{array} \right), \quad \beta_1 = \left( \begin{array}{c} * \\ + \end{array} \right), \quad \beta_2 = \left( \begin{array}{c} + \\ * \end{array} \right)\]

\[
a = 0, \quad \alpha_1 = \left( \begin{array}{c} + \\ * \\ 0 \end{array} \right), \quad \alpha_2 = \left( \begin{array}{c} 0 \\ * \\ + \end{array} \right)\]

$m, \mu_1, \mu_2$ are Lévy measures on $\mathbb{R}_{\geq 0}^2$

$\mu_1, \mu_2$ can have infinite variation

‘Classical’ examples: CIR, CIR with jumps, OU processes, Watanabe example . . .
Further examples

2D CIR with dependent jumps

\[ dX_t^1 = -\lambda_1 (X_t^1 - \theta_1) dt + 2\eta_1 \sqrt{X_t^1} dW_t^1 + \alpha_1 dZ_t, \]
\[ dX_t^2 = -\lambda_2 (X_t^2 - \theta_2) dt + 2\eta_2 \sqrt{X_t^2} dW_t^2 + \alpha_2 dZ_t, \]

where \( Z \) is a subordinator with a known cumulant generating function.
Further examples

2D CIR with dependent jumps

\[ \begin{align*}
    dX_t^1 &= -\lambda_1 (X_t^1 - \theta_1) dt + 2\eta_1 \sqrt{X_t^1} dW_t^1 + \alpha_1 dZ_t, \\
    dX_t^2 &= -\lambda_2 (X_t^2 - \theta_2) dt + 2\eta_2 \sqrt{X_t^2} dW_t^2 + \alpha_2 dZ_t,
\end{align*} \]

where \( Z \) is a subordinator with a known cumulant generating function.

E.g. compound Poisson process with exponentially distributed jumps:

\[ \begin{align*}
    F(u_1, u_2) &= \lambda_1 \theta_1 u_1 + \lambda_2 \theta_2 u_2 + \ell \frac{u_1 \alpha_1 + u_2 \alpha_2}{\frac{1}{\mu} - u_1 \alpha_1 - u_2 \alpha_2}, \\
    R_i(u_1, u_2) &= -\lambda_i u_i + 2\eta_i^2 u_i^2, \quad i = 1, 2.
\end{align*} \]

- Explicit solution of the Riccati equations when \( \lambda_1 = \lambda_2 \)
Further examples II

Stochastic volatility/intensity on $\mathbb{R}^2_{\geq 0}$

$$
\begin{align*}
\text{d}X_t &= -\lambda_1 (X_t - \theta_1) \text{d}t + 2\eta_1 \sqrt{X_t} \text{d}W^1_t + \text{d}Z_t, \\
\text{d}V_t &= -\lambda_2 (V_t - \theta_2) \text{d}t + 2\eta_2 \sqrt{V_t} \text{d}W^2_t,
\end{align*}
$$

where the intensity of $Z$ is an affine function of $V$: $\Lambda_t = \ell_0 + \ell_1 \cdot V_t$.

Functional characteristics

$$
\begin{align*}
F(u_1, u_2) &= \lambda_1 \theta_1 u_1 + \lambda_2 \theta_2 u_2 + \ell_0 \frac{u_1}{\mu - u_1}, \\
R_1(u_1, u_2) &= -\lambda_1 u_1 + 2\eta_1^2 u_1^2, \\
R_2(u_1, u_2) &= -\lambda_2 u_2 + 2\eta_2^2 u_2^2 + \ell_1 \frac{u_1}{\mu - u_1},
\end{align*}
$$

- Explicit solution of the Riccati equations when $2\eta_1^2 = \lambda_1 \mu$ and $\lambda_2 = \lambda_1 / 2$
  (see Polyanin and Zaitsev, 2003 for the explicit solution for $\psi^2$)
Approximations of characteristic functions

- often the Riccati equations cannot be solved explicitly
- approximations of the characteristic function based on a series representation of its holomorphic extension
- developed in Belomestny, Kampen and Schoenmakers (2009)
- Consider a $d$-dimensional affine process $X$ with characteristic function

$$\hat{p}(t, x, z) := \mathbb{E}_x \left[ e^{i \langle z, X_t \rangle} \right]$$

- Then BKS show that

$$\ln \hat{p}(t, x, z) = \ln \left( \sum_{r \geq 0} h_{r,0}(z; \eta_z)(1 - e^{-\eta_z t})^r \right) + i \langle z, x \rangle$$

$$+ \left\langle x, \sum_{r \geq 1} \frac{h_r(z; \eta_z)(1 - e^{-\eta_z t})^r}{\sum_{r \geq 0} h_{r,0}(z; \eta_z)(1 - e^{-\eta_z t})^r} \right\rangle, \quad x \in D, z \in \mathbb{R}^d,$$
where we denote
\[ f_z(x) := e^{i\langle z, x \rangle}, \quad z \in \mathbb{R}^d, \]

- and for each multi-index \( \beta \) we compute algebraically

\[ b_\beta(x, z) := i^{-|\beta|} \partial_{z^\beta} \frac{Af_z(x)}{f_z(x)} =: b^0_\beta(z) + \sum_{\kappa, |\kappa|=1} b^1_{\beta, \kappa}(z)x^\kappa, \]

- \( h \)'s are given by the following recursion (\( \gamma \) multi-index)

\[
(r + 1)h_{r+1, \gamma} = \sum_{|\beta| \leq r - |\gamma|} \eta^{-1}_{z}(\gamma + \beta) h_{r, \gamma + \beta} b^0_\beta \\
+ \sum_{|\kappa|=1, \kappa \leq \gamma} \sum_{|\beta| \leq r+1 - |\gamma|} \eta^{-1}_{z}(\gamma - \kappa + \beta) h_{r, \gamma - \kappa + \beta} b^1_{\beta, \kappa} + rh_{r, \gamma},
\]
BKS show that suitable choices of $\eta_z$ are

$$\eta_z \geq C(1 + \|z\|^2) \quad \text{in case of pure affine diffusions},$$

$$\eta_z \geq M e^{\zeta \|z\|} \quad \text{for affine jump processes with thinly tailed large jumps.}$$

As a natural approximation for the characteristic function we thus consider

$$\ln \hat{\rho}_K(t, x, z) = \ln \left( \sum_{r=0}^{K} h_{r,0}(z; \eta_z)(1 - e^{-\eta_z t})^r \right) + i \langle z, x \rangle$$

$$+ \left\langle x, \frac{\sum_{r=1}^{K} h_{r}(z; \eta_z)(1 - e^{-\eta_z t})^r}{\sum_{r=0}^{K} h_{r,0}(z; \eta_z)(1 - e^{-\eta_z t})^r} \right\rangle, \quad K \in \mathbb{N}$$
**Calibration to caplet data**

\( M \) maturities, \( 2M \) drivers (JCIR + CIR)

\[ X = ((X^{1,1}, X^{2,1}), \ldots, (X^{1,M}, X^{2,M})) \]

structured via

\[
\begin{align*}
    u_x^{M/\delta_x} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & (0, 0) & (0, 0) & \bar{u}_{M/\delta_x} \end{pmatrix} \\
    u_x^{(M-1)/\delta_x} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & (0, 0) & \bar{u}_{(M-1)/\delta_x} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix} \\
    u_x^{(M-2)/\delta_x} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & \bar{u}_{(M-2)/\delta_x} & \bar{v}_{(M-1)/\delta_{x-1}} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix} \\
    \vdots &= \vdots \\
    u_x^{1/\delta_x} &= \begin{pmatrix} \bar{u}_1/\delta_x & \bar{v}_2/\delta_{x-1} & \ldots & \bar{v}_{(M-2)/\delta_{x-1}} & \bar{v}_{(M-1)/\delta_{x-1}} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix}
\end{align*}
\]

and

\[
\begin{align*}
    v_x^{M/\delta_{x-1}} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & (0, 0) & (0, 0) & \bar{v}_{M/\delta_{x-1}} \end{pmatrix} \\
    v_x^{(M-1)/\delta_{x-1}} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & (0, 0) & \bar{v}_{(M-1)/\delta_{x-1}} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix} \\
    v_x^{(M-2)/\delta_{x-1}} &= \begin{pmatrix} (0, 0) & \ldots & (0, 0) & \bar{v}_{(M-2)/\delta_{x-1}} & \bar{v}_{(M-1)/\delta_{x-1}} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix} \\
    \vdots &= \vdots \\
    v_x^{1/\delta_{x-1}} &= \begin{pmatrix} \bar{v}_1/\delta_{x-1} & \bar{v}_2/\delta_{x-1} & \ldots & \bar{v}_{(M-2)/\delta_{x-1}} & \bar{v}_{(M-1)/\delta_{x-1}} & \bar{v}_{M/\delta_{x-1}} \end{pmatrix}
\end{align*}
\]
Calibration to caplet data II

![Graphs showing implied volatility for different maturities and strikes with model and market data overlaid.](image-url)
Calibration to caplet data III

- Furthermore, simultaneous calibration to caplets of multiple tenors

- In practice the dependence structure is determined from quotes for liquid multi-tenor instruments such as basis swaptions

- Example: 3m and 6m-caplets with maturities 1, \ldots, M \Rightarrow 4M drivers
Summary

We have presented

- a class of tractable LIBOR models for the multiple-curve framework
- explicit examples (multi-dimensional, correlated)
- numerical methods for swaption pricing
- calibration to caplets
- approximations via series expansions of the holomorphic transforms (numerical implementation in progress)

Based on:

Z. G., A. Papapantoleon, J. Schoenmakers, D. Skovmand
Affine LIBOR models with multiple curves: theory, examples and calibration (in preparation)
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Thank you for your attention