Scattering by a semi-infinite lattice of cylinders

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Introduction

- **Scattering by periodic arrangements of cylinders.**

Elasticity	Composite materials.
Water waves	Large offshore structures
Electromagnetism	Antennas, Photonic crystals,. . .

- **The Helmholtz equation** (Two-dimensional time-harmonic scattering.)

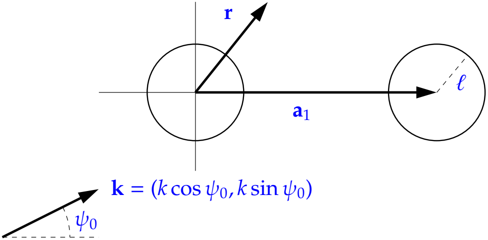
$$\nabla^2 u + k^2 u = 0$$

- **Wavefunctions** (separable solutions of the Helmholtz equation)

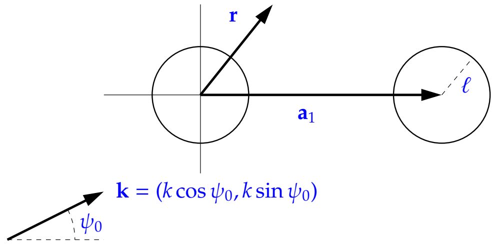
$$\mathcal{H}_q(\mathbf{r}) = H_q(kr)e^{iq\theta} \qquad \mathcal{J}_q(\mathbf{r}) = J_q(kr)e^{iq\theta}$$

- **The multipole expansion method** (linear combination of multipoles).
  Representation of the scattered field by one cylinder

$$u_{sc}(\mathbf{r}) = \sum_{q=-\infty}^{\infty} A^q \mathcal{H}_q(\mathbf{r})$$



The total field: $u(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{q=-\infty}^{\infty} A_0^q \mathcal{H}_q(\mathbf{r}), + \sum_{q=-\infty}^{\infty} A_1^q \mathcal{H}_q(\mathbf{r} - \mathbf{a}_1),$



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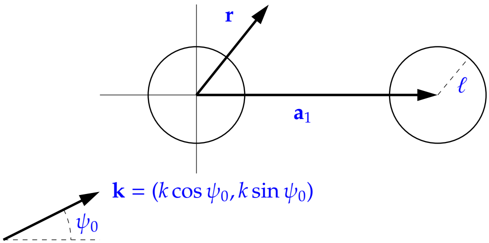
$$A_n^p + Z_p \sum_{q=-\infty}^{\infty} A_n^q H_{q-p}(ka_1) = -iZ_p e^{in\mathbf{k}\cdot\mathbf{a}_1} e^{-ip\psi_0}, \quad n = 0, 1, \quad p \in \mathbb{Z}.$$

with

$$Z_p = \begin{cases} J(k\ell))/H(k\ell) & \text{Dirichlet boundary condition} \\ J'(k\ell))/H'(k\ell) & \text{Neumann boundary condition.} \end{cases}$$

Scattering by two cylinders



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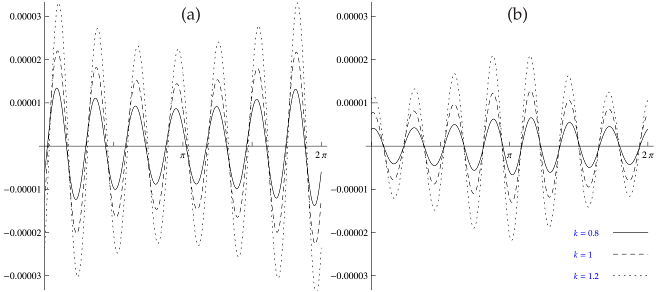
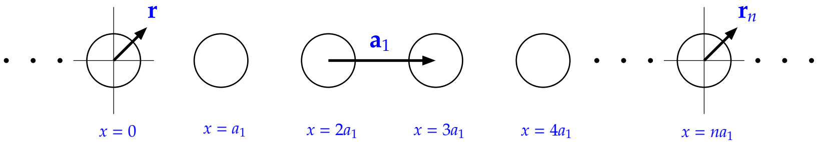
The truncation parameter Q

- *Row*(1953) Verified that for $Q = 6$ the calculations are in excellent agreement with experiments!
- *Linton & Evans* (1990) found that taking $Q = 6$ gave results with acccuracy of four significant figures.



- *Kleinmann & Vainberg*(1994)
  For $k\ell << 1$, take $Q = 0$ for DBC and $Q = 1$ for NBC with an error of order $(k\ell)^2$

Scattering by an infinite grating of cylinders



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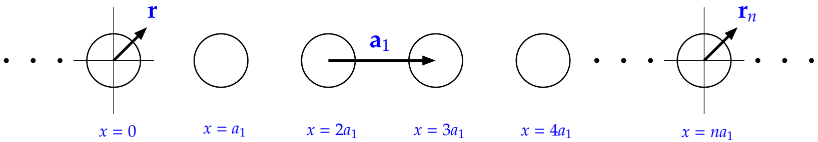
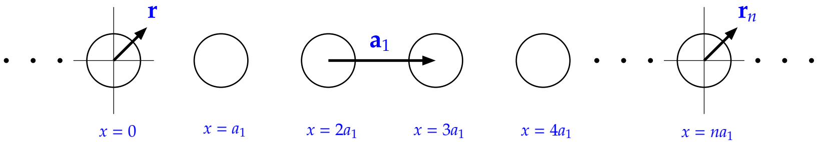


- The total field: $u(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{j=-\infty}^{\infty} u_j(\mathbf{r})$, with $u_j(\mathbf{r}) = \sum_{q=-Q}^{Q} A_j^q \mathcal{H}_q(\mathbf{r} - j\mathbf{a}_1)$,
- The quasi-periodicity condition: $A_n^q = e^{in\mathbf{k}\cdot\mathbf{a}_1} A_0^q, \quad \forall n \in \mathbb{Z}$.

Scattering by an infinite grating of cylinders



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$$A_0^p + Z_p \sum_{q=-Q}^{Q} A_0^q \sigma_{q-p}^{(1)}(k\cos\psi_0) = -i^p Z_p e^{-ip\psi_0}$$

Lattice Sums

$$\sigma_q^{(1)}(\beta) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} e^{ija_1\beta} \mathcal{H}_q(-j\mathbf{a}_1).$$

The Wiener–Hopf technique

- In 1931, Norbert Wiener (1894-1964) and Eberhard Hopf (1902-1983), presented a method to solve integral equations of the form

$$\int_0^\infty k(x-y)f(y)\,dy = g(x), \quad 0 < x < \infty.$$

- The discrete WHT is used to solve an infinite system of algebraic equations

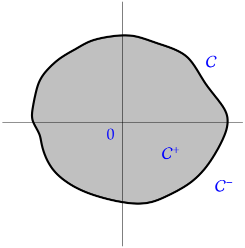
$$\sum_{m=0}^{\infty} K_{n-m}A_m = T_n, \quad n = 0, 1, 2 \ldots, .$$

$[K_{n-m}]_{n,m=0,1,\ldots}$ Toeplitz matrix.

The Wiener–Hopf equation

$$K(z)A^+(z) = T^+(z) + T^-(z), \qquad z \in \mathcal{C}$$

- '$+$' indicates analyticity in $\mathcal{C}^+$.
- '$-$' indicates analyticity in $\mathcal{C}^-$.



Three Fundamental Steps

The factorisation: $K(z) = K^+(z)K^-(z)$, $K^\pm(z)$ are analytic **and zero free** in $\mathcal{C}^\pm$.

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The sum-split: $\dfrac{T^+(z)}{K^-(z)} = D^+(z) + D^-(z)$, $D^\pm(z)$ are analytic in $\mathcal{C}^\pm$.

$$K^+(z)A^+(z) - D^+(z) = D^-(z) + \frac{T^-(z)}{K^-(z)} \tag{2}$$

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$J(z)$ is an entire function (analytic continuation of both sides)

Liouville's theorem: If the RHS$\to 0$ as $z \to \infty$, then $J(z) \equiv 0$

$$A^+(z) = \frac{D^+(z)}{K^+(z)} \quad \text{and} \quad A_n = \frac{1}{2\pi i}\oint_{z\in C} \frac{D^+(z)}{K^+(z)} z^{-n-1}\,dz, \qquad n = 0, 1, 2,$$

Wiener–Hopf decompositions

- Rational functions: trivial decomposition.
- Meromorphic functions:

Factorisation: Weierstrass product theorem
Sum decomposition: Mittag-Leffler theorem

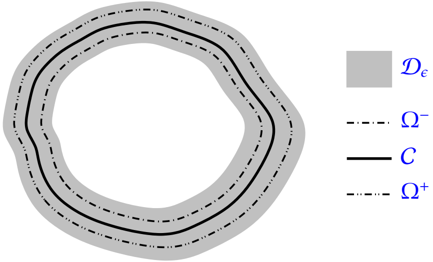
Cauchy's Integral Formula

$$f^{\pm}(z) = \pm\frac{1}{2\pi i}\oint_{\Omega^{\pm}} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \mathbb{C} - \Omega^{\pm}.$$

$$f(z) = f^{+}(z) + f^{-}(z)$$

$$f^{\pm}(z) = \exp\left(\pm\frac{1}{2\pi i}\oint_{\Omega^{\pm}} \frac{\log f(\zeta)}{\zeta - z}\, d\zeta\right),$$

$$f(z) = f^{+}(z)f^{-}(z)$$



- Approximate factorisations: Padé Approximants.

Remark: WH decomposition are constructed in an ad-hoc manner!

Matrix Wiener–Hopf equation

Matrix Wiener–Hopf equation

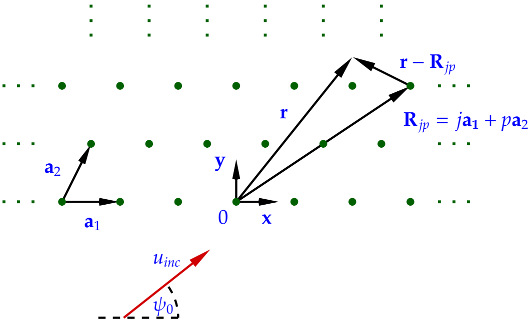
$$\mathbf{K}(z)\mathbf{A}^{+}(z) = \mathbf{T}^{+}(z) + \mathbf{T}^{-}(z), \qquad z \in \mathcal{C}$$

- $\mathbf{K}(z)$ is an $n \times n$ matrix.
- $\mathbf{A}^{+}(z)$, $\mathbf{T}^{-}(z)$, and $\mathbf{T}^{+}(z)$ are $n \times 1$ matrices.

Matrix Wiener-Hopf factorisation (MWHF)

$$\mathbf{K}(z) = \mathbf{K}^{+}(z)\mathbf{K}^{-}(z), \qquad \det\left(\mathbf{K}^{\pm}(z)\right) \neq 0 \text{ in } \mathcal{C}^{\pm}.$$

- Gohberg & Krein proved existence of MWHF.
- MWHF are extremely hard! (even for a rational matrix).
- There are methods to construct a MWHF. However, no practical applications!



$$u_{jp}(\mathbf{r}) = \sum_{q=-Q}^{Q} A_{jp}^{q} \mathcal{H}_q(\mathbf{r} - \mathbf{R}_{jp}),$$

$$u_{sc}(\mathbf{r}) = \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} u_{jp}(\mathbf{r}),$$

Quasi-periodicity property

$$A_{jp}^{q} = e^{ij\mathbf{k}\cdot\mathbf{a}_1} A_{0p}^{q}$$

The infinite system of algebraic equations

$$iZ_\mu^{-1}A_{0n}^\mu + i\sum_{q=-Q}^{Q}\sum_{p=0}^{\infty} A_{0p}^q \mathcal{S}_{n-p}^{q-\mu} = -ie^{in\mathbf{k}\cdot\mathbf{a}_2}e^{i\mu(\pi/2-\psi_0)}, \quad \mu=-Q,\ldots Q, \quad n=0,1,\ldots,$$

where

$$\mathcal{S}_0^q = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} e^{ij\mathbf{k}\cdot\mathbf{a}_1}\mathcal{H}_q(-j\mathbf{a}_1),$$

$$\mathcal{S}_n^q = \sum_{j=-\infty}^{\infty} e^{ij\mathbf{k}\cdot\mathbf{a}_1}\mathcal{H}_q(n\mathbf{a}_2 - j\mathbf{a}_1), \quad n\in\mathbb{Z}-\{0\}.$$

The scattering angles

$$\cos\psi_j = \cos\psi_0 + \frac{2\pi j}{ka_1}, \quad \text{and} \quad \sin\psi_j = i\gamma(\cos\psi_j)$$

where, $\gamma(t) = \begin{cases} \sqrt{t^2-1}, & |t|>1, \\ -i\sqrt{1-t^2}, & |t|\le 1. \end{cases}$

$$\mathcal{M} = \left\{m\in\mathbb{Z} \; : \; \left|\cos\psi_0 + 2\pi m/ka_1\right| < 1\right\},$$

$$\mathcal{N} = \left\{n\in\mathbb{Z} \; : \; \left|\cos\psi_0 + 2\pi n/ka_1\right| > 1\right\},$$

$$\mathbf{K}(z)\mathbf{A}^{+}(z) = \mathbf{T}^{+}(z) + \mathbf{T}^{-}(z), \qquad z \in \mathcal{C}$$

The kernel is the $(2Q+1)\times(2Q+1)$ matrix

$$\mathbf{K}(z) = \left[K_{\mu q}(z)\right]_{\mu=-Q,\dots Q;q=-Q\dots Q},$$

and the other terms are the column vectors

$$\mathbf{A}^{+}(z) = \left[A_q^{+}(z)\right]_{q=-Q,\dots,Q},$$

$$\mathbf{T}^{+}(z) = \left[T_q^{+}(z)\right]_{q=-Q,\dots,Q},$$

$$\text{and} \quad \mathbf{T}^{-}(z) = \left[T_q^{-}(z)\right]_{q=-Q,\dots,Q}.$$

$$K_{\mu q}(z) = iZ_q^{-1}\delta_{\mu q} + i\sum_{j=-\infty}^{\infty} \mathcal{S}_j^{q-\mu} z^j$$

$$T_\mu^{+}(z) = e^{i\mu(\pi/2-\psi_0)}\frac{ip_0}{z-p_0}.$$

$$A_{0n}^{q} = \frac{1}{2\pi i}\oint_{\mathcal{C}} A_q^{+}(z)z^{-n-1}dz,$$

$q = -Q,\dots,Q$, and $n \in \mathbb{Z}$.

The elements of the kernel

$$K_{\mu q}(z) = i(Z_q^{-1}\delta_{\mu q} + \mathcal{S}_0^{q-\mu}) + i\sum_{j=1}^{\infty} \mathcal{S}_j^{q-\mu} z^j + i\sum_{j=1}^{\infty} \mathcal{S}_{-j}^{q-\mu} z^{-j}, \qquad (3)$$

$$K_{\mu q}(z) = i\left(Z_q^{-1}\delta_{\mu q} + \mathcal{S}_0^{q-\mu}\right) - \frac{2i^{\mu-q+1}}{ka_1}\sum_{j=-\infty}^{\infty} \frac{1}{\sin\psi_j}\left(\frac{e^{i(q-\mu)\psi_j} z}{z-p_j} - \frac{e^{i(\mu-q)\psi_j}}{z\tau_j - 1}\right). \qquad (4)$$

- $|p_j = |\tau_j| = 1$ for $j \in \mathcal{M}$, and $p_j^* = \tau_j$ for $j \in \mathcal{N}$
- Also since $i\mathcal{S}_n^{q-\mu} = \frac{1}{2\pi i}\oint_C K_{\mu q}(z) z^{-n-1} dz, \quad \forall n \in \mathbb{Z}.$

The location of the poles with respect to $\mathcal{C}$

$$p_j \in \mathcal{C}^- - \mathcal{C}, \quad \text{and} \quad \tau_j^{-1} \in \mathcal{C}^+ - \mathcal{C}, \quad \forall j \in \mathbb{Z},$$

Symmetry

$$\left(K_{\mu q}(1/z^*)\right)^* = K_{q\mu}(z) \quad \text{or} \quad (\mathbf{K}(1/z^*))^* = \mathbf{K}(z)$$

$$K_{\mu q}(z) = i(Z_q^{-1}\delta_{\mu q} + \mathcal{S}_0^{q-\mu}) + iG^{+}_{q-\mu}(\mathbf{0},\beta(z)) + iG^{-}_{q-\mu}(\mathbf{0},\beta(z))$$

where

$$G_q^{\pm}(\mathbf{r},\beta) = \sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty} e^{i\mathbf{R}_{j,\mp m}\cdot\beta}\mathcal{H}_q(\mathbf{r}-\mathbf{R}_{j,\mp m}), \quad q \in \mathbb{Z},$$

and

$$\beta(z) = (k\cos\psi_0, \lambda(z)) \quad \text{with} \quad \lambda(z) = \frac{i\log z - \eta_1 k\cos\psi_0}{\eta_2}.$$

Approximation

$$\mathcal{K}_{\mu q}(z) = i\left(Z_q^{-1}\delta_{\mu q} + \mathcal{S}_0^{q-\mu}\right) - \frac{2i^{\mu-q+1}}{ka_1}\sum_{j=-\nu}^{\nu}\frac{1}{\sin\psi_j}\left(\frac{e^{i(q-\mu)\psi_j}z}{z-p_j} - \frac{e^{i(\mu-q)\psi_j}}{z\tau_j - 1}\right).$$

The approximation parameter $\nu \geq \max\left\{|x| : \; x \in \mathcal{M}\right\}$

The Wiener–Hopf technique modified

- We sum-split the kernel into plus and minus matrices:

$$\mathbf{K}(z) = \mathbf{K}^+(z) + \mathbf{K}^-(z), \qquad \mathbf{K}^\pm(z) = \left[\mathcal{K}^\pm_{\mu q}(z)\right]_{\mu=-Q,\dots,Q;\ q=-Q,\dots,Q},$$

  where

$$\mathcal{K}^+_{\mu q}(z) = i\left(Z_q\delta_{\mu q} + \mathcal{S}_0^{q-\mu}\right) - \frac{2i^{\mu-q+1}}{ka_1}\sum_{j=-\nu}^{\nu}\frac{e^{i(q-\mu)\psi_j}z}{\sin\psi_j(z-p_j)},$$

$$\mathcal{K}^-_{\mu q}(z) = \frac{2i^{\mu-q+1}}{ka_1}\sum_{j=-\nu}^{\nu}\frac{e^{i(\mu-q)\psi_j}}{\sin\psi_j(z\tau_j-1)}.$$

- WH equation: $\mathbf{K}^+(z)\mathbf{A}^+(z) + \mathbf{K}^-(z)\mathbf{A}^+(z) - \mathbf{T}^+(z) = \mathbf{T}^-(z)$.
- move the poles to the RHS with the aid of

$$\mathbf{M}^-(z) = \left[\frac{2i}{ka_1}\sum_{j=-\nu}^{\nu}\frac{e^{i\mu(\psi_j+\pi/2)}}{\sin\psi_j(z\tau_j-1)}\sum_{q=-Q}^{Q}e^{-iq(\psi_j+\pi/2)}A_q^+(\tau_j^{-1})\right]_{\mu=-Q,\dots,Q}$$

- WH equation: $\mathbf{K}^+(z)\mathbf{A}^+(z) + \mathbf{K}^-(z)\mathbf{A}^+(z) - \mathbf{M}^-(z) - \mathbf{T}^+(z) = \mathbf{T}^-(z) - \mathbf{M}^-(z)$.
- Liouville's theorem dictates:

$$\mathbf{T}^-(z) = \mathbf{M}^-(z)$$

Summary of what we know so far

- The elements of $\mathbf{T}^{-}(z)$ are given by

$$T_{\mu}^{-}(z) = \frac{2i}{ka_1} \sum_{j=-\nu}^{\nu} \frac{e^{i\mu(\psi_j+\pi/2)}}{\sin \psi_j (z\tau_j - 1)} X_j, \quad \mu = -Q, \ldots, Q,$$

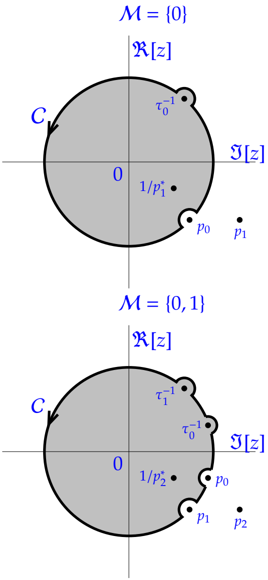
where

$$X_j = \sum_{q=-Q}^{Q} e^{-iq(\psi_j+\pi/2)} A_q^{+}(\tau_j^{-1}), \quad j = -\nu, \ldots \nu,$$

- The elements of $\mathbf{A}^{+}(z)$ are rational functions analytic inside the unit circle.

The objective

Form a system of $2\nu + 1$ algebraic equations to determine X_j.



The kernel in terms of a polynomial matrix

$$\mathbf{K}(z) = \frac{1}{\mathcal{P}(z)\mathcal{T}(z)}\mathbf{P}(z),$$

where $\mathbf{P}(z)$ is a polynomial matrix and

$$\mathcal{P}(z) = \prod_{n=-\nu}^{\nu}(z - p_n), \qquad \mathcal{T}(z) = \prod_{n=-\nu}^{\nu}(z - \tau_n^{-1}),$$

$D(z)$ denotes the determinant of $\mathbf{P}(z)$

- $\deg(D(z)) = 2(2Q+1)(2\nu+1)$,
- $D(z) = 0 \Leftrightarrow D(1/z^*) = 0$,
- $D(z) = \mathcal{P}^{2Q}(z)\mathcal{T}^{2Q}(z)D_1(z)$,

where $D_1(z)$ is a polynomial of degree $4\nu + 2$.

The adjugate of $\mathbf{P}(z)$

$$\operatorname{adj}\left(\mathbf{P}(z)\right) = \mathcal{P}^{2Q-1}(z)\mathcal{T}^{2Q-1}(z)\mathbf{F}(z),$$

for some polynomial matrix $\mathbf{F}(z)$.

- The WH equation:

$$\frac{1}{\mathcal{P}(z)\mathcal{T}(z)}\mathbf{P}(z)\mathbf{A}^{+}(z) = \mathbf{T}^{+}(z) + \mathbf{T}^{-}(z).$$

- Multiply with the adjugate of $\mathbf{P}(z)$ to obtain

$$D_1(z)\mathbf{A}^{+}(z) = \mathbf{F}(z)\Big(\mathbf{T}^{+}(z) + \mathbf{T}^{-}(z)\Big). \tag{1}$$

- $D_1(z_m) = 0 \Leftrightarrow D_1(1/z_m^*) = 0$ for $m = -\nu, \ldots \nu$. Therefore if $|z_m| > 1$,

The system of equations for X_j

$$\mathbf{F}(1/z_m^*)\Big(\mathbf{T}^{+}(1/z_m^*) + \mathbf{T}^{-}(1/z_m^*)\Big) = 0.$$

- Once we determine X_j, then $\mathbf{A}^{+}(z)$ is known and

$$A_{0n}^{\mu} = \frac{1}{2\pi i}\oint_C A_{\mu}^{+}(z)\, z^{-n-1}\, dz, \quad \mu = -Q, \ldots, Q, \quad n = 0, 1, \ldots,$$

$$A_{0n}^{\mu} = -\sum_{m=-\nu}^{\nu} z_m^{-n-1}\rho_m^{\mu}, \quad \mu = -Q, \ldots, Q, \quad n = 0, 1, \ldots,$$

with

$$\rho_m^{\mu} = \operatorname*{Res}_{z=z_m} A_{\mu}^{+}(z), \quad \mu = -Q, \ldots, Q.$$

The far field pattern

The scattered field in the lower half plane

$$u_{sc}(\mathbf{r}) \sim \sum_{j\in\mathcal{M}} \frac{2X_j}{ka_1 \sin\psi_j} e^{ik(x\cos\psi_j + y\sin\psi_j)} \quad \text{as} \quad y \to -\infty,$$

The scattered field in the upper half plane

- If the determinant of the kernel is zero free on the unit circle, then

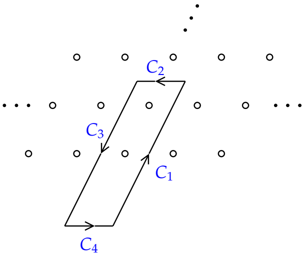
$$u_{sc}(\mathbf{r}) \sim -e^{ik(x\cos\psi_0 + y\sin\psi_0)} \quad \text{as} \quad y \to +\infty,$$

- If the determinant of the kernel has a pair of zeros z_0 and z_0' on the unit circle, and z_0 corresponds to a pole of $\mathbf{A}^+(z)$, then

$$u_{sc}(\mathbf{r}) \sim -\frac{1}{z_0} \sum_{q=-Q}^{Q} \rho_0^q G_q^{(2)}(\mathbf{r}, \beta(z_0)) - e^{ik(x\cos\psi_0 + y\sin\psi_0)} \quad \text{as} \quad y \to +\infty,$$

  where

$$G_q^{(2)}(\mathbf{r}, \beta) = \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} e^{i\mathbf{R}_{jm}\cdot\beta} \mathcal{H}_q(\mathbf{r} - \mathbf{R}_{jm}), \quad q \in \mathbb{Z}.$$



- Let C be a parallelogram with vertices

$$\pm\frac{1}{2}\,\mathbf{a_1} \pm \left(N + \frac{1}{2}\right)\mathbf{a_2}.$$

- $I_j = -\dfrac{P_0\omega}{2}\Im \displaystyle\int_{C_j} u(\mathbf{r}) \left(\frac{\partial u(\mathbf{r})}{\partial n}\right)^* ds,$

The principal of conservation of energy:

$$I_1 + I_2 + I_3 + I_4 = 0 \Rightarrow I_2 + I_4 = 0.$$

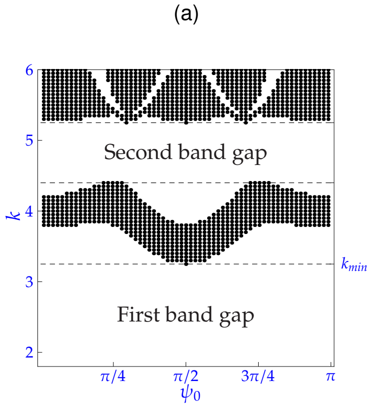
Calculations indicate that this is satisfied up to 6 significant figures.

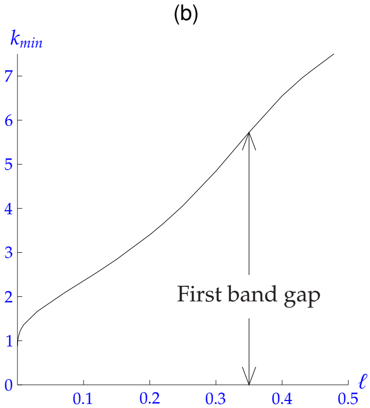
Using I_2 to determine which zero on the unit circle corresponds to a pole of $\mathbf{A}^+(z)$

Given a pair of zeros z_0 and z_0' on the unit circle

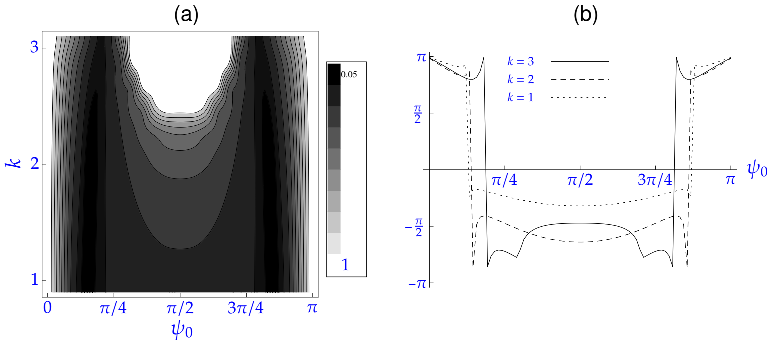
- $I_2(z_0') < 0$ means that energy is propagating $\downarrow$ across C_2. (Unphysical)
  $\mathbf{A}^+(z)$ must be analytic at this point.
- $I_2(z_0) > 0$ means that energy is propagating $\uparrow$ across C_2. (Right!)
  $\mathbf{A}^+(z)$ has a pole at this point.

(a) $\ell = 0.187$


(a)
6
5
4
3
2
k
Second band gap
First band gap
k_{min}
π/4
π/2
3π/4
π
ψ_0

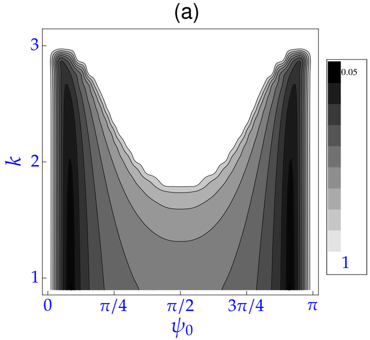

(b)
k_{min}
7
6
5
4
3
2
1
0
First band gap
0.1
0.2
0.3
0.4
0.5
ℓ

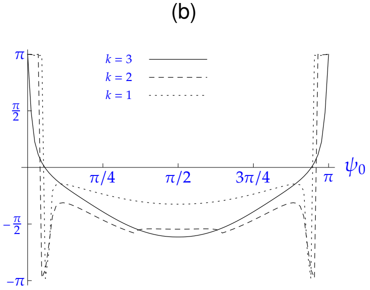
$\ell = 0.25$



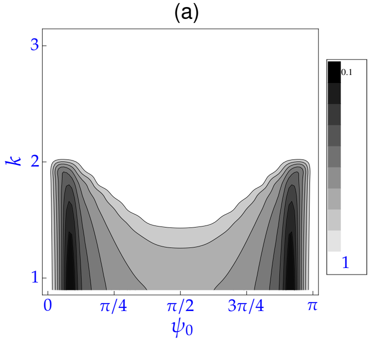
The band structure of square lattice (sound hard cylinders)

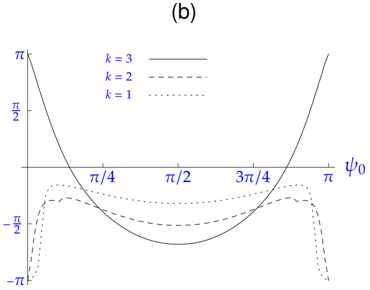
$\ell = 0.42$





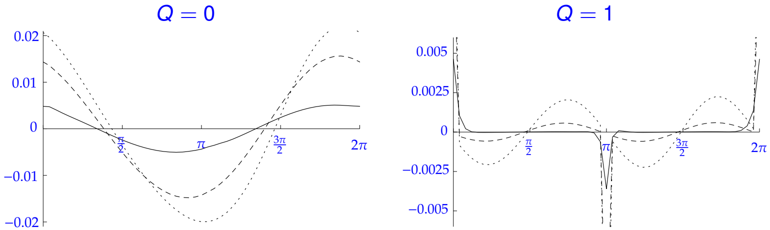
$\ell = 0.472$





Conclusions

- The method is efficient at low frequencies.
  $\ell = 0.01$ (solid line), $\ell = 0.05$ (dashed line) and $\ell = 0.1$ (dotted line)



- Higher frequencies Q and ν need to be sufficient large
  - Increasing Q causes the dimension of the matrix kernel to increase.
  - Increasing ν introduces new zeros. ($|z_\nu| >> 1$ and $|1/z_\nu| << 1$).
  - More than one pair of zeros on the unit circle introduces greater complexity.

Thank you!