Multiple-scattering theory and its applications

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Where are we?

Map of Saudi Arabia showing the location of KAUST in Jeddah.
More about us

Since September 2009

~800 Students (only graduate students)

~100 Faculty members

~450 Research scientist and Post-docs

~70 Nationalities, Western culture

3 Divisions, 9+3 Research Centers

~25% Female students
Outline

• Multiple-scattering theory

• Applications
  – In calculations of random media
    • Time-reversal
    • Wave transport
  – In calculations of periodic media
    • Band structures
    • Effective medium
  – In design of metamaterials
Multiple-Scattering Theory (MST)

A general picture

**System:** a collection of scatterers

For any scatterer \( i \), **two** parts contribute to incident wave

\[
\vec{u}_{i}^{in} (\vec{\rho}_i) = \vec{u}_{i}^{in(0)} (\vec{\rho}_i) + \sum_{j \neq i} \vec{u}_{j}^{sca} (\vec{\rho}_j)
\]
Single Scatterer

• Time harmonic scalar wave equation \((\nabla^2 + k^2)\phi = 0\)

• General solutions (2D for example)

\[
\phi(\vec{\rho}) = \sum_m \left[ a_m J_m(k\rho) + b_m H_m^{(1)}(k\rho) \right] e^{im\theta}
\]

• Single scatterer (Mie scattering)

In medium I: \((\rho < r_s)\)

\[
\phi^<(\vec{\rho}) = \sum_m \left[ c_m J_m(k_s\rho) \right] e^{im\theta}
\]

In medium II: \((\rho > r_s)\)

\[
\phi^>(\vec{\rho}) = \sum_m \left[ a_m J_m(k_0\rho) + b_m H_m^{(1)}(k_0\rho) \right] e^{im\theta}
\]

Boundary conditions: Continuities on the interface \((\rho = r_s)\)

\[
\phi^>\bigg|_{r=r_s} = \phi^<\bigg|_{r=r_s}, \quad \frac{\partial \phi^>}{\partial n}\bigg|_{r=r_s} = \frac{\partial \phi^<}{\partial n}\bigg|_{r=r_s}
\]

Mie coefficient:

\[
D_m = \frac{b_m}{a_m} = \frac{k_s J'_m(k_s r_s) J_m(k_0 r_s) - k_0 J_m(k_s r_s) J'_m(k_0 r_s)}{k_0 H'_m(k_0 r_s) J_m(k_s r_s) - k_s H_m(k_0 r_s) J'_m(k_s r_s)}
\]
Mie coefficient: Physical meaning

Solution in medium II (outside the scatterer):

\[ \phi^* (\bar{r}) = \sum_m a_m J_m (k_0 \rho) e^{im\theta} + b_m H_m^{(1)} (k_0 \rho) e^{im\theta} \]

- **Total**
- **Incident**
- **Scattered**

Mie coefficient: bridge connecting scattered wave to incident wave

\[ e^{i k \cdot r} = \sum_m i^m J_m (k_0 \rho) e^{im\theta} \]

**T Matrix**

\[ t_{mm'} = D_m \delta_{mm'} \]

\[ b_m = \sum_{m'} t_{mm'} a_{m'} \]
Graf’s addition theorem: convert the scattered wave from \( j \) to incident wave on \( i \)

\[
H_m^{(1)}(k \rho_j) e^{im\phi''} = \begin{cases} 
\sum_{n=-\infty}^{+\infty} J_{n-m}(k \rho_{ij}) e^{-i(n-m)\phi'} H_n^{(1)}(k \rho_i) e^{in\phi}, & \rho_i > \rho_{ij} \\
\sum_{n=-\infty}^{+\infty} H_n^{(1)}(k \rho_{ij}) e^{-i(n-m)\phi'} J_n(k \rho_i) e^{in\phi}, & \rho_i < \rho_{ij}
\end{cases}
\]

\[
\sum_m b_m H_m^{(1)}(k \rho_j) e^{im\phi''} = \sum_m \sum_{n=-\infty}^{+\infty} b_m g_{mn} J_n(k \rho_i) e^{in\phi}
\]

incident on \( i \)
The incident wave on a scatterer $i$:

$$\tilde{u}_i^{in}(\tilde{\rho}_i) = \tilde{u}_i^{in(0)}(\tilde{\rho}_i) + \sum_{j \neq i} \tilde{u}_j^{sca}(\tilde{\rho}_j)$$

$$\sum_m a_m^i J_m(k_0 \rho_i)e^{im\phi} = \sum_m a_m^{i0} J_m(k_0 \rho_i)e^{im\phi} + \sum_{j \neq i} \sum_n \sum_{m=-\infty}^{+\infty} b_n^j g_{mn}^j J_m(k \rho_i)e^{im\phi}$$

$$a_m^i = a_m^{i0} + \sum_{j \neq i} \sum_n b_n^j g_{mn}^j + b_m = \sum_{m'} t_{mm'} a_{m'}$$

It is a matrix-vector product system $AX = b$. The size of $A$ is related to number of scatterers and material contrast. For large $A$, fast algorithm is needed.

unknowns, can be solved self-consistently.
Pros and Cons

Advantages

• accounts fully for all the multiple scattering effects
• is an exact theory without approximation
• suitable for high contrast between the scatterer and the background

Disadvantages

• dense matrix
• highly symmetric scatterers* (scattering matrix)
Application I: to Random Media

Numerical experiments of time reversal

source emits signals

receiver records the signals, time reverses it, re-emits it

Time-Reversal Cavity (TRC):
array completely surrounds the source

Time-Reversal Mirror (TRM):
a limited angular area, limiting reversal and focusing quality
About time reversal

Passive array

Active array
Resolution

- 2D homogeneous media
- a sinc source with a carrier frequency $\omega_0$ and bandwidth $B \ll \omega_0$
- $\omega_0 c_0 a^2 \ll |\hat{y}|$ & $a \ll |\hat{y}|$
Increase multi-path

Strong-scattering media:
1) periodic medium (photonic crystal) Cylinder scatterers: radius = 0.3, ε = 11.3.
2) random medium, move the positions of the scatterers randomly (no touch)
Working Frequencies

Transmission ($\Gamma X$)

Gap

Band

Working Frequencies
If there is no scattering medium between the source and the receiver:

Results: Homogeneous
Results: with Random Media

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Results: with Periodic Media

Gap

11 sensors

21 sensors

41 sensors

Band

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Cross-Range: Array Size Effect

$y_0 \lambda_0 / a =$

4.8 (11 sensors)  
2.4 (21 sensors)  
1.2 (41 sensors)

The larger the array size, the better the image
Cross-Range: Medium

The best:
random medium (gap & band)

The worst:
periodic (gap), homogeneous (band)
Range: Array Size Effect

The larger the array size, the better the image

\[ \frac{2\pi c_0}{B} \]

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Range: Medium Effect

The worst: homogeneous (gap & band)
The best: periodic (band) random (gap)
Multiple-scattering calculation is an effective method to solve the wave equation with inhomogeneities.

- Numerical experiments of time reversal
  - The larger the array size, the better the image
  - Strong-scattering media enhance the resolution in general (effective aperture)
  - Range resolution is affected by the array size in the “near” field
  - Periodic medium differs from the random medium
Application II: to Periodic Media

G Matrix

- for periodic systems

\[ b^i_m = \sum_{m'} t_{m'm} \left[ a^{i0}_{m'} + \sum_{j \neq i} \sum_n e^{i \mathbf{K} \cdot \mathbf{R}_j} g^j_{nm} b^i_{m'} \right] = \sum_{m'} t_{m'm} \sum_n G_{nm} b^i_{m'} \]

- lattice sum:

- structure factor

- Evaluation of G matrix

\[ G_{nm'} = S(n - m') \]

\[ S(n) = \frac{4i^{n+1}k_0}{\Omega J_{n+1}(k_0a)} \sum_h \frac{J_{n+1}(Q_ha)}{Q_h(k_0^2 - Q_h^2)} e^{i\phi_n} - \left( \frac{H_1^{(1)}(k_0a)}{\pi k_0a} + \frac{2i}{\pi k_0a} \right) \delta_{n,0}, \]

\[ S(-n) = -S^*(n) \]

- The MST formulism for a periodic system

\[ \det |I - TG| = 0 \]
A typical photonic band structure with a square lattice

\[ c = \frac{\omega}{k} \]
Evaluating the effective parameters

System: a square array of solid cylinders embedded in liquid host, the filling ratio of the solid is $f$

- At the low frequency limit $\omega \to 0$ to the leading order in $\omega$

$$
\det \begin{vmatrix}
\frac{\rho_0 + \rho_s}{\rho_0 - \rho_s} + \frac{x^2 f}{1-x^2} & \frac{ixf}{1-x^2} & -\frac{f}{1-x^2} \\
-\frac{ixf}{1-x^2} & \frac{\kappa_s}{\kappa_s - \kappa_0} + \frac{x^2 f}{1-x^2} & \frac{ixf}{1-x^2} \\
-\frac{f}{1-x^2} & -\frac{ixf}{1-x^2} & \frac{\rho_0 + \rho_s}{\rho_0 - \rho_s} + \frac{x^2 f}{1-x^2}
\end{vmatrix} = 0
$$

- Effective parameters

$$
\frac{1}{\kappa_{\text{eff}}} = \frac{1-f}{\kappa_0} + \frac{f}{\kappa_s} \quad \rho_{\text{eff}} = \frac{(\rho_s + \rho_0) + (\rho_s - \rho_0)f}{(\rho_s + \rho_0) - (\rho_s - \rho_0)f} \rho_0
$$

$x = c_{\text{eff}} / c_1$,
If the filling ratio becoming larger and larger

\[ x^2 = \left( \frac{c_{\text{eff}}}{c_1} \right)^2 = \frac{\kappa_2}{p\kappa_1 + (1-p)\kappa_2} \cdot \frac{(\rho_2 + \rho_1) - (\rho_2 - \rho_1)(f + g)}{(\rho_2 + \rho_1) + (\rho_2 - \rho_1)(f - g)} \]

geometric factor,
Results

(a) Fe in Water Square

(b) Al in Air Hexagonal

\[ \rho_{\text{eff}} / \rho_1 \]

\[ c_{\text{eff}} / c_1 \]

\[ \text{filling ratio (f)} \]
Tight-packing: Critical Limit

(a) \( \log(D_1/D_{\text{eff}} - D_1/D_t) \) vs. \( \log(f_c - f) \) for different numbers of particles \( N = 1, 2, 3, 4, 5, 11, 18, 21 \). The red lines represent the COMSOL simulation with a slope of 0.52.

(b) Same as (a) but for different numbers of particles.
Conclusion II

- MST in the low frequency limit is able to give the analytic formula of the effective medium parameters.

- For solid-liquid systems with high filling ratios of solid inclusions:
  - effective mass is geometry dependent except for the tight-packing limit.
  - effective bulk modulus is given by the Wood’s formula.
Electromagnetic metamaterials are artificially structured materials that are designed to interact with and control electromagnetic waves... its electromagnetic properties are often beyond those of any known naturally occurring materials.

http://people.ee.duke.edu/~drsmith/metamaterials.html

e.g.

R. A. Shelby et al., Science 292, 77 -79 (2001)

Metamaterials: What’s the Secret?

• Artificially designed subwavelength materials
  – Described by a few effective parameters, e.g. $\varepsilon, \mu$
  – Unit length $a << \lambda$

• Homogenization on microstructure resonances

Microstructures

Structure resonance homogenization

= $\varepsilon, \mu$

Effective medium

Material parameters that can describe the macroscopic properties
\[ n = \sqrt{\varepsilon} \sqrt{\mu} \]

- Both \( \varepsilon \) and \( \mu \) are negative \( \rightarrow n < 0 \): negative bands
- Only one negative in \( \varepsilon \) and \( \mu \) \( \rightarrow n \) imaginary: gap

How can those unusual values of parameters become true?

- Resonances in the microstructure
- Develop effective medium theory that can describe resonances
- Design metamaterials by introducing proper resonances

\( \varepsilon < 0, \mu > 0 \) (metal)
\( \varepsilon > 0, \mu > 0 \) (dielectric)
\( \varepsilon < 0, \mu < 0 \) (LHM)
\( \varepsilon > 0, \mu < 0 \)
Elastic waves:

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \nabla \cdot (\mu \nabla u_i) + \nabla \cdot \left( \mu \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_i} \left( \lambda \nabla \cdot \vec{u} \right) \]

- **mass density**
- **shear modulus** \( \mu = 0 \) for liquids
- **lamé constant**
- **bulk modulus** \( \kappa = \lambda + \frac{2}{d} \mu \)

**Two types of waves & three material parameters.**

<table>
<thead>
<tr>
<th>Longitudinal: ( n_l = \sqrt{\rho/(\lambda + 2\mu)} )</th>
<th>Transverse: ( n_t = \sqrt{\rho/\mu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_e + \mu_e &gt; 0 )</td>
<td>( \kappa_e + \mu_e &lt; 0 )</td>
</tr>
<tr>
<td>( \mu_e &gt; 0 )</td>
<td>( \mu_e &lt; 0 )</td>
</tr>
</tbody>
</table>

**Values:**

<table>
<thead>
<tr>
<th>( \rho_e )</th>
<th>( n_{le} ), ( n_{te} )</th>
<th>( l: \text{gap} ), ( n_{te} &gt; 0 )</th>
<th>( n_{le} &gt; 0 ), ( t: \text{gap} )</th>
<th>( l, t: \text{gap} )</th>
</tr>
</thead>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho_e &lt; 0 )</td>
<td>( l, t: \text{gap} )</td>
<td>( n_{le} &lt; 0 ), ( t: \text{gap} )</td>
<td>( l: \text{gap} ), ( n_{te} &lt; 0 )</td>
<td>( n_{le} &lt; 0 ), ( n_{te} &lt; 0 )</td>
</tr>
</tbody>
</table>
(a) $\rho < 0$, $\kappa > 0$ and $\mu > 0$ (gap)


(b) $\rho < 0$, $\kappa < 0$, $\mu > 0$ ($n_1 < 0$)

**monopolar resonance**


**Helmholtz Resonator**

Motivation

How about $\mu < 0$?

<table>
<thead>
<tr>
<th>$\kappa_e + \mu_e &gt; 0$</th>
<th>$\kappa_e + \mu_e &lt; 0$</th>
<th>$\kappa_e + \mu_e &gt; 0$</th>
<th>$\kappa_e + \mu_e &lt; 0$</th>
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</tr>
</tbody>
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<th>$\rho_e &gt; 0$</th>
<th>$n_{le} &gt; 0, n_{te} &gt; 0$</th>
<th>$l : gap, n_{te} &gt; 0$</th>
<th>$n_{le} &gt; 0, t : gap$</th>
<th>$n_{le} &lt; 0, n_{te} &lt; 0$</th>
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<td>$\rho_e &lt; 0$</td>
<td>$l, t : gap$</td>
<td>$n_{le} &lt; 0, t : gap$</td>
<td>$l : gap, n_{te} &lt; 0$</td>
<td>$n_{le} &lt; 0, n_{te} &lt; 0$</td>
</tr>
</tbody>
</table>

We will design elastic metamaterials with negative bands ($\mu < 0$, also with $\rho < 0$, $\kappa < 0$)

- both $n_l$ and $n_t < 0$
- "Incompressible-solid-like": only support transverse waves
- "Fluid-like": only support longitudinal waves
Effective medium theory for elastic metamaterials:

\[
\frac{(\kappa_0 - \kappa_e)}{(\mu_0 + \kappa_e)} = \frac{4\tilde{D}_0^{ll}(s)}{i\pi r_0^2 k_{l0}^2}
\]

\[
\frac{(\rho_0 - \rho_e)}{\rho_0} = -\frac{8\tilde{D}_1^{ll}(s)}{i\pi r_0^2 k_{l0}^2}
\]

\[
\frac{\mu_0(\mu_0 - \mu_e)}{(\kappa_0\mu_0 + (\kappa_0 + 2\mu_0)\mu_e)} = \frac{4\tilde{D}_2^{ll}(s)}{i\pi r_0^2 k_{l0}^2}
\]


Design I

$\mu < 0, \rho < 0$

Microstructure

![Diagram of microstructure with water, rubber, and foam layers, showing radii $r_s = 0.32a$ and $r_w = 0.24a$.]

Effective medium parameters

![Graph showing effective medium parameters $\rho$, $\kappa$, $\mu$, and $\kappa + \mu$ as functions of $fa/c_{t0}$, with points $a = 0.167$ and $b = 0.216$.]

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Band structure and transmission

\[ \rho < 0, \quad \kappa + \mu < 0 \]

\[ \rho < 0, \quad \mu < 0 \]
negative refraction

Snell’s law: \[ k_t \sin i = k_{i0} \sin \theta_{i0} = k_{t0} \sin \theta_{t0} \]

Plug in \( k_t \)

\[ \theta_{t0} = 58^\circ \quad \theta_{i0} = 99^\circ \]

Negative refraction induced **mode conversion** on the interface.
(analogue to Brewster angle but in a more complex manner.)

From single mass LRM

rubber → 5mm lead

epoxy →

Hard core enhances the dipolar resonances which can induce negative mass density

Can we use many hard objects to enhance quadrupolar resonances?

Multi-mass locally resonant materials
System design

can enhance both **monopolar** and **quadrupolar** resonances
Band structure and states

gap with 2 negative bands

monopolar resonance

quadrupolar resonance

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Transmission (\( \Gamma_{IM} \))

**Transverse input** (--- u - - - v)

![Graph 1](image)

**Longitudinal input** (--- u - - - v)

![Graph 2](image)

Transmission (\( \Gamma_{IX} \))

**Transverse input** (--- u - - - v)

![Graph 3](image)

**Longitudinal input** (--- u - - - v)

![Graph 4](image)

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“fluid-like” solid
“Super-anisotropic” solid: “incompressible-solid-like” in ΓM direction and “fluid-like” in ΓX direction
Effective Medium Properties

MST offers a guideline in the engineering of metamaterials and serves as a tool in the calculation of the wave propagation.

Two examples for negative shear modulus:
- Fluid-like: only allows shear wave
- Super-anisotropic: different wave along different directions
- Negative shear band
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The End

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In the long-wavelength limit, \(|m| \leq 2, \quad m', m = -2, -1, 0, 1, 2\)

\[
\det \left| \sum_{m'} t_{\alpha \beta mn} S(\beta, m'-m) - \delta_{mm'} \right| = 0
\]

\[
S(\beta, n) = \frac{4i^n k_{\beta 0}}{\Omega J_{n+1}(k_{\beta 0}a)} \sum_h \frac{J_{n+1}(Q_h a)}{Q_h (k_{\beta 0}^2 - Q_h^2)} e^{in\phi_h} - \left( H_1^{(1)}(k_{\beta 0}a) + \frac{2i}{\pi k_{\beta 0}a} \right) \delta_{n,0}, \quad (0 \leq n \leq 4)
\]

\[
S(\beta, -n) = -S^*(\beta, n)
\]

\[
(Q_h, \phi_h) \leftrightarrow \tilde{Q}_h = \tilde{K} + \tilde{K}_h
\]

\[
\sum_h \frac{J_{n+1}(Q_h a)}{Q_h (k_{\beta 0}^2 - Q_h^2)} e^{in\phi_h} = \sum_{h(\tilde{K}_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k_{\beta 0}^2 - K_h^2)} e^{in\phi_{K_h}} + \frac{J_{n+1}(Ka)}{K(k_{\beta 0}^2 - K^2)} e^{in\phi_K}
\]
**Triangular Lattice: Isotropic**

\[
\sum_{h} \frac{J_{n+1}(Q_h a)}{Q_h (k^2_{\beta 0} - Q_h^2)} e^{i n \phi_h} = \sum_{h(K_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k^2_{\beta 0} - K_h^2)} e^{i n \phi_{K_h}} + \frac{J_{n+1}(Ka)}{K(k^2_{\beta 0} - K^2)} e^{i n \phi_K}
\]

\[S(\beta, 0) = \frac{8i}{\sqrt{3}a^2 (k^2_{\beta 0} - K^2)}.
\]

\[
\text{det} \sum_{m} t_{\alpha \beta mm'} \left( \text{S} \left( \frac{\beta}{2}, \frac{m}{2} \right) - \delta_{mm'} \right) = 0 \quad \iff \quad \text{roots } K \sim \omega
\]

\[n = 0 \quad e^{i n \phi_{K_h}} = 1
\]

\[\sum_{h(K_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k^2_{\beta 0} - K_h^2)} e^{i n \phi_{K_h}} \sim \omega^{0}
\]

\[\frac{J_{n+1}(Ka)}{K(k^2_{\beta 0} - K^2)} e^{i n \phi_K} \sim \omega^{1}
\]

\[(K_{h1}, \phi_{kh1})
\]

\[S(\beta, n) = \frac{8i^{n+1}K^n}{\sqrt{3}a^2 k_{\beta 0} (k^2_{\beta 0} - K^2)} e^{i n \phi_K}, \quad (0 < n \leq 4)
\]

\[\forall (K_{h1}, \phi_{kh1}), \quad \exists (K_{h1}, \phi_{kh1} + N\pi / 3), \quad N = 1, 2, 3, 4, 5
\]

\[\Rightarrow \sum_{N=0}^{5} e^{i n N\pi/3} = 0
\]

\[\sum_{h(K_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k^2_{\beta 0} - K_h^2)} e^{i n \phi_{K_h}} = 0
\]

\[n \neq 0
\]
roots: 

\[ K_{1}^{tri} = \sqrt{F_1(\tilde{D}_1^{ll}) F_2(\tilde{D}_2^{ll})}, \quad K_{2}^{tri} = \sqrt{F_1(\tilde{D}_1^{ll}) F_3(\tilde{D}_2^{ll}, \tilde{D}_0^{ll})} \]

\[
F_1(\tilde{D}_1^{ll}) = - \frac{8i\tilde{D}_1^{ll}(\kappa_0 + \mu_0) - \sqrt{3}}{2} a^2 \omega^2 \rho_0 \\
F_2(\tilde{D}_2^{ll}) = - \frac{4\tilde{D}_2^{ll}(\kappa_0 + \mu_0)(\kappa_0 + 2\mu_0) + i \frac{\sqrt{3}}{2} a^2 \omega^2 \mu_0 \rho_0}{\mu_0 \left( 4\tilde{D}_2^{ll} \kappa_0 (\kappa_0 + \mu_0) - i \frac{\sqrt{3}}{2} a^2 \omega^2 \mu_0 \rho_0 \right)} \\
F_3(\tilde{D}_2^{ll}, \tilde{D}_0^{ll}) = - \frac{\left( 4\tilde{D}_2^{ll} (\kappa_0 + \mu_0) + i \frac{\sqrt{3}}{2} a^2 \omega^2 \rho_0 \right) \left( 8\tilde{D}_2^{ll} (\kappa_0 + \mu_0)(\kappa_0 + 2\mu_0) + i \frac{\sqrt{3}}{2} a^2 \omega^2 \mu_0 \rho_0 \right)}{(\kappa_0 + \mu_0) \left( 32\tilde{D}_0^{ll} \tilde{D}_2^{ll} \mu_0 (\kappa_0 + \mu_0)^2 - i \frac{\sqrt{3}}{2} a^2 \omega^2 \rho_0 \left( 4\tilde{D}_2^{ll} \kappa_0 (\kappa_0 + \mu_0) - i \frac{\sqrt{3}}{2} a^2 \omega^2 \mu_0 \rho_0 \right) \right)}
\]

\[ K_{1,2}^{tri} : \phi_K \text{ independent} \rightarrow \text{isotropic dispersion relations} \]

\[ K_{1}^{tri} = K_l = \omega \sqrt{\rho_e / \mu_e} \quad K_{2}^{tri} = K_l = \omega \sqrt{\rho_e / (\mu_e + \kappa_e)} \]

Square Lattice: Anisotropic

\[
\sum_{h} \frac{J_{n+1}(Q_h a)}{Q_h (k_{\beta 0}^2 - Q_h^2)} e^{in\phi_h} = \sum_{h(k_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k_{\beta 0}^2 - K_h^2)} e^{in\phi_{kh}} + \frac{J_{n+1}(Ka)}{K(k_{\beta 0}^2 - K^2)} e^{in\phi_K}
\]

\[n = 0\] similar to the triangular case

\[S(\beta, 0) = \frac{4i}{a^2(k_{\beta 0}^2 - K^2)}.
\]

\[n \neq 0\]

\[\forall (K_{h1}, \phi_{kh1}), \exists (K_{h1}, \phi_{kh1} + N\pi / 2), N = 1, 2, 3\]

\[1 \leq n \leq 3\]

\[\sum_{N=0}^{3} e^{inN\pi/2} = 0\]

\[\sum_{h(k_h \neq 0)} \frac{J_{n+1}(K_h a)}{K_h (k_{\beta 0}^2 - K_h^2)} e^{in\phi_{kh}} = 0\]

\[S(\beta, n) = \frac{4i^{n+1}K^n}{a^2k_{\beta 0}^n(k_{\beta 0}^2 - K^2)} e^{in\phi_K}\]

\[n = 4\]

\[\sum_{N=0}^{3} e^{inN\pi/2} = 4\]

\[S(\beta, 4) = \left(\gamma_\beta + \frac{4iK^4}{a^2k_{\beta 0}^4(k_{\beta 0}^2 - K^2)}\right) e^{i4\phi_K}\]

\[16 \cdot 2^5 \cdot 5!i \sum_{h_j=0}^{N} \sum_{h_i=1}^{N} J_5 \left(2\pi \sqrt{h_i^2 + h_j^2}\right) e^{i4\arctan(h_j/h_i)}\]

\[K_{1,2}^{\text{squ}} : \phi_K \text{ dependent} \rightarrow \text{anisotropic dispersion relations}\]
Effective Medium Theory

Two commonly encountered microstructures:

1. Dispersed-inclusion
2. Symmetric

System:

\[ \frac{\pi r_s^2}{\pi r_0^2} = p \]

Structural unit:

\[ \kappa_s \mu_s \rho_s \]

\[ \kappa_0 \mu_0 \rho_0 \]

P. Sheng, in *Homogenization and Effective Moduli of Materials and Media*, (Springer-Verlag, New York, 1986);
Structural unit:

Solution in the effective medium (III):

\[
\phi^{III}_{\alpha=1,t} = \sum_{m} \left( a_{\alpha em} J_{m} (k_{\alpha e} r) + b_{\alpha em} H^{(1)}_{m} (k_{\alpha e} r) \right) e^{im\theta}
\]

B.C. on the interface between II and III

\[
b_{10m} = D^{ll}_{m} (e)a_{10m} + D^{lt}_{m} (e)a_{10m}
\]

B.C. on the interface between I and II

\[
b_{10m} = D^{ll}_{m} (s)a_{10m} + D^{lt}_{m} (s)a_{10m}
\]

Effective medium condition

\[
D_{m}^{\alpha\beta} (e) = D_{m}^{\alpha\beta} (s)
\]
Effective medium condition: \[ D_{m}^{\alpha\beta}(e) = D_{m}^{\alpha\beta}(s) \]

The long-wavelength limit: \[ k_{l0} r_0, \ k_{i0} r_0, \ k_{le} r_0, \ k_{le} r_0 << 1 \]

\[
\frac{\kappa_0 - \kappa_e}{\kappa_0 + \kappa_e} = \frac{4\tilde{D}_0^{ll}(s)}{i\pi r_0^2 k_{l0}^2} \\
\frac{\rho_0 - \rho_e}{\rho_0} = \frac{8\tilde{D}_1^{ll}(s)}{i\pi r_0^2 k_{l0}^2} \\
\frac{\mu_0(\mu_0 - \mu_e)}{\kappa_0 \mu_0 + (\kappa_0 + 2\mu_0) \mu_e} = \frac{4\tilde{D}_2^{ll}(s)}{i\pi r_0^2 k_{l0}^2}
\]

The quasi-static limit: \[ k_{l0} r_0, \ k_{i0} r_0, \ k_{le} r_0, \ k_{le} r_0, \ k_{ls} r_s, \ k_{ts} r_s << 1 \]

\[
\frac{\kappa_0 - \kappa_e}{\kappa_0 + \kappa_e} = p \frac{\kappa_0 - \kappa_s}{\kappa_0 + \kappa_s} \\
\frac{\rho_0 - \rho_e}{\rho_0} = p(\rho_0 - \rho_s) \\
\frac{\mu_0(\mu_0 - \mu_e)}{\kappa_0 \mu_0 + (\kappa_0 + 2\mu_0) \mu_e} = p \frac{\mu_0 - \mu_s}{\kappa_0 \mu_0 + (\kappa_0 + 2\mu_0) \mu_s}
\]

\( k_{ls} r_s \) and \( k_{ts} r_s \) can be very large can be applied to metamaterials
