On optimal investment, consumption, option pricing, and turnover under small transaction costs
heuristics and rigorous results

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Outline

1 The problem
   - Proportional transaction costs
   - Fixed transaction costs

2 Results
   - Proportional transaction costs
   - Fixed transaction costs

3 Heuristics and verification
   - Basics of stochastic control
   - Proportional transaction costs
   - Fixed transaction costs

4 Conclusion
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     • Fixed transaction costs

4 Conclusion
The setup
Itô process with proportional transaction costs

- Risky asset:
  \[ dS_t = b_t S_t dt + \sigma_t S_t dW_t \]

- Wealth process:
  \[ X_t^\varepsilon(\psi^\varepsilon, k^\varepsilon) = x + \int_0^t \psi_s^\varepsilon dS_s - \int_0^t k_s^\varepsilon ds + \psi_t - \int_0^t \varepsilon_s d\|\psi^\varepsilon\|_s, \]

where
- \( \psi_t^\varepsilon \) trading strategy (= number of shares),
- \( k_t^\varepsilon \) consumption rate,
- \( x \) initial endowment,
- \( \Psi_t \) random endowment stream, e.g. \( \Psi_t = 0 \)
- \( \varepsilon_t = \varepsilon \mathcal{E}_t \) proportional transaction costs, e.g. \( \mathcal{E}_t = S_t \)

\( \varepsilon \) is supposed to be “small.”
Maximising expected utility
under small proportional transaction costs

Goal:

\[ \max_{(\psi^\varepsilon, k^\varepsilon)} E \left[ \int_0^T u_1(t, k_t^\varepsilon) dt + u_2(X_T^\varepsilon(\psi^\varepsilon, k^\varepsilon)) \right] \]

with possibly random utility functions \( u_1(t, \cdot), u_2(\cdot) \)

More precisely: determine leading-order correction to frictionless problem (\( \varepsilon = 0 \)) for small costs \( \varepsilon \)
Consider a contingent claim with payoff $H$ at time $T$.

Look for option premium $\pi$ such that

$$\max_\psi E[u(X_T(\psi))] = \max_\psi E [u(X_T(\psi) + \pi - H)].$$

$\pi$ is the utility indifference price.

Leading-order correction due to small transaction costs?
Related problems ct’d
Mean-variance portfolio optimisation (Markowitz 1952, Tobin 1958, Merton 1972, …)

- Equivalent families of problems, leading to so-called efficient portfolios:
  - \( \max_{\psi} E[X_T(\psi) - X_0] \) subject to \( \text{Var}[X_T(\psi)] \leq s^2 \)
  - \( \min_{\psi} \text{Var}[X_T(\psi)] \) subject to \( E[X_T(\psi) - X_0] \geq m \)
  - \( \max_{\psi} E[-(c - X_T(\psi) + X_0)^2] =: U(c) \)
  - \( \max_{\psi} SR(\psi) \) for the Sharpe ratio

\[
SR(\psi) = \frac{E[X_T(\psi) - X_0]}{\sqrt{\text{Var}[X_T(\psi)]}}
\]

- Relation between optimal values:
  - \( E[X_T(\psi^*) - X_0] = sSR^* \)
  - \( \text{Var}[X_T(\psi^*)] = \frac{(m-x)^2}{SR^*} \)
  - \( U(c) = cU(1) \)
  - \( SR^* = \sqrt{\frac{-1}{U(1)} - 1} \)

- Leading-order correction due to small transaction costs?
Related problems ct’d

Growth optimal portfolio (Kelly 1956, Latané 1959, …)

- The growth-optimal portfolio maximises the long-term growth rate of wealth

  \[
  \lim_{T \to \infty} \sup \frac{1}{T} \log(X_T(\psi))
  \]

  almost surely.

- Existence and leading-order correction under small transaction costs?
Known general structure

and some references

- There is a no-trade region around the frictionless optimiser.
- Do nothing while portfolio is inside no-trade region.
- Do infinitesimal trades at boundary of no-trade region.
- Width of the no-trade region is of order $\varepsilon^{1/3}$.
- (Certainty equivalent of) utility loss is of order $\varepsilon^{2/3}$.
- Some references: Magill and Constantinides (1976), Constantinides (1986), Dumas and Luciano (1991), Taksar et al. (1988), Davis and Norman (1990), Shreve and Soner (1994) and many more
width of no-trade corridor: \( \Delta NT_t = 3 \sqrt[3]{12 R_t \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t} \)

certainty equivalent loss: \( \Delta CE = E^Q \left[ \int_0^T \frac{(\Delta NT_t)^2}{8 R_t} d\langle S \rangle_t \right] \)
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Itô process with fixed transaction costs

- **Risky asset:**
  \[ dS_t = b_t^S dt + \sigma_t^S dW_t \]

- **Wealth process:**
  \[ X_t^\varepsilon(\psi^\varepsilon, k^\varepsilon) = x + \int_0^t \psi^\varepsilon_s dS_s - \int_0^t k^\varepsilon_s ds + \Psi_t - \sum_{s \leq t} \varepsilon_s 1\{\Delta \psi^\varepsilon_s \neq 0\}, \]

where

- \( \psi^\varepsilon_t \) piecewise constant trading strategy (= number of shares),
- \( k^\varepsilon_t \) consumption rate,
- \( x \) initial endowment,
- \( \Psi_t \) random endowment stream, e.g. \( \Psi_t = 0 \)
- \( \varepsilon_t = \varepsilon \varepsilon_t \) fixed transaction costs, e.g. \( \varepsilon_t = 1 \)

- \( \varepsilon \) is supposed to be “small.”
Maximising expected utility
under small fixed transaction costs

Goal:

\[
\max_{(\psi^\varepsilon, k^\varepsilon)} E \left[ \int_0^T u_1(t, k^\varepsilon_t) dt + u_2(X_T^\varepsilon(\psi^\varepsilon, k^\varepsilon)) \right]
\]

with possibly random utility functions \( u_1(t, \cdot), u_2(\cdot) \)

More precisely: determine leading-order correction to frictionless problem \( (\varepsilon = 0) \) for small costs \( \varepsilon \)

in this talk: \( u_2(x) = -\exp(-\rho x) \), no consumption, \( \varepsilon_t = \varepsilon \)
There is a no-trade region around the frictionless optimiser.

Do nothing while portfolio is inside no-trade region.

Jump to frictionless optimum if the boundary of the no-trade region is reached.

Width of the no-trade region is of order $\varepsilon^{1/4}$.

(Certainty equivalent of) utility loss is of order $\varepsilon^{1/2}$.

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Leading-order asymptotics

- Asymptotic no-trade region: \([\bar{NT}_t - \Delta NT_t, \bar{NT}_t + \Delta NT_t]\),

\[
\bar{NT}_t = \varphi_t + \varphi'_t(X^\varepsilon_t(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)),
\]

\[
\Delta NT_t = 3\sqrt{\frac{3R_t d\langle \varphi \rangle_t}{2 d\langle S \rangle_t}} \varepsilon_t
\]

- Asymptotic consumption rate:

\[
\kappa^\varepsilon_t = \kappa_t + \kappa'_t(X^\varepsilon_t(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)),
\]

where

- \(\varphi'_t\) derivative of frictionless optimal portfolio wrt. time-\(t\) wealth,
- \(\kappa'_t\) derivative of frictionless optimal consumption wrt. time-\(t\) wealth,
- \(R_t\) indirect risk tolerance of the frictionless problem
Indirect risk tolerance $R_t$

of the frictionless problem

\begin{align*}
R_t := -\frac{U'(t, X_t)}{U''(t, X_t)}
\end{align*}

for indirect utility

\begin{align*}
U(t, x) := \sup_{(\psi_t, k_t)_{t \in [t, T]}} E_t \left[ \int_t^T u_1(s, k_s) ds + u_2 \left( x + \int_t^T \psi_s dS_s - \int_t^T k_s ds + \psi_t \right) \right]
\end{align*}

Compare Kramkov and Sîrbu (2006), Soner and Touzi (2012)

$R$ yields $\varphi'$, $\kappa'$ via

\begin{align*}
\varphi'_t = \frac{d\langle R, S \rangle_t}{R_t d\langle S \rangle_t}, \quad \kappa'_t = \frac{r_t}{R_t}
\end{align*}

with direct risk tolerance

\begin{align*}
r_t := -\frac{u'_1(t, \kappa_t)}{u''_1(t, \kappa_t)}.
\end{align*}
Indirect risk tolerance $R_t$

as solution to a BSDE

$R$ is solution to quadratic BSDE

\[ dR_t = \left( \frac{\zeta_t^T \zeta_t}{R_t} - \frac{(\sigma_t^T \zeta_t)^2}{R_t \sigma_t^T \sigma_t} - r_t \right) dt + \zeta_t dW_t^Q, \quad R_T = -\frac{u'_2(X_T)}{u''_2(X_T)} \]

if

- $Q$ dual minimiser for frictionless utility maximisation problem,
- filtration generated by $d$-dimensional Brownian motion $W^Q$,
- $dS_t = \sigma_t dW^Q_t$. 

Indirect risk tolerance $R_t$
for standard utility functions without random endowment

- Exponential utility $u_2(x) = -e^{-px}$ and $u_1(t, x) = -e^{\delta(T-t)}e^{-px}$:

$$R_t = \frac{1+T-t}{p},$$
$$\varphi'_t = 0, \quad \kappa'_t = \frac{1}{1+T-t},$$
$$\mathbb{NT}_t = \varphi_t$$

- Power utility $u_2(x) = \frac{x^{1-p}}{1-p}$ and $u_1(t, x) = -e^{\delta(T-t)}\frac{x^{1-p}}{1-p}$:

$$R_t = \frac{x_t}{p},$$
$$\varphi'_t = \frac{\varphi_t}{x_t}, \quad \kappa'_t = 1,$$

no-trade region centered in relative terms.
Leading-order asymptotics continued

- Asymptotic certainty equivalent of utility loss

\[ E_Q \left[ \int_0^T \frac{(\Delta NT_t)^2}{2R_t} d\langle S\rangle_t \right] \]

with dual minimiser \( Q \) of frictionless problem

\( (2/3 \text{ loss due to transaction costs, } 1/3 \text{ loss due to displacement, cf. Rogers 2004}) \)

- Implied trading volume

\[ \|\varphi^\varepsilon\|_T \approx \int_0^T \sqrt[3]{\frac{16}{3R_t}} \left( \frac{d\langle \varphi\rangle_t}{d\langle S\rangle_t} \right) \frac{1}{\varepsilon_t} d\langle S\rangle_t \]
Related results
Utility indifference pricing for exponential utility

- In general:
  \[ \pi^\varepsilon(H) \approx \pi^0(H) + \frac{p}{2} \left( E_{Q^H} \left[ \int_0^T (\Delta N T^H_t)^2 dt \right] - E_Q \left[ \int_0^T (\Delta N T_t)^2 dt \right] \right) \]
  with \( Q, Q^H \) minimal entropy martingale measures relative to \( P \) resp. \( P^H \) with density \( \frac{dP^H}{dP} = \frac{e^{-pH}}{E[e^{-pH}]} \)

- In complete markets with negligible pure investment:
  \[ \frac{\pi^\varepsilon(nH)}{n} = E_Q[H] + \left( \frac{9pn\varepsilon^2}{32} \right) E_Q \left[ \int_0^T |\Gamma^H_t S_t|^{4/3} \frac{d\langle S\rangle_t}{S_t^2} \right] \]
  with \( \Gamma^H \) defined as the Gamma of the claim.
Use general results for quadratic utility $u(x) = -(c + X_0 - x)^2$.

Get correction terms for $U(1)$, $SR^*$ etc.
Use no trade region of optimal portfolio for logarithmic utility

Long term growth rate loss due to transaction costs:

\[
\lim_{T \to \infty} \frac{1}{T} \log(X_T(\varphi)) - \lim_{T \to \infty} \frac{1}{T} \log(X_\varepsilon(\varphi_\varepsilon)) \\
\approx \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{(\Delta \pi_t)^2}{2} \frac{d\langle S \rangle_t}{S_t^2}
\]

with \(\Delta \pi_t\) being the half width of the no trade region in fractions of wealth.
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Leading-order asymptotics
for exponential utility of terminal wealth

- Asymptotic no-trade region: \([\varphi_t - \Delta N T_t, \varphi_t + \Delta N T_t]\),
  
  \[
  \Delta N T_t = 4 \sqrt{\frac{12}{p}} \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon
  \]

- Asymptotic certainty equivalent loss:
  
  \[
  E_Q \left[ \int_0^T \frac{p}{6} (\Delta N T_t)^2 d\langle S \rangle_t \right]
  \]

  with minimal entropy martingale measure \(Q\) of frictionless problem
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Some basics of stochastic control

Dynamic programming approach

- **Goal:**
  \[
  \max_{\psi} E[u(X_T(\psi))] =: J_0
  \]

- **Value process:**
  \[
  J(t, \psi) := \operatorname{ess sup} \left\{ E[u(X_T(\psi))|\mathcal{F}_t] : \tilde{\psi} = \psi \text{ on } [0, t] \right\}
  \]

  \[
  J(T, \psi) = u(X_T(\psi)),
  \]

  \((J(t, \psi))_{t \in [0, T]}\) supermartingale for all \(\psi\),

  martingale for the optimiser \(\psi^*\)

- **Upper bound:**
  If \(\tilde{J}(T, \psi) = u(X_T(\psi))\), \((\tilde{J}(t, \psi))_{t \in [0, T]}\) supermartingale for all \(\psi\),
  then \(J_0 \leq \tilde{J}(0, \star)\).

  ▶ Proof: \(\tilde{J}(0, \psi^*) \geq E[\tilde{J}(T, \psi^*)] = E[u(X_T(\psi^*))] = J_0\)
Some basics of stochastic control
Dynamic programming approach ct’d

- **Verification:**
  If \( \tilde{J}(T, \psi) = u(X_T(\psi)) \),
  \( (\tilde{J}(t, \psi))_{t \in [0, T]} \) supermartingale for all \( \psi \), martingale for some \( \psi^* \),
  then \( J_0 = \tilde{J}(0, \star) \).
  
  - **Proof:**
    \[
    \tilde{J}(0, \psi^*) = E[\tilde{J}(T, \psi^*)] = E[u(X_T(\psi^*))] \\
    \geq E[\tilde{J}(T, \psi)] = E[u(X_T(\psi))]
    \]

- **Derivation:**
  **Ansatz:** \( \tilde{J}(t, \psi) = v(t, X_t(\psi)) \) & Itô’s formula → HJB equation
Some basics of stochastic control
Dynamic programming approach ct’d (later!)

- **Approximation:**
  If \( \tilde{J}(T, \psi) = u(X_T(\psi)) + o(\varepsilon^{1/2}) \),
  \( (\tilde{J}(t, \psi))_{t \in [0, T]} \) has drift \( \leq o(\varepsilon^{1/2}) \) for all \( \psi \),
  has drift \( o(\varepsilon^{1/2}) \) for some \( \psi^* \),
  then \( E[u(X_T(\psi))] \geq J_0 + o(\varepsilon^{1/2}) \).

  - **Proof:**
    \[
    \tilde{J}(0, \psi^*) = E[\tilde{J}(T, \psi^*)] + o(\varepsilon^{1/2}) = E[u(X_T(\psi^*))] + o(\varepsilon^{1/2}) \\
    \geq E[\tilde{J}(T, \psi)] + o(\varepsilon^{1/2}) = E[u(X_T(\psi))] + o(\varepsilon^{1/2})
    \]

- **Fixed costs:**
  Consider the certainty equivalent \( CE(t, \psi) := u^{-1}(\tilde{J}(t, \psi)) \).
  **Ansatz:** \( CE^\varepsilon(t, \psi) = CE^0(t, \psi) + f^\varepsilon_t(\Delta \psi_t) \) with
  \( f^\varepsilon_t(\delta) = \alpha_t \delta^4 - \beta_t \delta^2 - \gamma_t \) and \( \alpha_t, \beta_t, \gamma_t \) such that approximate (super-)martingale property holds.
Some basics of stochastic control
Convex duality approach

Goal:

$$\max_{\psi} E[u(X_T(\psi))]$$

Equivalent martingale measures (EMM’s) $Q$ and their density process $Z$: $Z_0 = 1$, $Z$ martingale, $ZS$ martingale

Upper bounds: for any EMM $Z$

$$E[u(X_T(\psi))] = E[u(X_T(\psi) + Z_T X_T(\psi)] - Z_0 X_0$$

$$\leq E[\nu(Z_T)] - X_0$$

with $\nu(y) = \sup_x (u(x) + xy)$ (Legendre transform of $-u$).

Equality for minimiser $Z^*$ (corresponding e.g. to MEMM $Q$)
Some basics of stochastic control
Convex duality approach ct’d

- Verification: if $\psi^*$ such that $Q$ with $\frac{dQ}{dP} = \frac{u'(X_T(\psi^*))}{E[u'(X_T(\psi^*))]}$ EMM, then $\psi^*$ is optimal:

$$E[u(X_T(\psi))] \leq E[u(X_T(\psi^*)) + u'(X_T(\psi^*)) (X_T(\psi) - X_T(\psi^*))]$$

$$= E[u(X_T(\psi^*))] + E[u'(X_T(\psi^*))] E_Q[X_T(\psi) - X_T(\psi^*)]$$

$$= E[u(X_T(\psi^*))]$$

- Derivation: look for $\psi^*, Z^*$ such that $Z^*, Z^* S$ are martingales, $Z_T^* = u'(X_T(\psi^*)) / E[u'(X_T(\psi^*))]$ (related to FBSDE)

- Approximate optimality: look for $\psi^*, Z^*$ such that

$$E[u(X_T(\psi^*))] = E[v(Z_T^*)] - X_0 + o(\varepsilon^{2/3})$$
Some basics of stochastic control
Convex duality approach ct’d (later!)

- Proportional transaction costs (shadow prices): $Z, \tilde{S}$ instead of $Z; S \leq \tilde{S} \leq \bar{S}; Z\tilde{S}$ martingale

- Relation value process/shadow prices: Consider value process as function of wealth in assets 0,1: $J(t, X^0_t, X^1_t)$

\[
\tilde{S}_t = \frac{\partial_3 J(t, X^0_t, X^1_t)}{\partial_2 J(t, X^0_t, X^1_t)}, \quad Z_t = \partial_2 J(t, X^0_t, X^1_t)
\]

- Small costs:
  
  Ansatz: Consider $\Delta S := \tilde{S} - S, \Delta \psi := \psi^\epsilon - \psi$.
  
  Ansatz: $\Delta S_t = f_t(\Delta \psi)$ with $f_t(\delta) = \alpha_t \delta^3 - \gamma_t \delta$ and $\alpha_t, \gamma_t$ such that martingale properties hold.
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Shadow prices approach
(Cvitanić and Karatzas 1996, Loewenstein 2000, . . .)

- General principle:
  portfolio optimization in market with transaction costs
  \[\iff\]
  portfolio optimization without transaction costs for some shadow process within bid-ask bounds
Shadow prices approach
(Cvitanić and Karatzas 1996, Loewenstein 2000, ...)

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Shadow prices approach

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Shadow prices approach
(Cvitanić and Karatzas 1996, Loewenstein 2000, ...)

- General principle:
  portfolio optimization in market with transaction costs

  ≡

  portfolio optimization without transaction costs for some shadow process within bid-ask bounds
Shadow price approach
in the terminal wealth case $u_1 = 0$ without random endowment

Look for

- strategy $\varphi^\varepsilon_t$,
- price process $\tilde{S}^\varepsilon_t \in [S_t, \overline{S}_t]$,
- process $Z^\varepsilon_t$,

which satisfy

- $Z^\varepsilon_T = u'_2(x + \varphi^\varepsilon \cdot \tilde{S}^\varepsilon_T)$,
- $Z^\varepsilon$ martingale,
- $Z^\varepsilon \tilde{S}^\varepsilon$ martingale,
- $\varphi^\varepsilon$ changes only when $\tilde{S}^\varepsilon_t \in \{S_t, \overline{S}_t\}$.

Then $\varphi^\varepsilon$ is the optimal portfolio.
Optimality of $\varphi^\varepsilon$

informally

For any competitor $\psi$ we have

$$E[u(X_T(\psi))] \leq E[u(x + \psi \cdot \tilde{S}^\varepsilon_T)]$$

$$\leq E[u(x + \varphi^\varepsilon \cdot \tilde{S}^\varepsilon_T)) + E[u'(x + \varphi^\varepsilon \cdot \tilde{S}^\varepsilon_T)((\psi - \varphi^\varepsilon) \cdot \tilde{S}^\varepsilon_T)]$$

$$= E[u(X_T(\varphi^\varepsilon))] + E[Z^\varepsilon_T((\psi - \varphi^\varepsilon) \cdot \tilde{S}^\varepsilon_T)]$$

$$= E[u(X_T(\varphi^\varepsilon))].$$

(Second inequality follows from $u(y) \leq u(x) + u'(x)(y - x)$.)
Approximate solution

Look for

- strategy $\varphi^\varepsilon$,
- price process $\tilde{S}^\varepsilon_t \in [S_t, \bar{S}_t]$,
- process $Z^\varepsilon$,

which satisfy

- $Z^\varepsilon_T = u'(x + \varphi^\varepsilon \cdot \tilde{S}^\varepsilon_T) + O(\varepsilon)$,
- $Z^\varepsilon$ has drift $o(\varepsilon^{2/3})$,
- $Z^\varepsilon \tilde{S}^\varepsilon$ has drift $O(\varepsilon^{2/3})$,
- $\varphi^\varepsilon$ changes only when $\tilde{S}_t^\varepsilon \in \{S_t, \bar{S}_t\}$.

Then $\varphi^\varepsilon$ is optimal to the leading-order.
Approximate optimality of $\varphi^\varepsilon$

informally

For any competitor $\psi^\varepsilon$ with $\psi^\varepsilon - \varphi = o(1)$ we have

$$E[u(X_T(\psi^\varepsilon))] \leq E[u(x + \psi^\varepsilon \cdot \tilde{S}_T^\varepsilon)]$$

$$\leq E[u(x + \varphi^\varepsilon \cdot \tilde{S}_T^\varepsilon)] + E\left[u'(x + \varphi^\varepsilon \cdot \tilde{S}_T^\varepsilon)((\psi^\varepsilon - \varphi^\varepsilon) \cdot \tilde{S}_T^\varepsilon)\right]$$

$$= E[u(X_T(\varphi^\varepsilon))] + E\left[Z_T^\varepsilon((\psi^\varepsilon - \varphi^\varepsilon) \cdot \tilde{S}_T^\varepsilon)\right] + o(\varepsilon^{2/3})$$

$$= E[u(X_T(\varphi^\varepsilon))] + o(\varepsilon^{2/3}).$$

Compare with

$$E[u(X_T(\varphi^\varepsilon))] - E[u(x + \varphi \cdot \tilde{S}_T)] = O(\varepsilon^{2/3}).$$
How to find an approximate solution
The ansatz

- How to find \( \phi^\epsilon, \tilde{S}^\epsilon, Z^\epsilon \), no-trade bounds \( \Delta \varphi^\pm \)?
- Write

\[
\begin{align*}
\phi^\epsilon &= \phi + \Delta \varphi, \\
\tilde{S}^\epsilon &= S + \Delta S, \\
Z^\epsilon &= Z(1 + K),
\end{align*}
\]

where \( \phi, Z \) are optimal for the frictionless problem.

- Ansatz:

\[
\begin{align*}
\Delta S &= f(\Delta \varphi, \ldots), \\
f(x) &= \alpha x^3 - \gamma x
\end{align*}
\]
Some references
on verifying first-order approximations

- Based on dynamic programming:
  - Careful analysis and expansion of the value function:
    Shreve and Soner (1994), Janeček and Shreve (2004, 2010),
  - Homogenisation approach:
    Soner and Touzi (2013), Possamai, Soner, Touzi (2012), Altarovici,
    Moreau, Muhle-Karbe, Soner (2014), Possamai and Royer (2014),
    . . .

- Based on shadow prices:
  - Expanding the exact solution:
    Gerhold, Muhle-Karbe, Schachermayer (2012), Gerhold, Guasoni,
    Muhle-Karbe, Schachermayer (2014), Guasoni and Muhle-Karbe
    (2013), . . .
  - Here (joint work with Shen Li):
    find explicit nearly optimal candidates for primal and dual problem
A rigorous theorem

Setup

- Itô process

\[ dS_t = b^S_t \, dt + \sigma^S_t \, dW_t \]

- Exponential utility \( u(x) = -e^{-px} \) of terminal wealth

- Transaction costs \( \varepsilon_t = \varepsilon S_t \)

- Maximise

\[ \text{CE}(X_T(\psi)) = -\frac{1}{p} \log E[\exp(-pX_T(\psi))] \]
A rigorous theorem

1. Existence of the process $\Delta \varphi$ as solution to a Skorohod problem with reflection.
2. Definition of $\Delta \tilde{S}$ as a function of $\Delta \varphi$.
3. Leading-order approximation of the corresponding expected utility by Taylor expansion.
4. Consider MMM $\tilde{Z}^\varepsilon$ for $\tilde{S}^\varepsilon$ relative to $Q$.
5. Compute leading-order approximation of Lagrange dual function at $\tilde{Z}^\varepsilon$. Observe that lower and upper bound coincide to leading-order.
6. Replace $\Delta \varphi$ in expression for certainty equivalent loss by $\Delta NT$, using the ergodic property of reflected Brownian motion.

Draw a picture!
There exists EMM for $S$ with finite relative extropy. 
\( \leadsto \) MEMM $Q$ and frictionless optimiser $\varphi$ exist.

$\varphi$ and $\rho := \frac{d\langle \varphi \rangle}{d\langle S \rangle} > 0$ are Itô processes.

Set $Y := \log(S)$, $\pi := \varphi S$, $\eta := \rho S^4$.

$$E_Q \left[ \sup_{t \in [0,T]} |g|_t^n \right] < \infty, \quad n \in \mathbb{N}$$

for processes $g \in \{\pi, \eta, 1/\eta, b^\pi, b^\eta, c^Y, 1/c^Y, c^\pi, c^\eta\}$.

$$E_Q \left[ \exp(|9\rho \int_0^T \varphi_t dS_t|) \right] < \infty$$

$c^\pi, c^Y, c^\eta$ are continuous.
Regularity conditions in the literature
which fail if $d\langle \varphi \rangle_t / dt = 0$

Bichuch (2011):

**Assumption 3.1** We will assume that the contingent claim payoff function $g : [0, \infty) \to [0, \infty)$ is four times continuously differentiable, and such that $g(S) - g'(S)S, S^2g''(S), S^3g^{(3)}(S)$ and $S^4g^{(4)}(S)$ are all bounded for $S \in [0, \infty)$.

**Assumption 3.2** We will also assume that $\frac{\varepsilon^{-T}(\mu-r)}{\gamma \sigma^2} - \sup_{S \in [0,\infty)} S^2|g''(S)| \geq \varepsilon_1$, for some $\varepsilon_1 > 0$. The supremum above is finite by Assumption 3.1.

Soner and Touzi (2013):

equation is nondegenerate away from the origin. For later use we record that there exist two constants $0 < \alpha_* \leq \alpha^*$ so that

$$0 < \alpha_* \leq \frac{\alpha(s, z)}{z} \leq \alpha^* \quad \forall s, z \in \mathbb{R}_+.$$  

We will not attempt to verify the above hypothesis. However, in the power utility
A rigorous theorem

Statement

- Define $\Delta NT$ as before.
- There exists $\varphi^\varepsilon = \varphi + \Delta \varphi = \varphi^{\varepsilon \uparrow} - \varphi^{\varepsilon \downarrow}$ with values in $[\varphi - \Delta NT, \varphi - \Delta NT]$, where $\varphi^{\varepsilon \uparrow}, \varphi^{\varepsilon \downarrow}$ increase only at the boundary.
- Set

$$
\tau^\varepsilon := \inf \left\{ t \in [0, T] : |X_t(\varphi^\varepsilon) - (x + \varphi \cdot S_t)| > 1 \text{ or } |X_t(\varphi^\varepsilon)| > \varepsilon^{-4/3} \right\}
$$

- Then $\varphi^{\varepsilon 1}_{[0, \tau^\varepsilon]}$ is optimal to the leading-order with certainty equivalent loss as stated earlier.

Draw another picture!
Relaxed regularity conditions
by trading more carefully

- There exists EMM for $S$ with finite relative extropy.
  $\leadsto$ MEMM $Q$ and frictionless optimiser $\varphi$ exist.

- $\varphi$ and $\rho := \frac{d\langle \varphi \rangle}{d\langle S \rangle}$ are Itô processes.

$$E_Q \left[ \int \left( (1 + S_t^2)c_t^\varphi \right)^{\frac{5}{12}} dt \right] < \infty,$$

$$E_Q \left[ \int \left( (1 + |S_t|)b_t^\varphi \right)^{\frac{5}{12}} dt \right] < \infty,$$

$$E_Q \left[ \int (c_t^S)^3 dt \right] < \infty,$$

$$E_Q \left[ \sup_{t \in [0, T]} (1 + |\varphi_t|)|S_t| \right] < \infty,$$

$$E_Q \left[ \exp(|9p_0^T \varphi_t dS_t|) \right] < \infty$$

$c^\varphi, c^S, c^\rho, 1/c^S, b^\varphi, b^\rho$ are continuous.
Outline

1 The problem
   - Proportional transaction costs
   - Fixed transaction costs

2 Results
   - Proportional transaction costs
   - Fixed transaction costs

3 Heuristics and verification
   - Basics of stochastic control
   - Proportional transaction costs
   - Fixed transaction costs

4 Conclusion
Main conclusions
concerning leading-order asymptotics

- **Proportional costs:**
  - Robust formulas for no-trade region, consumption rate, utility loss, implied trading volume
  - In some sense myopic results
  - Important ingredients of frictionless problem:
    - indirect risk tolerance $R_t$
    - activity rate or generalized squared gamma $\frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t}$
  - 2/3 loss due to transaction costs, 1/3 loss due to displacement
  - Verification is sometimes painful.

- **Fixed transaction costs:**
  - Similar results
  - Different rates