Multivariate Shortfall Risk Allocation

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Workshop on Stochastic Methods in Finance and Physics
Crete
Importance of Risk Assessment Instruments

**Relevance:** Economically sound

- Multivariate (Diversification, Monetary, ...)
- Risk Contribution, Risk Allocation, Sensitivity, Systemic Risk

**Performance:** Easy and efficient implementation

- Simple, understandable class of risk measures adequate for optimization
- Numerical methods (curse of dimensionality)
Motivation
Systemic Risk and Risk Allocation

Given $d$ risk factors $L := (L_1, \ldots, L_d)$, (Banks, Clearing Members of CCP, etc.)

1. Compute the overall monetary risk

$$R(L_1, \cdots, L_d)$$

amount of liquidity for the whole system to overcome unexpected stress situations.

What is a sound but computable risk assessment $R$?

2. Allocate this amount $R(L)$ among the risk factors

$$\sum RA_k(L) = R(L), \quad RA_k(L) \text{ allocation for risk factor } k$$

How much should I charge one or the other risk factor? (Systemic Risk)

3. Give me numbers!

How to compute those? (implementable for the industry)
Univariate Risk Allocation

$L$ represents the loss and profit of a financial asset (Random Variable)

**Definition (Monetary Risk Measure)**

A *monetary risk measure* $R$ is a risk measure, that is

- **Diversifying**: Diversification of two losses $L_1$ and $L_2$ is less risky than the worse
  
  $$R(\lambda L_1 + (1 - \lambda) L_2) \leq \max \{R(L_1), R(L_2)\}$$

- **Monotone**: $R(L_1) \leq R(L_2)$ if loss $L_1$ is greater than loss $L_2$;
  which is *monetary*:

- **Monetary**: $R(L - m) = R(L) - m$ for every amount of cash $m$.

Monetary and diversification implies that $R$ is **convex**

$$R(\lambda L_1 + (1 - \lambda) L_2) \leq \lambda R(L_1) + (1 - \lambda) R(L_2)$$
**Univariate Risk Allocation**

Example: Shortfall Risk Measure (Foellmer and Schied)

- **Expected loss of** $L$
  \[ E[\ell(L)] \]

- Monetary allocation $m$ is **acceptable** if
  \[ E[\ell(L - m)] \leq c \]
  for some threshold $c$

- Minimize the acceptable allocation costs $\sim$
  \[ R(L) := \inf \{ m : E[\ell(L - m)] \leq c \} \]
Univariate Risk Allocation
Risk Allocation: A First Take on

Up to now: focus on overall risk, univariate.

\[ R(L_1 + \cdots + L_d) \]
\[ d \text{ risk factors, overall loss profile } L = \sum L_k \]

Financial crisis: concerns about allocation of this overall risk

Respective risk shares \( R\!A_k(L) \) for member 1, \ldots, \( d \)?

Properties: What about the economical features

- **Full allocation:** \( \sum R\!A_k(L) = R(L); \)
- **Riskless allocation:** \( R\!A_k(m) = m; \)
- **Causal Responsability:** \( R(L + \Delta L_k) = R\!A_k(L + \Delta L_k); \)
- **Additivity:** \( R\!A_1(L) + R\!A_2(L) = R\!A_1(L_1 + L_2, 0, L_3, \ldots, L_d); \)
Motivation  Univariate Risk Allocation  Multivariate Shortfall Risk Measure  Risk Allocation, Sensitivity  Implementation

Univariate Risk Allocation
Risk Allocation: First Take On, Several Methods

**Naive way** (kitchen made allocation proportional to size, etc.) or **Risk contribution based**

\[
RC_i(L) := \lim_{\varepsilon \searrow 0} \frac{R \left( \sum_{k \neq i} L + \varepsilon L_i \right) - R \left( \sum_{k \neq i} L_k \right)}{\varepsilon}
\]

Possible Allocation Procedures:

- **Euler type:** \( RA_i = \alpha_i R(L) \) where \( \alpha_i = \frac{RC_i}{\sum RC_j} \) (Euler type)

- **Additive type:** \( RA_i = RC_i - \mu s_i \) where \( s_i \) is constant depending on the size of risk factor \( i \) and \( \mu \) chosen such that \( R(L) = \sum RA_i \) (Brunnemeier, Cheridito)

- ...
Multivariate Shortfall Risk Measure

A New Approach

New take on

1 Allocate first

2 Aggregate then

See also Feinstein et al. and Biagini et al.
Multivariate Shortfall Risk Measure
A New Approach

Definition
A function \( \ell: \mathbb{R}^d \to (-\infty, \infty] \) is called a loss function if

- \( \ell \) is convex and lower semi-continuous;
- \( \ell(x) \leq \ell(y) \) whenever \( x_k \leq y_k \) for every \( k \);
- \( \ell(0) = 0 \) and \( \ell(x) \geq \sum x_k \)

Examples:

1. \( \ell(x) = h(\sum x_k) \);
2. \( \ell(x) = \sum h(x_k) \);
3. \( \ell(x) = \alpha \sum h_1(x_k) + \beta h_2(\sum x_k) \)

for a one dimensional loss function \( h: e^x - 1; -\ln(1 - x); x + (x^+)^2 / 2. \)

Typically: \( \ell(x) = \sum x_k + \frac{1}{2} \sum (x_k^+)^2 + \alpha \sum_{k<j} x_k^+ x_j^+, \quad 0 \leq \alpha \leq 1 \)
Typically: \[ \ell(x) = \sum x_k + \frac{1}{2} \sum (x_k^+)^2 + \alpha \sum_{k<j} x_k^+ x_j^+, \quad 0 \leq \alpha_{kj} \leq 1 \]
Multivariate Shortfall Risk Measure
A New Approach

- System of risk factors $L_1, \ldots, L_d$ (in some multidimensional Orlicz Heart)

- The loss $L = (L_1, \ldots, L_d)$ is acceptable at threshold $c > 0$ if
  \[ E[\ell(L)] = E[\ell(L_1, \ldots, L_d)] \leq c \]

- monetary allocation $m = (m_1, \ldots, m_d)$ is acceptable if
  \[ E[\ell(L - m)] = E[\ell(L_1 - m_1, \ldots, L_d - m_d)] \leq c \]

Set of acceptable monetary allocations:
\[ A(L) := \left\{ m \in \mathbb{R}^d : E[\ell(L_1 - m_1, \ldots, L_d - m_d)] \leq c \right\} \]
One can consider $L \mapsto A(L) \subseteq \mathbb{R}^d$, set-valued function.

**Proposition**

For $L^1, L^2$ two $d$ dimensional losses it holds

- $A(L)$ is convex, monotone and closed;

- $A(L^1) \supseteq A(L^2)$ whenever $L^1 \leq L^2$;

- $A(\alpha L^1 + (1 - \alpha)L^2) \supseteq \alpha A(L^1) + (1 - \alpha)A(L^2)$, for any $\alpha \in (0, 1)$;

- $A(L^1 - m) = A(L^1) - m$, for any $m \in \mathbb{R}^d$;

- $\emptyset \neq A(L) \neq \mathbb{R}^d$.

In other terms, $A$ is a set-valued convex risk measure mapping in the complete lattice of subsets of upwards directed closed convex sets of $\mathbb{R}^d$. $\sim$ Hamel, Rudlof, etc.
Multivariate Risk Measure

Acceptable Allocations

\[ A(X) \]

\[ \rho = 1.00 \]
\[ \rho = 0.25 \]
\[ \rho = 0.00 \]
\[ \rho = -0.25 \]
\[ \rho = -0.75 \]
\( \sum m_k \) is the aggregate cost of an acceptable monetary allocation \( m \in A(L) \)

Minimize the costs:

\[
R(L) := \inf \left\{ \sum m_k : E \left[ \ell(L_1 - m_1, \ldots, L_d - m_d) \right] \leq c \right\} \\
= \inf \left\{ \sum m_k : m \in A(L) \right\}
\]

**Theorem**

- \( R \) is a monetary risk measure.

- it is continuous, sub-differentiable and it admits the dual representation

\[
R(L) = \max_Q \{ E_Q [L] - \alpha(Q) \}
\]

where the penalty function is given by

\[
\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left( c + E \left[ \ell^* \left( \lambda \frac{dQ}{dP} \right) \right] \right).
\]
Up to now

\[ R(X) = \inf \left\{ \sum m_k : E \left[ \ell (L_1 - m_1, \ldots, L_d - m_d) \right] \leq c \right\} \]

questions:

- Does there exists an optimal allocation?
- Is it unique?
- Can it be characterized (Computation)?
- Behavior with respect to interdependence structure?
Risk Allocation, Sensitivity
Risk Allocation: Existence Problem I

Example

Let

\[ \ell(x, y) = \begin{cases} 
  x + y + y/(1 - y) & y < 1 \\
  \infty & \text{otherwise}
\end{cases} \]

which is a loss function. For \( c = (5/12, 1/4) \), it follows that

\[ A(0) = \left\{ m \in \mathbb{R}^2 : m_2 > -1 \text{ and } m_1 \geq -m_2 - m_2/(1 + m_2) \right\} \]

and therefore

\[ R(0) = \inf_{m_2 > -1} \{-m_2 - m_2/(1 + m_2) + m_2\} = \inf_{m_2 > -1} -\frac{m_2}{1 + m_2} = -1. \]

However, the infimum is not attained.
Example

Let

\[ \ell(x) = h \left( \sum x_k \right) \]

for some one dimensional loss function.

- It can be shown that there exists an optimal allocation \( m^* \in \mathbb{R}^d \) for any \( L \).

- However for any other zero-sum allocation, i.e. \( n \in \mathbb{R}^d \) with \( \sum n_k = 0 \), then \( m^* + n \) is optimal as well!

- In two dimensions with two banks \( L^1, L^2 \), with optimal allocation (50M €, 10M €).

  Instead of requiring 50M € from the first bank and 10M € from the second bank

  You could as well require 500M € from the first bank and give 440M € to the second...

You can forget about selling this to the industry...
**Definition**

ℓ is **unbiased** if for every zero-sum allocation u

\[ ℓ(\alpha u) = 0 \text{ for any } \alpha > 0 \text{ implies that } ℓ(-\alpha u) = 0 \text{ for any } \alpha > 0 \]

**Theorem**

If ℓ is an unbiased loss function, then:

- for every L, there exists an optimal risk allocation \( m^* \)
- \( m^* \) is characterized by

\[ 1 = \lambda^* E \left[ \nabla ℓ (L - m^*) \right] \quad \text{and} \quad E [ℓ (L - m^*)] = c. \]

where \( \lambda^* > 0 \) is a Lagrange multiplier.

- If ℓ is strictly convex outside \( \mathbb{R}^d \) along zero-sums allocation, then the optimal allocation is unique.
Risk Allocation, Sensitivity

Risk Allocation: good loss functions?

Unique optimal risk allocation denoted \( m^* = RA(L) = (RA_1(L), \ldots, RA_d(L)) \)

Existence, uniqueness depends on \( \ell \). For instance

\[ \ell(x) = \alpha h \left( \sum x_k \right) + (1 - \alpha) \sum h(x_k) \]

- if \( \alpha = 1 \) existence is ensured but we can rearrange as we want by zero-sum allocations: BAD!!!.
- if \( \alpha = 0 \) existence and uniqueness is ensured, BUT!!!!

**Theorem**

If \( \ell(x) = \sum h(x_k) \) for some loss function \( \ell \), then:

- for every \( L \), there exists a unique optimal risk allocation \( m^* \)
- However, for every \( L' \) such that \( L_k \sim L'_k \), then it holds

\[ RA(L) = RA(L') \quad \text{and naturally} \quad R(L) = R(L') \]

In the case where \( \alpha = 0 \), the interdependence structure is therefore irrelevant!
Risk Allocation, Sensitivity
Risk Allocation: good loss functions?

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Goal: Study the sensitivity of the risk allocation, in particular in function of the interdependence structure.

**Definition**

The *marginal risk contribution* of $Z$ to $L$:

$$R(L; Z) := \limsup_{\varepsilon \searrow 0} \frac{R(L + \varepsilon Z) - R(L)}{\varepsilon}.$$ 

The *risk allocation marginals* of $L$ in the direction of $Z$:

$$RA_k(L; Z) = \limsup_{\varepsilon \searrow 0} \frac{RA_k(L + \varepsilon Z) - RA_k(L)}{\varepsilon}, \quad k = 1, \ldots, d.$$
Theorem

Assuming that \( \ell \) delivers only unique optimal risk allocations, then:

\[
R(L; Z) = \lambda E \left[ Z \nabla \ell (L - RA(L)) \right]
\]

for \( \lambda \) the optimal Lagrange optimizer.

Furthermore

\[
\begin{bmatrix}
RA(L; Z) \\
\lambda'
\end{bmatrix} = \begin{bmatrix}
\lambda E \left[ \nabla^2 \ell(X - RA(L)) \right] & -1/\lambda \\
1 & 0
\end{bmatrix}^{-1} \begin{bmatrix}
\lambda E \left[ \nabla^2 \ell(L - RA(L))Z \right] \\
R(L; Z)
\end{bmatrix}
\]

where \( \lambda' = \limsup_{\varepsilon \searrow 0} (\lambda(L + tZ) - \lambda(L))/t \).

As soon as the risk allocation \( RA(L) \) and \( \lambda \) are computed, then the risk allocation marginals \( R(L; Z) \) are (almost) immediately known.
Example

Impact loss \( Z = (Z_1, 0) \) to the first risk factor \( L_1 \) where

\[
\ell(x_1, x_2) = x_1 + x_2 + \frac{1}{2} \left[ (x_1^+)^2 + (x_2^+)^2 \right] + \alpha x_1^+ x_2^+, \quad 0 \leq \alpha \leq 1
\]

The marginal risk contribution with respect to impact \( Z = (Z_1, 0) \) yields:

\[
R(L; Z) = \lambda E[Z_1] + \lambda E[Z_1 (L_1 - m_1)^+] + \lambda \alpha E[Z_1 (L_2 - m_2)^+ | L_1 \geq m_1; L_2 \geq m_2]
\]

The marginal risk allocation yields:

\[
RA_1(L; Z) = \frac{(q - \alpha r) R(L; Z) + \lambda E[Z_1 | L_1 \geq m_1] - \lambda \alpha r \lambda E[Z_1 | L_1 \geq m_1; L_2 \geq m_2]}{p + q - 2\alpha r}
\]

\[
RA_2(L; Z) = \frac{(p - \alpha r) R(L; Z) - \lambda E[Z_1 | L_1 \geq m_1] + \lambda \alpha r \lambda E[Z_1 | L_1 \geq m_1; L_2 \geq m_2]}{p + q - 2\alpha r}
\]

where

\[
p = P[L_1 \geq m_1] \quad q = P[L_2 \geq m_2] \quad r = P[L_1 \geq m_1; L_2 \geq m_2]
\]
Risk Allocation, Sensitivity
Risk Allocation Sensitivity to External Impact

Example

Impact loss $Z = (Z_1, 0)$ to the first risk factor $L_1$ where

$$\ell(x_1, x_2) = x_1 + x_2 + \frac{1}{2} \left[ (x_1^+)^2 + (x_2^+)^2 \right] + \alpha x_1^+ x_2^+, \quad 0 \leq \alpha \leq 1$$

The marginal risk contribution with respect to impact $Z = (Z_1, 0)$ yields:

$$R(L; Z) = \lambda E[Z_1] + \lambda E[Z_1 (L_1 - m_1)^+] + \lambda \alpha E[Z_1 (L_2 - m_2)^+ | L_1 \geq m_1; L_2 \geq m_2]$$

The marginal risk allocation yields:

$$RA_1(L; Z) = \frac{(q - \alpha r)R(L; Z) + \lambda E[Z_1 | L_1 \geq m_1] - \lambda \alpha r \lambda E[Z_1 | L_1 \geq m_1; L_2 \geq m_2]}{p + q - 2\alpha r}$$

$$RA_2(L; Z) = \frac{(p - \alpha r)R(L; Z) - \lambda E[Z_1 | L_1 \geq m_1] + \lambda \alpha r \lambda E[Z_1 | L_1 \geq m_1; L_2 \geq m_2]}{p + q - 2\alpha r}$$

where

$$p = P[L_1 \geq m_1] \quad q = P[L_2 \geq m_2] \quad r = P[L_1 \geq m_1; L_2 \geq m_2]$$
Risk Allocation, Sensitivity
Risk Allocation Sensitivity to correlation: an Example

Example

Consider
\[
\ell(x_1, x_2) = \frac{1}{1 + \alpha} \left[ \frac{1}{2} e^{2x_1} + \frac{1}{2} e^{2x_2} + \alpha e^{x_1} e^{x_2} \right] - 1
\]

Let \( L = (L^1, L^2) \) be \( \mathcal{N}(\mu, \Sigma) \) where
\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix}
\]

From the first order conditions, computations yields
\[
RA_1(X) = \sigma_1^2 + \frac{1}{2} \ln \left( 1 + \alpha e^{\rho \sigma_1 \sigma_2 - \frac{1}{2} (\sigma_1^2 + \sigma_2^2)} \right) - \frac{1}{2} \ln \left( c(1 + \alpha) \right)
\]
\[
RA_2(X) = \sigma_2^2 + \frac{1}{2} \ln \left( 1 + \alpha e^{\rho \sigma_1 \sigma_2 - \frac{1}{2} (\sigma_1^2 + \sigma_2^2)} \right) - \frac{1}{2} \ln \left( c(1 + \alpha) \right)
\]

Hence, depending on \( \alpha \) the we get a systemic risk contribution:
\[
SRC(L) = \frac{1}{2} \ln \left( 1 + \alpha e^{\rho \sigma_1 \sigma_2 - \frac{1}{2} (\sigma_1^2 + \sigma_2^2)} \right) - \frac{1}{2} \ln \left( c(1 + \alpha) \right)
\]
Risk Allocation, Sensitivity
Risk Allocation Sensitivity: Systemic Risk Contribution

\[ SRC = \frac{1}{2} \ln \left( 1 + \alpha e^{\rho \sigma_1 \sigma_2 - \frac{1}{2} (\sigma_1^2 + \sigma_2^2)} \right) - \frac{1}{2} \ln \left( \tilde{c} (1 + \alpha) \right) \]

\[ SRC(L) = \frac{1}{2} \ln \left( 1 + \alpha e^{\rho \sigma_1 \sigma_2 - \frac{1}{2} (\sigma_1^2 + \sigma_2^2)} \right) - \frac{1}{2} \ln \left( \tilde{c} (1 + \alpha) \right) \]
Motivation
Univariate Risk Allocation
Multivariate Shortfall Risk Measure
Risk Allocation, Sensitivity
Implementation

Risk Allocation, Sensitivity
Risk Allocation Sensitivity: Systemic Risk Contribution

\[ L = (L_1, L_2, L_3) \] with covariance matrix

\[
\Sigma = \begin{bmatrix}
0.5 & 0.5\rho & 0 \\
0.5\rho & 0.5 & 0 \\
0 & 0 & 0.6
\end{bmatrix}
\]

<table>
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<th>( \rho )</th>
<th>( m_1^* )</th>
<th>( m_2^* )</th>
<th>( m_3^* )</th>
<th>( \lambda^* )</th>
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<td>&lt;</td>
<td>0.2065</td>
</tr>
</tbody>
</table>

**Table:** Trivariate case with systemic contribution. \( \sigma_1 = \sigma_2 = 0.5, \sigma_3 = 0.6 \).
The properties are fulfilled in a first order, but the systemic nature of the risk measurement call for deviation from it.

1. **Full Allocation:** \( \sum RA_k(L) = R(L) \); Ok

2. **Riskless Allocation:** \( RA_k(L) = L_k \) if \( X_k \) is deterministic; In first order yes

3. **Causal Responsability:** \( R(L + \Delta L_k) = RA_k(L + \Delta L_k) \); In first order yes

4. **Additivity:** \( RA_1(L) + RA_2(L) = RA_1(L_1 + L_2, 0, L_3, \ldots, L_d) \); depends a lot on the loss but usually \( \leq \)
What is to compute?

A priori not much, just some root findings

\[ 1 = \lambda E [\ell(L - m)] \quad \text{and} \quad E [\ell(L - m)] = c \]

Well...

In CCP roughly 60 members, on trading floors maybe more

We take

\[ \ell(x) = \sum x_k + \frac{1}{2} \sum (x_k^+)^2 + \alpha \sum_{k<j} x_k^+ x_j^+, \quad 0 \leq \alpha \leq 1 \]

First order condition yields...
Implementation

In reality

$$1/\lambda = 1 + E \left[ (L_1 - m_1)^+ \right] + \sum_{k=2}^{60} \alpha_{1k} E \left[ (L_k - m_k)^+ | L_1 \geq m_1 \right]$$

$$1/\lambda = 1 + E \left[ (L_2 - m_2)^+ \right] + \sum_{k=1, k \neq 2}^{60} \alpha_{2k} E \left[ (L_k - m_k)^+ | L_2 \geq m_2 \right]$$

$$\vdots$$

$$1/\lambda = 1 + E \left[ (L_{60} - m_{60}) \right] + \sum_{k=1}^{59} \alpha_{60k} E \left[ (L_k - m_k)^+ | L_{60} \geq m_{60} \right]$$

$$c = \sum_{k=1}^{60} E \left[ (L_k - m_k) \right] + \frac{1}{2} \sum_{k=1}^{60} E \left[ \left((L_k - m_k)^+\right)^2 \right] + \sum_{k<j} \alpha_{kj} E \left[ (L_k - m_k)^+ (L_j - m_j)^+ \right]$$

- 61 dimensional root finding (iterative computation)
- Each iteration, compute 3*60=180 simple integrals and 2*59*60=7080 double integrals...
Implementation

Computation

How to compute numerically each of those integrals at each step

- Monte Carlo: Stochastic root finding methods are pretty slow already in one dimensional case. Better methods for those high dimensional root finding?

- Fourier Methods: moment generating functions of marginals as well as the pairwise joint distributions (fat tails)
Implementation

Computation: Fourier

Proposition

For some "nice" $L$, then

$$E[\partial_k \ell(L - m)] = 1 - \frac{1}{2\pi} \int_{\mathbb{R}} e^{(iu_k - \beta_k)m_k} \frac{M_{L_k}(\alpha_k - iu_k)}{u_k + i\beta_k} du_k$$

$$+ \frac{\alpha}{(2\pi)^2} \sum_{j=1,j\neq k}^d \int_{\mathbb{R}^2} e^{iu - \alpha,m} \frac{M_{L_j,L_k}(\alpha - iu)}{(\alpha_k - iu_k)(u_j + i\alpha_j)^2} du_j du_k$$

and

$$E[\ell(X - m)] = \sum_{k=1}^d \left\{ E(L_k - m_k) + \frac{1}{\pi} \int_{\mathbb{R}} e^{(iu_k - \beta_k)m_k} \frac{M_{L_k}(\alpha_k - iu_k)}{i(u_k + i\beta_k)^3} du_k \right\}$$

$$+ \frac{\alpha}{(2\pi)^2} \sum_{1 \leq k < j \leq d} \int_{\mathbb{R}^2} e^{iu - \alpha,m} \frac{M_{L_j,L_k}(R - iu)}{(u_j + i\beta_j)^2(u_k + i\beta_k)^2} du_j du_k.$$
Implementation

Computation: Chebychev Interpolation

- A root finding methods is an iterative methods, time intensive.

- Chebychev interpolation method \( \sim \) take advantage of parallelism
  - Choose reasonable bounds \([-m, m] \subseteq \mathbb{R}^d\)
  - Divide each pairwise square in a grid of size \(N^2\).
  - In parallel, compute for each node \(m_k\) all integrals in parallel \(7080N^2\).
  - Interpolate with the Chebychev weights the obtained points.
  - Compute the root of the obtained interpolation \( \sim \) dividing CT by the amount of steps necessary to reach the optimal value.
Thank you!