Backward Stochastic Evolution Equations
A Wiener chaos approach

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The study of backward stochastic evolution equations (BSEEs) is an important field of stochastic analysis. In this talk we will present an approach on how the theory of the Wiener chaos expansion can be used for the construction of solutions of backward stochastic evolution equations.

The Wiener chaos expansion reduces a stochastic evolution equation to an infinite hierarchy of deterministic evolution equations (the propagator). The solution of the stochastic evolution equation can be reconstructed from knowledge of the solution of this deterministic system.

This allows not only the representation of known solutions in a convenient form, but also the construction of novel solutions (under weaker assumptions) the weighted Wiener chaos solutions. This permits a better understanding of the structure of solutions, the use of deterministic numerical techniques for their implementation, and the construction of generalized solutions to problems that may not admit solutions under the standard theory of stochastic differential equations.
The Wiener chaos expansion

Let $Y$ be a separable Hilbert space with inner product $(\cdot, \cdot)_{Y}$ and orthonormal basis $\{y_k\}_{k \in \mathbb{N}}$, and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$ a complete, filtered probability space.

**Definition**

A cylindrical Brownian motion, $W = \{W_y(t)\}_{0 \leq t \leq T, y \in Y}$ is a Gaussian family of random variables with the following properties: For each $y \in Y$, the stochastic process $t \mapsto W_y(t)$ is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$; and for every $(y, z) \in Y^2$, $(s, t) \in [0, T]^2$ we have

$$
\mathbb{E}[W_y(t)] = 0, \quad \mathbb{E}[W_y(t)W_z(s)] = (s \wedge t)(y, z)_Y.
$$

This cylindrical Brownian motion can be considered as a collection of real-valued and independent Brownian motions – in the sense that it can be represented in terms of standard, real-valued and independent Wiener processes $w_k(\cdot) := W_{y_k}(\cdot), k \in \mathbb{N}$ as $W_y(t) = \sum_{k \in \mathbb{N}} (y, y_k)_Y w_k(t), (t, y) \in [0, T] \times Y$. 
Let
\[ H_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad x \in \mathbb{R}, \]
be the Hermite polynomial of order \( n \in \mathbb{N}_0 \).

Let \( \mathcal{M} = \{ m_i(\cdot) \}_{i \in \mathbb{N}} \) be an orthonormal basis of \( L^2([0, T]) \) such that \( m_i(\cdot) \in L^\infty([0, T]), \ i = 1, 2, \cdots \) (e.g. the Fourier basis).

Define,
\[ \xi_\alpha := \frac{1}{\sqrt{\alpha!}} \prod_{(i,k) \in \mathbb{N}^2} H_{\alpha_i^k}(\xi_{ik}), \text{ where } \xi_{ik} := \int_0^T m_i(t)dw_k(t), \quad \alpha \in \mathcal{J}. \]

Here the multi-index \( \alpha \) denotes an element of
\[ \mathcal{J} := \left\{ (\alpha_i^k)_{(i,k) \in \mathbb{N}^2} : \alpha_i^k \in \mathbb{N}_0, \ |\alpha| := \sum_{i,k} \alpha_i^k < \infty \right\}, \text{ and } \alpha! := \prod_{i,k} \alpha_i^k! \cdot \]

The set \( \mathcal{J} \) is countable and, for each \( \alpha \in \mathcal{J} \), only finitely many \( \alpha_i^k \) are non-zero.
The $\{\xi_\alpha\}_{\alpha \in \mathcal{J}}$ are orthonormal,

$$\mathbb{E}[\xi_\alpha \xi_\beta] = \delta_{\alpha,\beta}$$

The Hilbert space consisting of all $\mathcal{F}_T$-measurable and $\mathbb{P}$-square integrable random variables $L^2_T(\mathbb{W})$, called Wiener Chaos space, can be constructed using the Cameron-Martin basis.
Theorem (Wiener-Cameron-Martin)

Any \( X \in \mathbb{L}^2_T(\mathbb{W}) \) admits a representation in the series

\[
X = \sum_{\alpha \in \mathcal{J}} x_\alpha \xi_\alpha
\]

where \( x_\alpha = \mathbb{E}[X \xi_\alpha] \) deterministic.

The following Parseval identity holds:

\[
\mathbb{E}[X^2] = \sum_{\alpha \in \mathcal{J}} x^2_\alpha,
\]

Conversely if

\[
X = \sum_{\alpha \in \mathcal{J}} x_\alpha \xi_\alpha
\]

for \( x_\alpha \in \mathbb{R} \) such that \( \sum_{\alpha \in \mathcal{J}} x^2_\alpha < \infty \) holds then \( X \in \mathbb{L}^2_T(\mathbb{W}) \).
This construction can be generalized (e.g. Rosovskii - Lototsky Ann. Prob. 2006) for square integrable random variables taking values in a general Hilbert space $\mathcal{X}$, where now

$$X = \sum_{\alpha \in J} x_\alpha \xi_\alpha$$

where $x_\alpha = \mathbb{E}[X \xi_\alpha] \in \mathcal{X}$ deterministic. The following Parseval identity holds:

$$\mathbb{E}[||X||_{\mathcal{X}}^2] = \sum_{\alpha \in J} ||x_\alpha||_{\mathcal{X}}^2,$$

and criterion for the convergence of the series

$$\sum_{\alpha \in J} x_\alpha \xi_\alpha$$

is

$$\sum_{\alpha \in J} ||x_\alpha||_{\mathcal{X}}^2 < \infty.$$
Suppose now that instead of characterizing a single random variable we want to characterize a whole family $X(t), t \in [0, T]$, a stochastic process.

If the process is such that $X(t)$ for all $t \in [0, T]$, is measurable with respect to the $\sigma$–algebra $\mathcal{F}_T = \sigma(W(r), r \leq T)$, then we apply Cameron-Martin theorem for every $t$ and obtain the series representation

$$X(t) = \sum_{\alpha \in \mathcal{J}} x_\alpha(t) \xi_\alpha, \quad x_\alpha(t) = \mathbb{E}[X(t) \xi_\alpha]$$
Related research

Rosovskii and Lototsky in a series of seminal works (e.g. RL, Ann. Prob. 2006) have studied the problem of whether the expansion coefficients \( \{ u_\alpha(t) \}_{\alpha \in \mathcal{J}} \) can be obtained directly from the evolution equation and gave a number of interesting results for the case of linear parabolic stochastic evolution equations of the form

\[
du = (Au + f) \, dt + \sum_{k \in \mathbb{N}} M_k u + g_k \, dw_k,
\]

where we consider \( W(t) = \sum_{k \in \mathbb{N}} w_k(t) y_k, \{ y_k \}_k, \text{ basis of } \mathcal{Y} \).

Their finding is that the deterministic amplitudes \( \{ u_\alpha \}_{\alpha \in \mathcal{J}} \) are the solutions of an infinite hierarchy of deterministic evolution equations of the form

\[
u'_\alpha = A u_\alpha + f_\alpha + \sum_{i,k} \sqrt{a^k_i} (M_k u_{a^{-i,k}} + g_k)) m_i, \quad \alpha \in \mathcal{J}
\]
This system is lower triangular.

If the solution of this system, the propagator, satisfies
\[ \sum_{\alpha \in \mathcal{J}} \| u_\alpha \|_{\mathcal{X}}^2 < \infty, \]
then the random field \( \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha \) coincides with the “traditional” square integrable solution of the stochastic evolution equation.

If the solution of the propagator satisfies \( \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| u_\alpha \|_{\mathcal{X}}^2 < \infty \), for some weight functions \( r_\alpha \), then the random field \( \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha \) corresponds to new types of solutions not derivable by the standard theory of stochastic evolution equations.

These solutions are called Wiener chaos solutions and correspond to solutions of the stochastic evolution equation under weaker assumptions on the data of the problem.
The Wiener Chaos approach is a beautiful construction leading to deep understanding of the solutions of SPDEs and to a wealth of new solutions.

Can it be extended to other types of stochastic evolution equations rather than forward SPDEs?

- Backward stochastic evolution equations
A modification of the Cameron-Martin basis

However, in most cases of interest in stochastic evolution equations we need to make sure that for all \( t \in [0, T] \), \( X(t) \) is measurable with respect to the \( \sigma \)-algebra \( \mathcal{F}_t = \sigma(W(r), r \leq t) \), and not \( \mathcal{F}_T = \sigma(W(r), r \leq T) \), which is larger!

The intuition behind that is that the values of the process up to time \( t \) must be known only by the evolution of the noise up to this time and not by the future values of time!

Adapted vs. anticipative processes — Causality issues.

The Wiener-Cameron-Martin expansion always guarantees that \( X(t) \in m - \mathcal{F}_T, \forall t \in [0, T] \) – However, this is not enough we need to make sure that \( X(t) \in m - \mathcal{F}_t, \forall t \in [0, T] \)

▶ We need a new basis!
A modification of the Cameron-Martin basis

A candidate for this basis is \( \{ \xi_\alpha(t) \}_{\alpha \in \mathcal{J}} := \{ \mathbb{E}[\xi_\alpha | \mathcal{F}_t] \}_{\alpha \in \mathcal{J}} \).

This set is adapted, but unfortunately no longer orthonormal!

Lemma

(i) The quantities \( \rho_{\alpha,\beta}(t) := \mathbb{E}[\xi_\alpha(t)\xi_\beta(t)], \ 0 \leq t \leq T \) satisfy the deterministic system of backward ordinary differential equations

\[
d\rho_{\alpha,\beta}(t) = \sum_{i,j,k} \sqrt{\alpha_i^k} \sqrt{\beta_j^k} \rho_{\alpha^-(i,k),\beta^-(j,k)}(t) m_i(t) m_j(t) dt, \quad \rho_{\alpha,\beta}(T) = \delta_{\alpha\beta}
\]

for \((\alpha, \beta) \in \mathcal{J}^2\). Here by \(\alpha^-(i, k)\) we denote the matrix

\[
(\alpha^-(i, k))_j^\ell = \begin{cases} 
\max(\alpha_i^k - 1, 0), & \text{if } i = j \text{ and } k = \ell \\
\alpha_j^k & \text{otherwise}
\end{cases}
\]

and \(m_i(\cdot)\) are the elements of the orthonormal basis \(\mathcal{M}\) for \(L^2([0, T])\) chosen in the construction of the Cameron-Martin basis \(\{\xi_\alpha\}_{\alpha \in \mathcal{J}}\).

(ii) The quantities \(\rho_{\alpha,\beta}(t)\) satisfy the bound \(\rho_{\alpha,\beta}(t) \leq 1\) for all \(t \leq T\).
A modification of the Cameron-Martin basis

However even though this is new “basis” is not orthogonal, we have the following:

**Lemma**

Suppose \( u(\cdot) : [0, T] \times \Omega \rightarrow \mathbb{R} \) is an \( \mathbb{F} \)–adapted process such that \( u(t) \in L^2_t(\mathbb{W}; \mathcal{X}) \) for every \( t \in [0, T] \).

Then \( u(t) \) admits the Wiener chaos expansion

\[
    u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha(t) \quad \text{with} \quad u_\alpha(t) = \mathbb{E}[u(t)\xi_\alpha(t)], \quad \xi_\alpha(t) := \mathbb{E}[\xi_\alpha|\mathcal{F}_t] \quad (1)
\]

and the Parseval-type identity \( \mathbb{E}[\|u(t)\|_\mathcal{X}^2] = \sum_{\alpha \in \mathcal{J}} (\|\mathbb{E}[u(t)\xi_\alpha(t)]\|_\mathcal{X})^2 \) holds.

Conversely, if we define the stochastic process

\[
    u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha(t)
\]

and the \( \{u_\alpha\}_{\alpha \in \mathcal{J}} \) are such that \( \sum_{\alpha \in \mathcal{J}} \|u_\alpha\|_\mathcal{X}^2 < \infty \) then the stochastic process \( u(\cdot) \in L^2_t(\mathbb{W}; \mathcal{X}) \).
Furthermore,

**Lemma**

For every $t \in [0, T]$, the family $\{ \xi_\alpha(t) \}_{\alpha \in \mathcal{J}}$ is total in $L^2_t(W)$; to wit, if $\mathbb{E}[X \xi_\alpha(t)] = 0$, $\forall \alpha \in \mathcal{J}$ holds for some $\mathcal{F}_t$–measurable random variable $X \in L^2_t(W)$, then $X = 0$ a.e.

**Lemma**

For every $\mathbb{F}$–adapted, square-integrable stochastic process $u(\cdot)$ with Wiener chaos expansion $u(\cdot) = \sum_{\alpha \in \mathcal{J}} u_\alpha(\cdot) \xi_\alpha$, the deterministic coefficients $u_\alpha(\cdot)$ satisfy the system of equations

$$u_\alpha(t) = \sum_{\beta \in \mathcal{J}} u_\beta(t) \rho_{\alpha,\beta}(t), \quad 0 \leq t \leq T \quad \text{for all } \alpha \in \mathcal{J}. \quad (2)$$
Remark

Because the \( \{ \xi_\alpha(t) \}_\alpha \in \mathcal{J} \) are not necessarily linearly independent, there may exist several expansions of the process \( u(\cdot) \), say

\[
u(t) = \sum_{\alpha \in \mathcal{J}} \bar{u}_\alpha(t) \xi_\alpha(t)
\]

with \( \bar{u}_\alpha(t) \neq u_\alpha(t) \), in addition to the expansion of (1).

In general we shall have \( u(t) \neq \sum_{\alpha \in \mathcal{J}} \bar{u}_\alpha(t) \xi_\alpha \), unless the coefficients \( \bar{u}_\alpha(t) \) satisfy the “compatibility condition” (2) of Lemma 6.

If this condition is satisfied, then \( u(t) \) admits an expansion in the basis \( \{ \xi_\alpha \} \) and an expansion in the basis \( \{ \xi_\alpha(t) \} \) with the same expansion coefficients \( \bar{u}_\alpha(t) \).
Consider now the formal “Fourier” series

\[ \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha, \quad u_\alpha \in \mathcal{X}, \quad \mathcal{X} \]

If the coefficients \( \{u_\alpha(t)\}_{\alpha \in \mathcal{J}} \) are such that \( \sum_{\alpha \in \mathcal{J}} \|u_\alpha\|_{\mathcal{X}}^2 < \infty \) for all \( t \), then this series defines for every fixed \( t \in [0, T] \) a square integrable (with respect to the probability measure) random element in \( \mathcal{X} \), and varying \( t \) over the interval \([0, T]\) we obtain an \( \mathcal{X} \)-valued stochastic process.
What if the numerical series \( \sum_{\alpha \in \mathcal{J}} \| u_\alpha \|_{X}^2 \) does not converge, but rather \( \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| u_\alpha \|_{X}^2 < \infty \) for a \( \{ r_\alpha \}_{\alpha \in \mathcal{J}}, r_\alpha < 1 \)?

Then we obtain generalized stochastic processes, “living” in weaker spaces than the space of square integrable random elements.

If on the other hand, \( \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| u_\alpha \|_{X}^2 < \infty \) for a \( \{ r_\alpha \}_{\alpha \in \mathcal{J}}, r_\alpha > 1 \), then we obtain generalized stochastic processes which are better behaved than square integrable random elements.

▶ These two types of spaces are called weighted Wiener chaos spaces.
Backward SEEs and their solutions

Definition

A linear \textit{backward stochastic evolution equation} (BSEE) for the pair of \(\mathcal{F}\)– progressively measurable stochastic processes \((u(\cdot), Z(\cdot))\) is an integral equation

\[
\Lambda - u(t) = \int_t^T (A u(s) + B Z(s) + f(s)) ds + \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T
\]

with \(u(t) \in V\) and \(Z(t) \in \mathbb{L}_2(K; H)\).

\[
\int_0^t Z(s) dW(s) = \sum_{k \in \mathbb{N}} \int_0^t Z(s; e_k) d\mathbf{w}_k(s) := \sum_{k \in \mathbb{N}} \int_0^t \mathbf{\zeta}_k(s) d\mathbf{w}_k(s),
\]

where \(\{e_k\}\) is an orthonormal basis of the Hilbert space \(K\), and

\[
\mathbf{\zeta}_k(s) := Z(s; e_k) = Z(s)e_k, \quad k \in \mathbb{N}
\]
Definition

We say that the pair of $\mathbb{F}-$progressively measurable stochastic processes $(u(\cdot), Z(\cdot)) \in L^2(\mathbb{W}, L^2([0, T]; V)) \times L^2(\mathbb{W}, L^2([0, T]; L^2(K; H)))$ is a weak solution of the backward stochastic evolution equation (3), if the equation (3) holds in $V'$, i.e., if for every $v \in V$ we have almost surely:

$$
(\Lambda, v)_H - (u(t), v)_H = \int_t^T (\langle A u(s), v \rangle + (B Z(s), v)_H + \langle f(s), v \rangle) \, ds + \sum_{k \in \mathbb{N}} \int_t^T (\delta_k(s), v)_H \, dW_k(s), \quad \forall \ 0 \leq t \leq T.
$$

(5)

Definition

Within the semigroup framework, we say that a weak solution of (3) is a strong solution, if for every $t \in [0, T]$ we have $u(t) \in D(A)$. By the density of $D(A)$ in $H$, this implies that (3) is satisfied a.s.
Wiener chaos and stochastic evolution equations

Since the Wiener chaos expansion may be used to generate any stochastic process, square integrable or not, it may also be used to generate solutions of backward stochastic evolution equations.

The question is the following:

If the stochastic process \( u(\cdot) \), taking values in \( \mathcal{X} \) is the solution of the stochastic evolution equation

\[
du = (Au + BZ + f) \, dt + Z \, dW(t)
\]

and

\[
u(t) = \sum_{\alpha \in \mathcal{J}} u_{\alpha}(t) \xi_{\alpha} \quad \text{or} \quad \nu(t) = \sum_{\alpha \in \mathcal{J}} \bar{u}_{\alpha}(t) \xi_{\alpha}(t)
\]

(where at this point this series is understood as a formal object)

what can we deduce concerning the amplitudes of the expansion \( \{ u_{\alpha} \} \) or \( \{ \bar{u}_{\alpha} \} \)?
The resembles the Faedo-Galerkin expansion which is used in the numerical or even analytical treatment of deterministic evolution equations.

This Galerkin expansion “separates” randomness which is included in the choice of the basis \( \{ \xi_\alpha \}_{\alpha \in \mathcal{J}} \), or \( \{ \xi_\alpha (t) \}_{\alpha \in \mathcal{J}} \) from the deterministic part of the solution which is included in the \( \{ u_\alpha (t) \}_{\alpha \in \mathcal{J}, t \in [0, T]} \) or the \( \{ \bar{u}_\alpha (t) \}_{\alpha \in \mathcal{J}, t \in [0, T]} \).

The stochastic part \( \{ \xi_\alpha \}_{\alpha \in \mathcal{J}} \), or \( \{ \xi_\alpha (t) \}_{\alpha \in \mathcal{J}} \) is determined only by the Wiener process and is thus independent of the stochastic evolution equation at hand!

As we shall see shortly, the part of the expansion that depends on the particular type of evolution equation are the deterministic coefficients \( \{ u_\alpha (t) \}_{\alpha \in \mathcal{J}, t \in [0, T]} \) or \( \{ \bar{u}_\alpha (t) \}_{\alpha \in \mathcal{J}, t \in [0, T]} \).
Representation of solutions using Wiener chaos expansions

In this section assuming the existence of square integrable solutions to BSEEs we investigate their representation in Wiener chaos expansion. We need “regularity” assumptions on the data.

**Assumption**

The following hold:

**A1:** The terminal condition $\Lambda$ admits the Wiener chaos expansion

$$\Lambda = \sum_{\alpha \in \mathcal{J}} \Lambda_\alpha \xi_\alpha$$

with deterministic coefficients $\Lambda_\alpha = \mathbb{E}[\Lambda \xi_\alpha] \in H$.

**A2:** The $\mathcal{F}$—progressively measurable process $f(\cdot) \in \mathbb{L}^2(\mathbb{W}, L^1([0, T]; V'))$ admits the Wiener chaos expansion

$$f(t) = \sum_{\alpha \in \mathcal{J}} f_\alpha(t) \xi_\alpha, \quad \forall \ t \in [0, T],$$

with deterministic coefficients $f_\alpha(\cdot) = \mathbb{E}[f(\cdot) \xi_\alpha] \in L^1([0, T]; V')$.

**A3** The data of the problem and its solution are sufficiently smooth in terms of Malliavin derivatives.
Proposition (Expansion I)

Suppose that

(i) the pair of $\mathcal{F}$–progressively measurable processes $(u(\cdot), Z(\cdot))$ is a weak solution of the BSEE, with $B = 0$, and

(ii) Regularity assumptions hold for the “data” of the BSEE and its solution.

Then each $u(t)$ has a Wiener chaos expansion of the form $u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$ in which the deterministic coefficients $u_\alpha(t) = \mathbb{E}[u(t)\xi_\alpha]$ satisfy in the weak sense the system of deterministic evolution equations

$$
\sum_{\beta \in \mathcal{J}} \Lambda_{\beta} \rho_{\alpha, \beta}(t) - u_\alpha(t) = \sum_{\beta \in \mathcal{J}} \left( \int_{t}^{T} (A\beta(s) + f_\beta(s)) \, ds \right) \rho_{\alpha, \beta}(t), \quad t \in [0, T]
$$

for each $\alpha \in \mathcal{J}$ whereas the process $Z(\cdot)$ is determined via (4) by the expansion, valid for every $k \in \mathbb{N}$ and $t \in [0, T]$:

$$
\zeta_k(t) = \sum_{\beta \in \mathcal{J}} \left( \Lambda_{\beta} - \int_{t}^{T} (A\beta(s) + f_\beta(s)) \, ds \right) \sum_{j \in \mathbb{N}} m_j(t) \sqrt{\beta_j^k} \xi_{\beta - (j, k)}(t).
$$
Proposition (Expansion II)

Suppose that

(i) the BSEE (3) admits a weak solution \((u(\cdot), Z(\cdot))\), and

(ii) Regularity assumptions hold.

Then \(u(\cdot)\) admits the Wiener chaos representation

\[
u(t) = \sum_{\alpha \in J} \bar{u}_\alpha(t) \xi_\alpha(t)\]

where the \(\bar{u}_\alpha(\cdot) \in L^2([0, T]; V)\) are non-random functions that satisfy in the weak sense the infinite system of deterministic backward differential equations

\[
\Lambda_\alpha - \bar{u}_\alpha(t) = \int_t^T (A\bar{u}_\alpha(s) + \bar{f}_\alpha(s)) \, ds, \quad 0 \leq t \leq T
\]

for \(\alpha \in J\). Furthermore, each \(Z(t)\) has the Wiener chaos expansion

\[
z_k(t) := Z(t)e_k = \sum_{\alpha \in J} \sum_{i \in \mathbb{N}} \bar{u}_\alpha(t) \sqrt{\alpha_i^k} \xi_{\alpha-(i,k)}(t) m_i(t), \quad k \in \mathbb{N}.
\]
Expansion II provides an easier representation for the solution than that of Expansion I: the $\bar{u}_\alpha(t)$ have a simpler form than $u_\alpha(t)$, and the former expansion guarantees by construction the adaptivity of the solution, an essential requirement for the definition of BSEEs. However, in general, some care should be taken with expansions of the form

$$u(t) = \sum_\alpha \bar{u}_\alpha(t) \xi_\alpha(t)$$

1. It is not the case that $\bar{u}_\alpha(t) = \mathbb{E}[u(t) \xi_\alpha]$, and there is no uniqueness for the $\bar{u}_\alpha(t)$.

2. Consequently, we have in general $u(t) \neq \sum_{\alpha \in \mathcal{J}} \bar{u}_\alpha(t) \xi_\alpha$ for $0 \leq t < T$.

3. Furthermore, only the stochastic process $\sum_{\alpha \in \mathcal{J}} \bar{u}_\alpha(t) \xi_\alpha(t)$ is adapted, and not $\sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha$ in general.
The two expansions are related.

**Proposition**

A solution \( \{ u_\alpha(\cdot) \}_{\alpha \in \mathcal{J}} \) of the system (8) may be obtained as

\[
 u_\alpha(\cdot) = \sum_{\beta \in \mathcal{J}} \bar{u}_\beta(\cdot) \rho_{\alpha,\beta}(\cdot),
\]

where the \( \{ \bar{u}_\beta(\cdot) \}_{\beta \in \mathcal{J}} \) solve the decoupled deterministic evolution equations (10).
We now give a converse proposition, which shows how to construct a weak solution to the BSEE (3) starting from (10), (11).

**Proposition**

Suppose that

(i) the functions $\bar{u}_\alpha(\cdot)$ solve the infinite system (10) of deterministic backward differential equations; that

(ii) the series

$$u(t) = \sum_{\alpha \in \mathcal{J}} \bar{u}_\alpha(t) \xi_\alpha(t), \quad 0 \leq t \leq T$$

$$\zeta_k(t) = \sum_{\alpha \in \mathcal{J}} \sum_{i \in \mathbb{N}} \bar{u}_\alpha(t) \sqrt{\alpha_i^k} \xi_{\alpha-i}(t) m_i(t), \quad k \in \mathbb{N}$$

converge in $L^2(\mathbb{W}; L^2([0, T]; V))$ and in $L^2(\mathbb{W}; L^2([0, T]; \mathbb{L}^2(K, H)))$ respectively; and that

(iii) Regularity assumptions hold for the “data” of BSEE (3) and for the stochastic process $u(\cdot)$ defined in (12).

Then the pair of stochastic processes $(u(\cdot), Z(\cdot))$ defined via (12), (4) constitute a weak solution of the BSEE (3).
The case where $B \neq 0$

**Theorem**

Suppose that the standing assumptions hold, and that the solution of the BSEE admits a weak solution $(u(\cdot), Z(\cdot))$, with Wiener chaos expansions
$$
\sum_{\alpha \in \mathcal{J}} \tilde{u}_\alpha(t) \xi_\alpha(t).
$$

Then the functions $\{\tilde{u}_\alpha(\cdot)\}_{\alpha \in \mathcal{J}}$ satisfy the system of deterministic backward equations

$$
\Lambda_\alpha - \tilde{u}_\alpha(t) = \int_t^T (A\tilde{u}_\alpha(s) + f_\alpha(s)) \, ds
$$

$$
+ \sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{N}} \int_t^T \sqrt{\alpha_i^k + 1} m_i(s) B_k \tilde{u}_{\alpha^i+k}(s) \, ds, \quad \alpha \in \mathcal{J}
$$

in the weak sense, and

$$
\zeta_k(t) = \sum_{\beta \in \mathcal{J}} \tilde{u}_\beta(t) \sum_{i \in \mathbb{N}} \sqrt{\beta_i^k} \xi_{\beta^i-k}(t) m_i(t), \quad 0 \leq t \leq T.
$$

We now obtain an upper triangular system.
Wiener chaos and backward stochastic evolution equations

Forward SEE
\[ \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \]

Backward SEE
\[ \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \]
Solvability of the propagator and generalized solutions

In this section we

1. Look at the solvability of propagator (the deterministic problems) for $u_\alpha(t)$ and thus provide conditions for solvability of BSEEs using Wiener chaos

2. Investigate the existence of **generalized solutions** for BSEEs using the deterministic equations for coefficients, in **weighted Wiener spaces**.
We shall work in terms of the expansion \( u(\cdot) = \sum_{\alpha \in \mathcal{J}} u_\alpha(\cdot) \xi_\alpha(\cdot) \), where the expansion coefficients satisfy deterministic backward equations of the form

\[
u_\alpha(t) = \Lambda_\alpha - \int_t^T (Au_\alpha(s) + f_\alpha(s) + G_\alpha(s, u)) \, ds, \quad 0 \leq t \leq T. \tag{13}\]

Here \( G_\alpha(t, u), \alpha \in \mathcal{J} \) is a family of linear operators of the form

\[
G_\alpha(t, u) = \sum_{\beta \in \mathcal{J}} c_{\alpha, \beta}(t) u_\beta(t) \tag{14}
\]

for appropriately chosen operators \( c_{\alpha, \beta}(t) : V \to V', \ 0 \leq t \leq T; \)
Solvability of the deterministic problems

Assumption

Let \((V, H, V')\) be a Gel'fand triple and \(V \times V \ni (\phi, \psi) \mapsto \mathcal{A}(\phi, \psi) \in \mathbb{R}\) a bilinear form with the following properties:

(i) there exists a constant \(\gamma > 0\) such that
\[
|\mathcal{A}(\phi, \psi)| \leq \gamma \|\phi\|_V \|\psi\|_V, \quad \forall \ (\phi, \psi) \in V^2.
\]

(ii) there exist constants \(k_0 \in \mathbb{R}, \lambda > 0\) such that
\[
\mathcal{A}(\phi, \phi) + k_0 \|\phi\|^2_H \geq \lambda \|\phi\|^2_V, \quad \forall \ \phi \in V.
\]

A linear operator \(A : V \to V'\) is defined then via the “Riesz representation”
\[
\mathcal{A}(\phi, \psi) = \langle A\phi, \psi \rangle.
\]

The general equation arising from the projection of BSEEs onto the Wiener chaos basis, is
\[
u_\alpha(\tau) - \Lambda_\alpha + \int_0^\tau A\nu_\alpha(t) \, dt = -\int_0^\tau f_\alpha(t) \, ds + \sum_{\beta \in \mathcal{J}} \int_0^\tau c_{\alpha\beta}(t) u_\beta(t) \, dt \tag{15}
\]
for \(\tau \geq 0, \ \alpha \in \mathcal{J}.
\]
Proposition

Assume the following:

(i) there exist a double sequence \( \{ \varepsilon(\alpha, \beta) \} \) \((\alpha, \beta) \in \mathcal{J}^2\) of positive numbers, a weight function \( \{ r_\alpha \} \) \(\alpha \in \mathcal{J}\), and a constant \( K \in (0, \infty) \) such that the coupling coefficients \( c_{\alpha, \beta}(t) \) satisfy the conditions

\[
\sum_{\beta \in \mathcal{J}} \frac{|c_{\alpha, \beta}(t)|}{\varepsilon(\beta, \alpha)} \leq K \quad \text{for all } \alpha \in \mathcal{J}, \quad t \in [0, T],
\]

(ii) the forcing term satisfies \( \sum_{\alpha \in \mathcal{J}} \int_0^T r_\alpha^2 \| f_\alpha(s) \|_V^2 \, ds < \infty \); and

(iii) the terminal condition has Wiener chaos expansion with coefficients \( \Lambda_\alpha \) such that \( \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| \Lambda_\alpha \|_H^2 < \infty \).

Then the solutions of the deterministic hierarchy (15) generate Fourier coefficients \( \{ u_\alpha(\cdot) \} \) \(\alpha \in \mathcal{J} \), such that for some positive constant \( C \) we have

\[
\sup_{0 \leq t \leq T} \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| u_\alpha(t) \|_H^2 \leq C \left( \sum_{\alpha \in \mathcal{J}} \int_0^T r_\alpha^2 \| f_\alpha(s) \|_V^2 \, ds + \sum_{\alpha \in \mathcal{J}} r_\alpha^2 \| \Lambda_\alpha \|_H^2 \right).
\]
If the assumptions of the above Proposition hold, then the "formal" expansion in (weighted) Wiener chaos converges and thus defines a generalized solution for the BSEE.

The choice of $r_\alpha$ is essential for our purposes.

The choice $r_\alpha = 1$, $\alpha \in \mathcal{J}$ corresponds to weak solutions for (15) and, using the Wiener chaos representations, to weak solutions for (3).

It can be shown that for the particular case $r_\alpha = 1$ coincide the propagator is well posed as long as the usual assumptions for square integrable solutions of BSEEs hold.

However, other choices for $r_\alpha$ can prove useful, either for studying the regularity of weak solutions, or for the definition of even weaker types of solutions.
Corollary (Malliavin differentiable solutions)

Assume that

(i) there exists a double sequence \( \{ \varepsilon(\alpha, \beta) \}_{(\alpha, \beta) \in J^2} \) such that conditions (16)-(17) hold with \( r_\alpha = \sqrt{|\alpha|} \),

(ii) the forcing term satisfies \( \sum_{\alpha \in J} \int_0^T |\alpha| \| f_\alpha(t) \|_V \, dt < \infty \), and

(iii) the terminal condition \( \Lambda \) has Wiener chaos expansion with coefficients \( \Lambda_\alpha \) such that \( \sum_{\alpha \in J} |\alpha| \| \Lambda_\alpha \|_H^2 < \infty \).

Then there exists a constant \( C \in (0, \infty) \), such that the solutions of (13) satisfy

\[
\max_{0 \leq t \leq T} \sum_{\alpha \in J} |\alpha| \| u_\alpha(t) \|_H^2 \leq C \left( \sum_{\alpha \in J} \int_0^T |\alpha| \| f_\alpha(s) \|_V \, ds + \sum_{\alpha \in J} |\alpha| \| \Lambda_\alpha \|_H^2 \right) < \infty.
\]

In particular, \( u(t) = \sum_{\alpha \in J} u_\alpha(t) \xi_\alpha(t) \in D_{1,2}(H) \) for all \( t \in [0, T] \).
Weighted Wiener chaos solutions

Definition

The stochastic process \( u(\cdot) = \sum_{\alpha \in \mathcal{J}} u_\alpha(\cdot) \xi_\alpha(\cdot) \) is called a \( QL^2(\mathbb{W}, H) \) weighted Wiener chaos solution of the BSEE (3) with weight \( Q = \{ q_1, q_2, \cdots \} \), if the functions \( u_\alpha(\cdot), \alpha \in \mathcal{J} \) satisfy the hierarchy of deterministic evolution equations (13) and
\[
\sup_{t \in [0, T]} \sum_{\alpha \in \mathcal{J}} (q^\alpha)^2 \| u_\alpha(t) \|^2_H < \infty.
\]

Proposition

Assume that

(i) there exists a real valued sequence \( Q = \{ q_1, q_2, \cdots \} \) such that conditions (16)-(17) hold with \( r_\alpha = q^\alpha := \prod_{i,k} q_{i,k}^\alpha \),
(ii) the forcing term \( f(\cdot) \) satisfies the condition \( f(\cdot) \in L^2([0, T], QL^2(\mathbb{W}; V')) \)
(iii) the final condition \( \Lambda \) satisfies the condition \( \Lambda \in QL^2(\mathbb{W}; H) \).

Then, BSEE (3) admits a \( QL^2(\mathbb{W}, H) \) weighted Wiener chaos solution.
Conclusions

- We have proposed a construction of solutions of BSEEs using the Wiener chaos expansion.

- We reduce the problem to the solution of an infinite hierarchy of deterministic evolution equations which play the rôle of the Fourier coefficients of the solution.

- Through the solvability of this hierarchy:
  1. We provide existence results and regularity results for solutions of BSEEs using the relevant theory for deterministic systems.
  2. We define and provide existence results for a new type of generalized solutions, the weighted Wiener chaos solutions.
THANK YOU FOR YOUR ATTENTION