Effects of thermal noise on transport in soliton ratchet systems

With

L. Morales, V. Stehr, P. Müller (Bayreuth)

A. Sanchez (Madrid), N.R. Quintero (Sevilla)

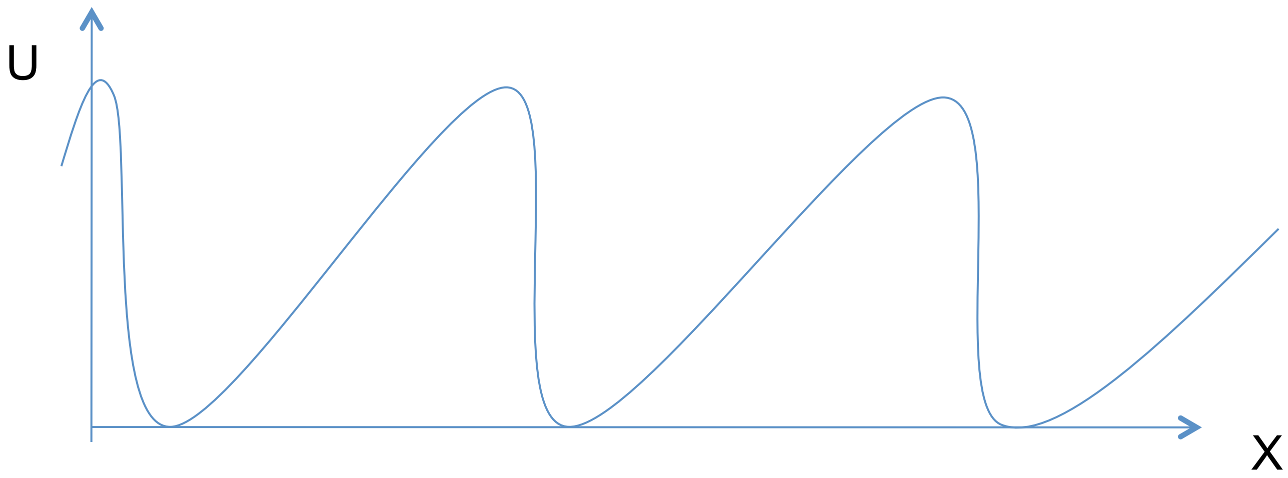
A.R. Bishop (Los Alamos)

J. Buceta, K. Lindenberg (San Diego)

1. Particle ratchets

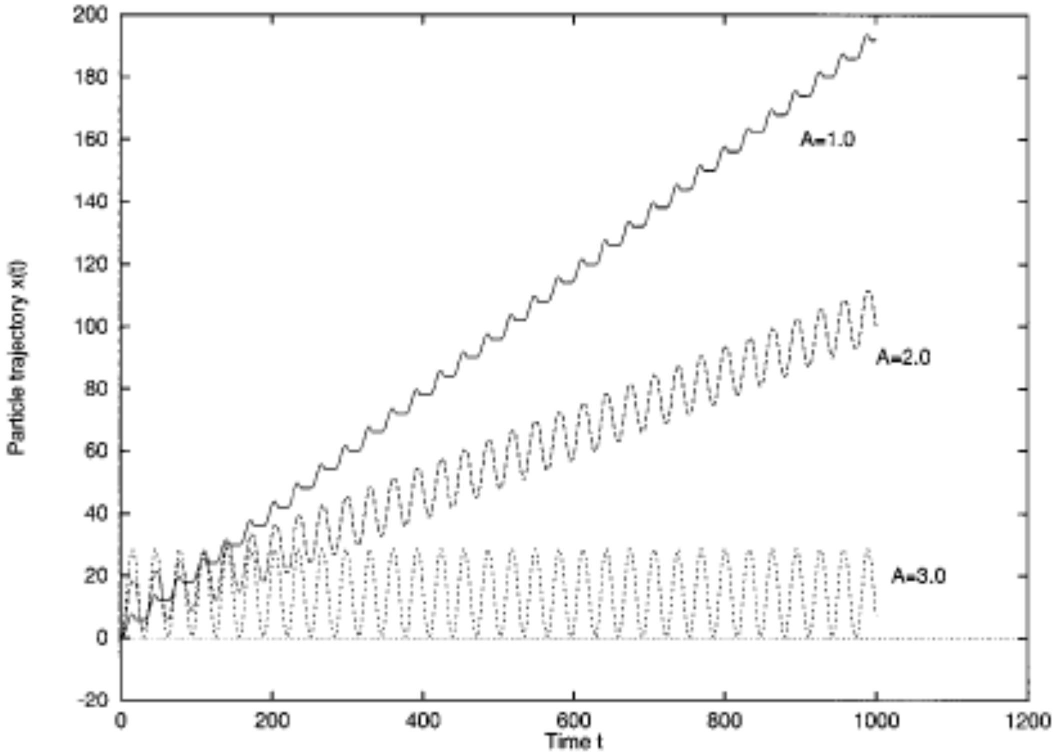
1.1 Equation of motion

Point-like particle in an asymmetric, periodic potential U(X), driven by an ac force, which has zero time average:



$$M\ddot{X} + \beta\dot{X} + \frac{dU}{dX} = A\cos(\omega t)$$

1.2 Overdamped motion: inertial term can be neglected



Unidirectional net motion, although force has zero time average, when time for motion from one well to the next one becomes commensurable with the period of the driver; i.e. when the particle motion is locked to the driver.

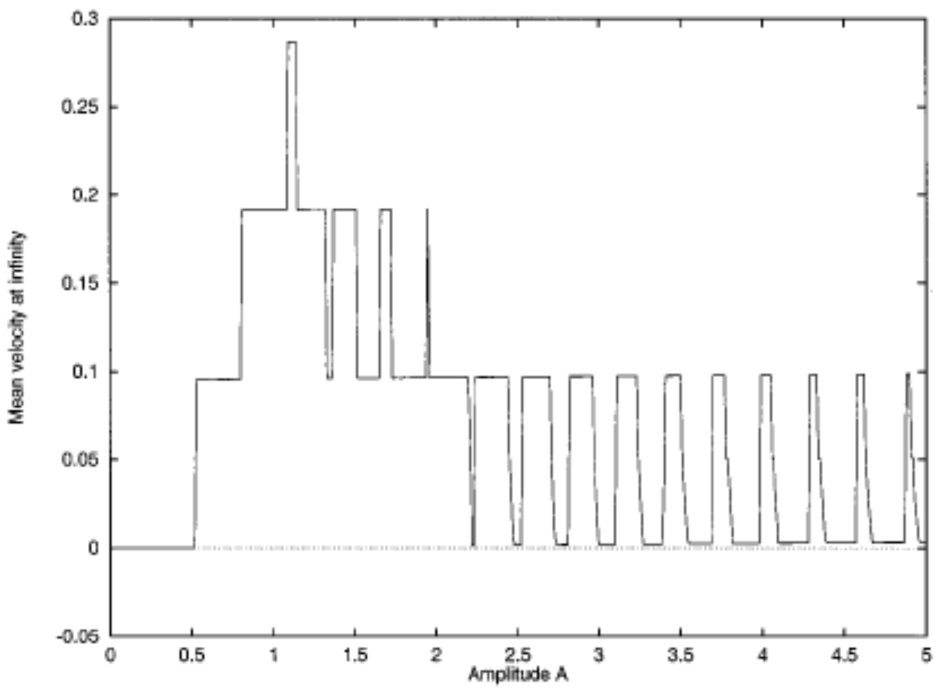
No ratchet effect, if amplitude A too small or too large.

Average velocity <v> constant for all parameter values for which locked solution is stable. <v> is "quantized":

$$\langle v \rangle = \frac{m}{n}\frac{L}{T} = \frac{m}{n} v_{step}, \text{ with m, n integers (no general proof)}$$

L= period of potential, T = 2**p** / **w**= period of driver

v_{step} = basic locking velocity



Dialynas, Lindenberg, Tsironis
PRE 56, 3976 (1997)

1.3 Underdamped motion

Dynamics exhibits additonal features:

a) Strong dependence on initial conditions

b) Bifurcations and chaotic regimes in plots of <v> vs. control parameter A

c) Current reversals: changes of sign of <v>

d) Hysteresis effects

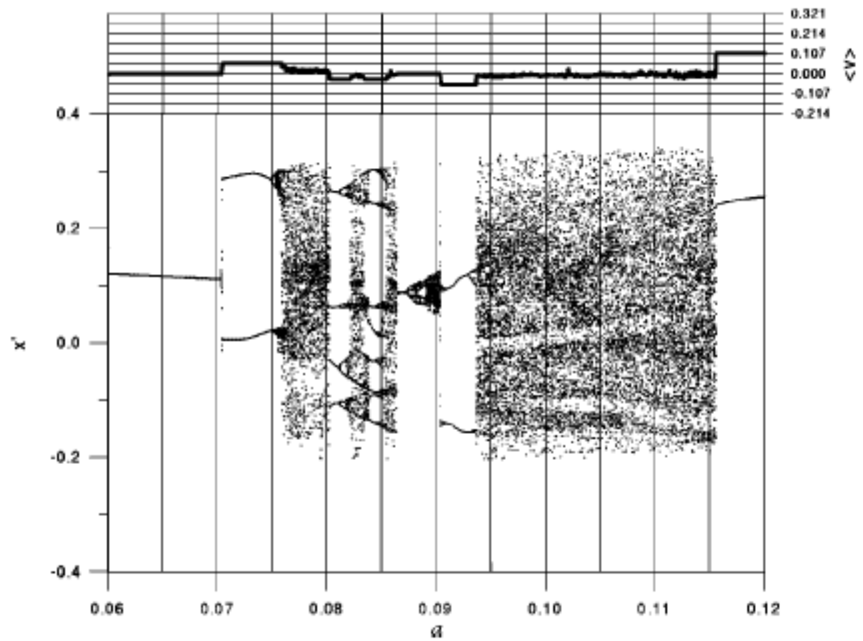


FIG. 2. Bifurcation diagram (lower graph) and mean velocity (upper graph) in the range of the forcing parameter $a \in (0.06, 0.12)$, including the interval discussed in Ref. [22]. The other system parameters are $b = 0.1$ and $\omega = 0.67$. Grid lines in the velocity plot correspond to multiples of half the driver-induced locking velocity, which is $v_\omega \cong 0.107$ with our choice of parameters. All variables here and in the following graphs are dimensionless.

Barbi, Salerno , PRE 62, 1988 (2000)

Jung, Kissner, Hänggi, PRL 76, 3436 (1996)

1.4 Other types of ratchets

e.g. , model for molecular motor: in living cells proteins kinesin and myosin move along tubulin and actin filaments, resp., converting chemical energy into mechanical work

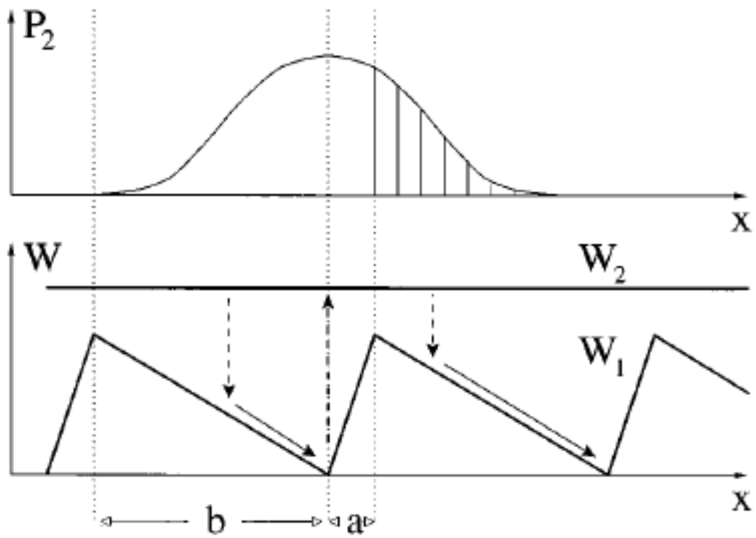


FIG. 6. Motion generation for W_2=const and f_{ext}=0: A particle trapped in state 1 is excited to state 2, where it diffuses freely. It returns to state 1 after a typical lifetime ω_2^{-1} when it has a Gaussian probability distribution P_2. With a probability proportional to the hatched area of the Gaussian distribution, it arrives at the next minimum of W_1 provided it has sufficient time in state 1 to slide to that minimum.

Jülicher et al., RMP 69, 1269 (1997)

2. Soliton ratchets

Solitons: nonlinear, coherent collective excitations, behave like particles.

Generalize particle ratchet systems to spatially extended nonlinear systems, in which solitons play a similar role as the point particles.
To get ratchets, either a spatial or a temporal symmetry must be broken!

E.g., nonlinear Klein-Gordon models, in particular sine-Gordon model:

$$\phi_{tt} - \phi_{xx} + \sin\phi = -\beta\phi_t + f(t)$$

1.) Spatially homogeneous -> temporal symmetry must be broken

E.g., $f(t) \neq -f(t+T/2)$: time shift symmetry broken by biharmonic driving:
$f(t) = \varepsilon_1 \sin(\omega t + \delta) + \varepsilon_2 \sin(m\omega t + \theta_2)$, with integer $m \geq 2$

Flach et al., PRL 88, 184101 (2002), Salerno, Zolotaryuk, PRE65,056603(2003)

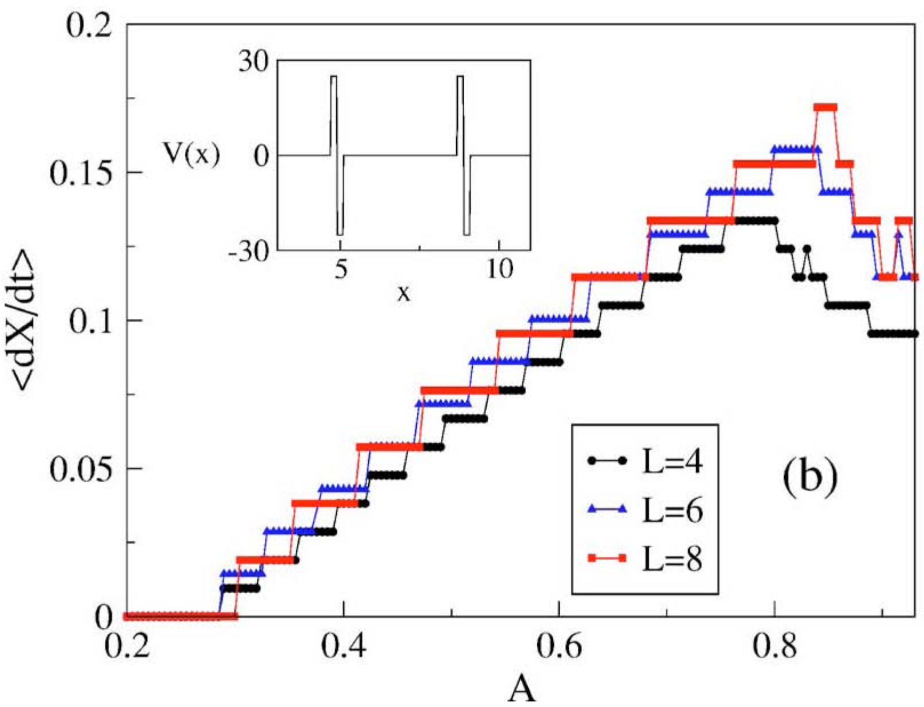
Long Josephson junctions: Ustinov et al., PRL 93, 087001 (2004)

2. Break spatial symmetry by introducing inhomogeneities

$$\boldsymbol{f}_{tt} - \boldsymbol{f}_{xx} + (1 + V(x))\sin\boldsymbol{f} = -\boldsymbol{bf}_t + f(t)$$

V(x): periods of length L, with asymmetric arrays of strongly localized inhomogeneities, e.g. narrow and high boxes.

Optimal choice: 2 boxes with opposite signs that touch each other.



$f(t) = A\sin(\boldsymbol{w}t)$

$\boldsymbol{w} = 0.015$

$\boldsymbol{b} = 1$

For ideal ratchet <v> = 0.5, critical velocity is 1.

Mertens, Morales, Bishop, Sanchez, Mueller, PRE 74, 066602 (2006)

3. Thermal noise

$$\phi_{tt} + \phi_t - \phi_{xx} + \sin(\phi)[1 + V(x)] = f(t) + \eta(x,t),$$

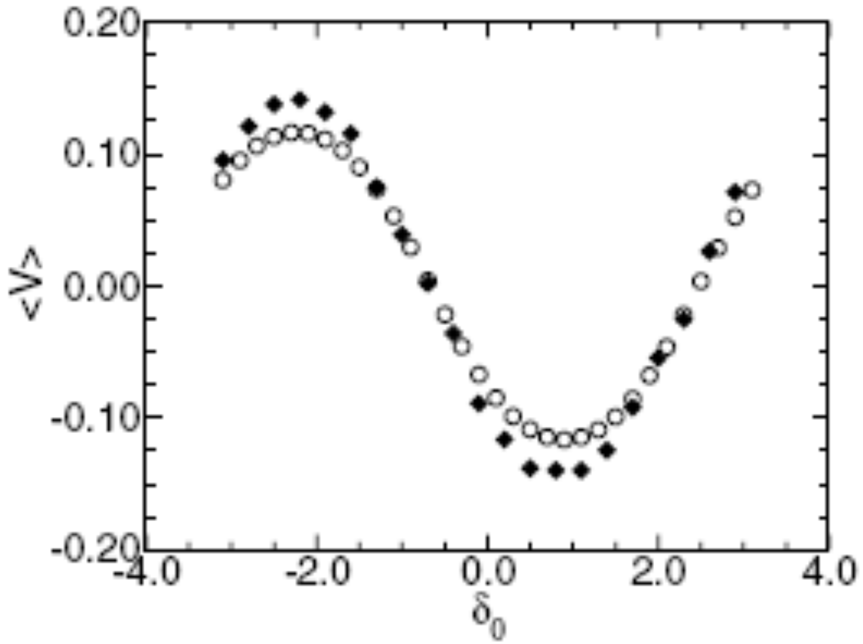
with

$$\langle \eta(x,t) \rangle = 0,$$

$$\langle \eta(x,t)\, \eta(x',t') \rangle = D\delta(x - x')\delta(t - t'),$$

and $D = \beta\ k_B T$, Gaussian white noise

a) Homogeneous system, biharmonic drive



Morales, Quintero, Mertens and Sanchez, PRL 91, 234102 (2003)

average soliton velocity <v> vs. initial phase

D = 0 (deterministic): empty circles
D = 0.03 (stochastic): full diamonds

Ratchet effect **robust,** even **enhanced by noise**. Reason ?
See Collective Coordinate Theory (below)

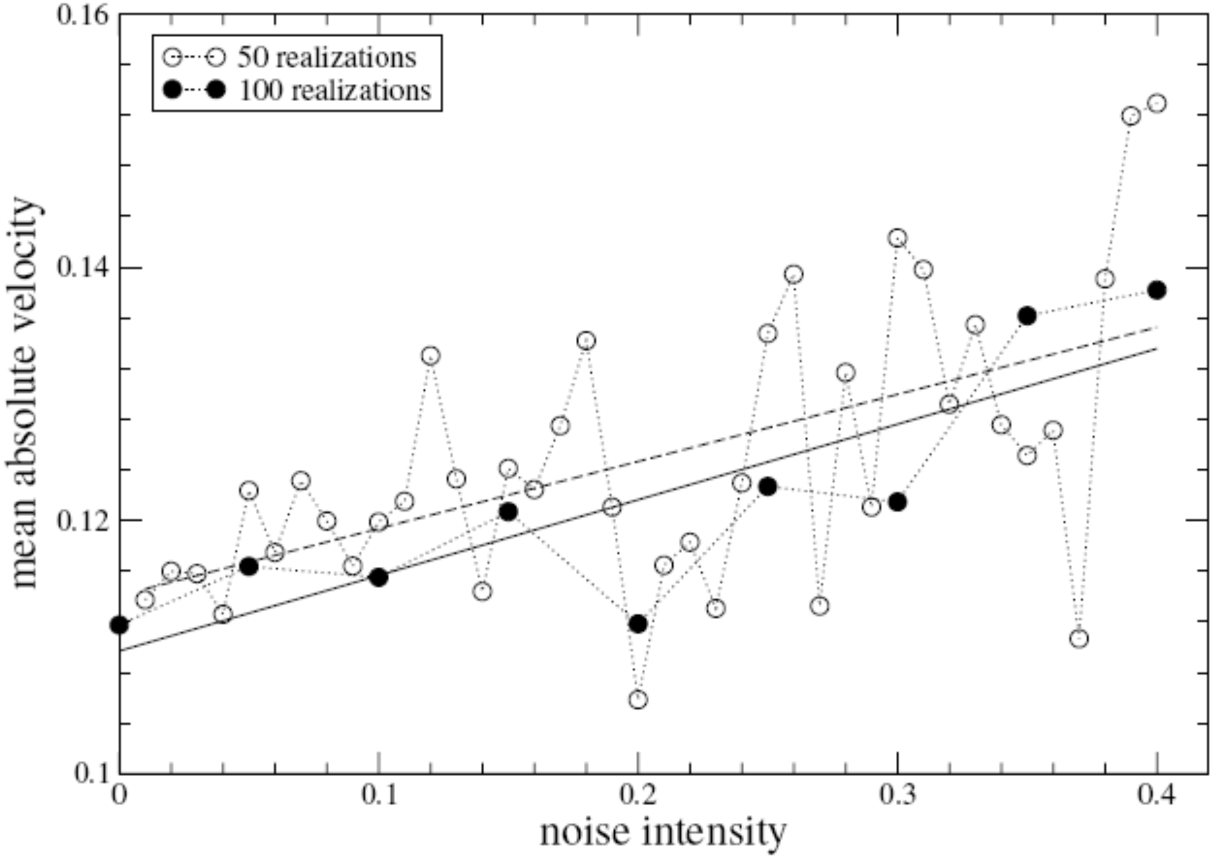


Fig. 4. Mean absolute value of the kink center velocity vs noise intensity. Parameters are $\epsilon_1 = \epsilon_2 = 0.2$, $\beta = 0.05$, $\delta = 0.1$, initial phase $\delta_0 = -0.31831$, and relative phase $\theta = -\pi/2$. Realizations in the averages are as indicated. The straight lines are linear fits of slopes 0.053 (dashed, for the 50 realizations set) and 0.060 (solid, for the 100 realizations set).

However, is there an **optimal** temperature?

Sanchez, Morales, Mertens, Quintero, Buceta, Lindenberg, Fluct. Noise Lett. 4, L571 (2004)

b) Inhomogeneous systems

strongly localized inhomogeneities, e.g. delta functions or narrow boxes

here an optimal temperature exists:

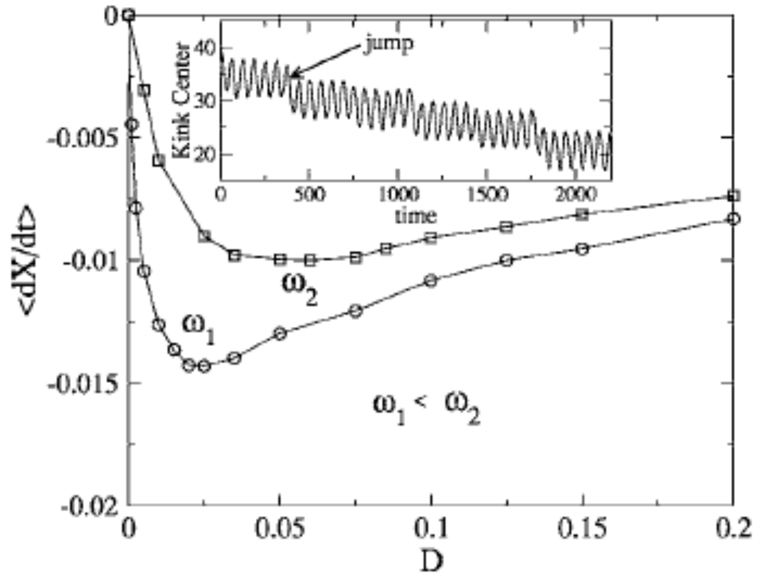


FIG. 8. Mean kink velocity $\langle dX/dt \rangle$ vs intensity of noise D. Circles, $\omega=0.1$ and $A=0.70$; squares, $\omega=0.11$ and $A=0.75$. Inset shows one realization for the motion of the kink center for $\omega=0.1$, $A=0.70$, $\delta_0=0$, and $D=0.005$.

4. Collective Coordinate Theory

Use 1-soliton solution of unperturbed sine-Gordon system for ansatz with 2 collective coordinates (M. Rice, 1983)

$$\phi(x,t) = \phi^{(0)}(x - X(t), l(t)) = 4 \arctan\left(\exp\left[\frac{x - X(t)}{l(t)} \right] \right)$$

x – vt with constant velocity v replaced by x – X(t),
where X(t) is the soliton position

Lorentz-contracted soliton width replaced by width l(t)

Insert into inhomogeneous, driven, noisy sine-Gordon equation

$$\phi_{tt} + \phi_t - \phi_{xx} + \sin(\phi)[1 + V(x)] = f(t) + \eta(x,t)$$

different methods all yields the same two ODEs:

$$M_0 l_0 \frac{\ddot{X}}{l} + \beta M_0 l_0 \frac{\dot{X}}{l} - M_0 l_0 \frac{\dot{X}\dot{l}}{l^2} = F^{ac} + F^{inh} + F^{st}$$

$$\alpha M_0 l_0 \frac{\ddot{l}}{l} + \beta \alpha M_0 l_0 \frac{\dot{l}}{l} + M_0 l_0 \frac{\dot{X}^2}{l^2} = K^{int}(l, \dot{l}, \dot{X}) + K^{inh} + K^{st}$$

where $\alpha = \pi^2/12,\ M_0 = 8,\ l_0 = 1$

$$F^{ac} = \int_{-\infty}^{\infty} dx\, f(t) \frac{\partial \phi^{(0)}}{\partial X} = -2\pi f(t)$$

driving force

$$F^{inh} = -\int_{-\infty}^{\infty} dx\, \sin(\phi^{(0)}) V(x) \frac{\partial \phi^{(0)}}{\partial X} = -\frac{\partial U}{\partial X}$$

force due to inhomogeneities, effective potential U(X,l)

$$F^{st} = \int_{-\infty}^{+\infty} dx\, \eta(x,t) \frac{\partial \phi^{(0)}}{\partial X}$$

stochastic force

similar forces in 2nd ODE

$$\langle F^{\mathrm{st}}(t)F^{\mathrm{st}}(t')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dx' \frac{\partial\phi^{(0)}(x,t)}{\partial X}\frac{\partial\phi^{(0)}(x',t')}{\partial X}$$

$$\times\langle\eta(x,t)\eta(x',t')\rangle = D\delta(t-t')\int_{-\infty}^{\infty} dx\left(\frac{\partial\phi^{(0)}}{\partial X}\right)^2$$

$$= D\delta(t-t')\frac{l_0}{l}M_0.$$

Correlation function for 1st stochastic force ,

similar result for 2nd force, cross correlation vanishes

$$M_0 l_0\frac{\ddot{X}}{l} + \beta M_0 l_0\frac{\dot{X}}{l} - M_0 l_0\frac{\dot{X}\dot{l}}{l^2} = -\frac{\partial U}{\partial X} - qf(t) + \sqrt{\frac{DM_0 l_0}{l}}\xi_1(t), \tag{B21}$$

$$\alpha M_0 l_0\frac{\ddot{l}}{l} + \beta\alpha M_0 l_0\frac{\dot{l}}{l} + M_0 l_0\frac{\dot{X}^2}{l^2} = -\frac{\partial U}{\partial l} + K^{\mathrm{int}}(l,\dot{l},\dot{X}) + \sqrt{\frac{D\alpha M_0 l_0}{l}}\xi_2(t) \tag{B22}$$

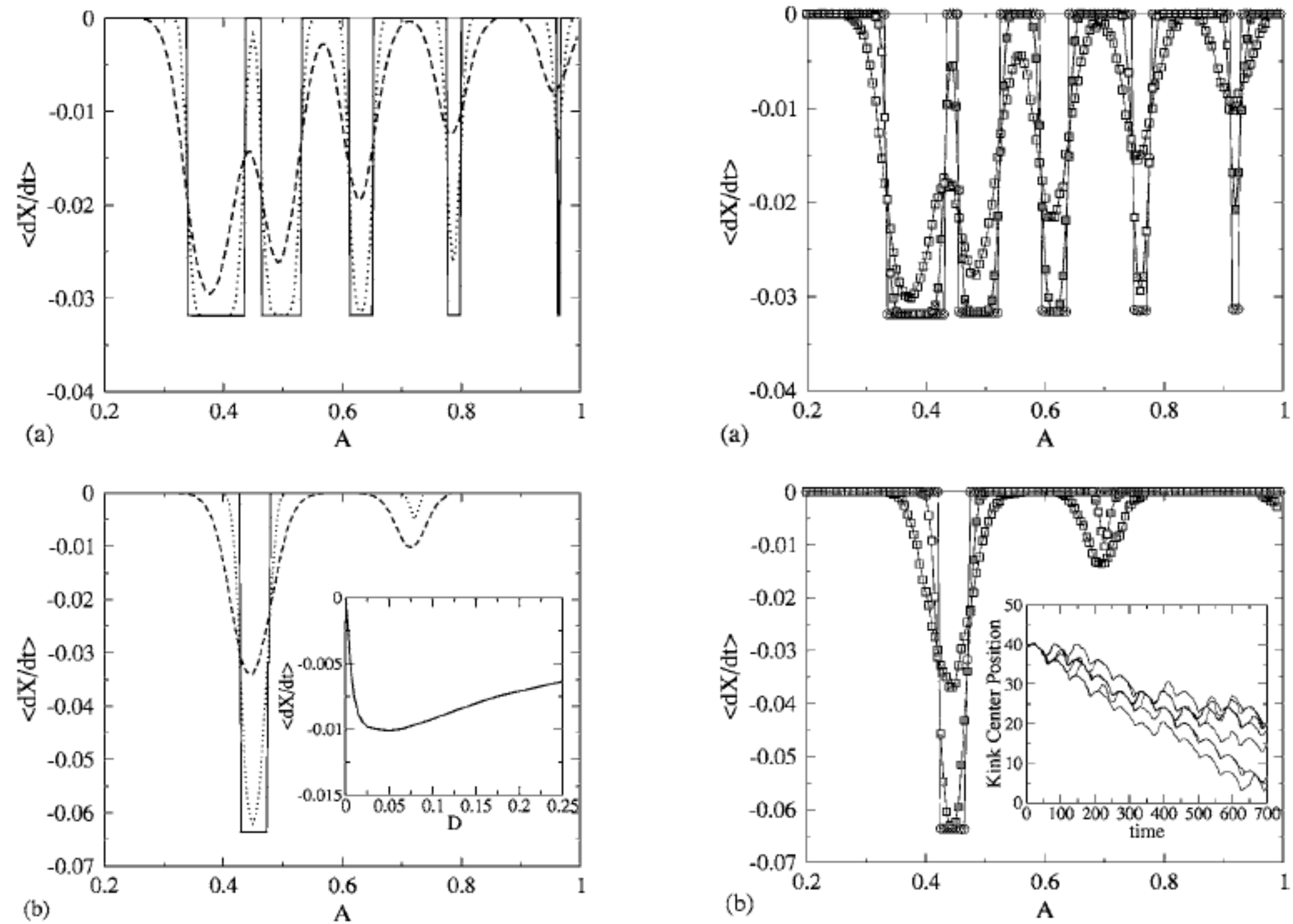
Two stochastic ODEs with two multiplicative noises, because of factor 1/Sqrt[l(t)], to be solved numerically

with $\langle\xi_i(t)\rangle=0$, $\langle\xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t')$, for $i,j=1,2$.

independent Gaussian white noises

CC-theory

Simulations



Two main effects of thermal noise:

Sharp „windows“ of the deterministic case are smeared out and broadened

New peak appears in region where no window existed in deterministic case

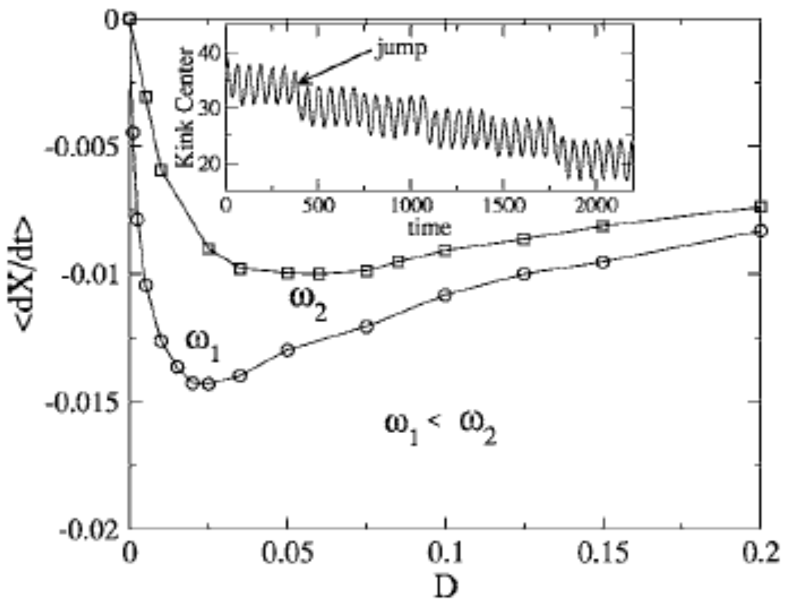


FIG. 8. Mean kink velocity $\langle dX/dt \rangle$ vs intensity of noise D. Circles, ω=0.1 and A=0.70; squares, ω=0.11 and A=0.75. Inset shows one realization for the motion of the kink center for ω=0.1, A=0.70, δ_0=0, and D=0.005.

New peak appears only for **very narrow frequency range**:

Similarity to Stochastic Resonance

For every frequency there exists an **optimal temperature**, which agrees with that of the CC-theory

Morales, Mertens, Sanchez,
PRE 72, 016612 (2005)

5. Additive inhomogeneities

So far multiplicative inhomogeneities

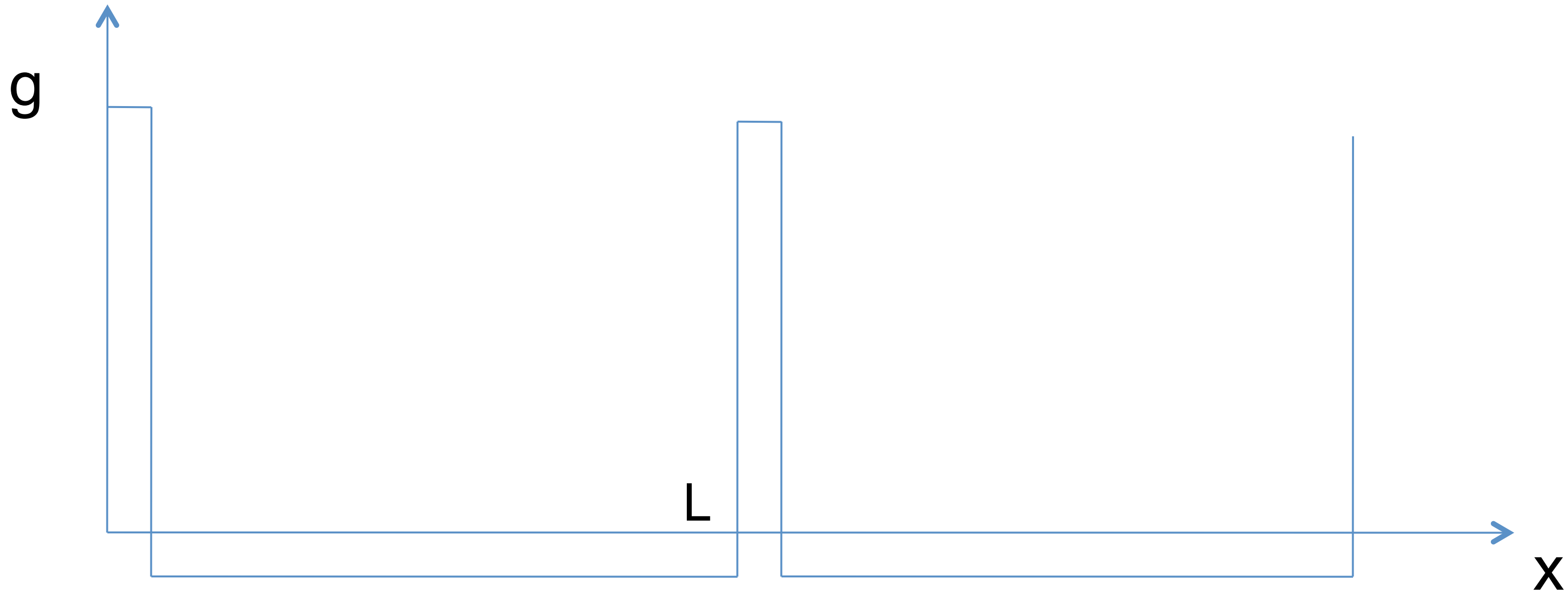
$$\phi_{tt} + \beta\phi_t - \phi_{xx} + [1 + V(x)]\sin\phi = A\sin(\omega t + \delta_0)$$

V(x): periodically repeated unit cells, each with asymmetric array of strongly localized inhomogeneities

Long Josephson junctions: microshorts and microresistors

Now additive inhomogeneities:

$$\phi_{tt} - \phi_{xx} + \sin\phi = -\beta\phi_t + f(t) + g(x)$$



where spatial average of g(x) must vanish.

Constant current is injected in a small region of an annular Josephson junction
and is extracted from the same electrode along the rest of the junction.
This yields large <v>.

Beck, Goldobin, Neuhaus, Siegel, Kleiner, Koelle, PRL 95, 090603 (2005)

1-CC theory --> 1 ODE for the soliton position X(t), solved numerically:

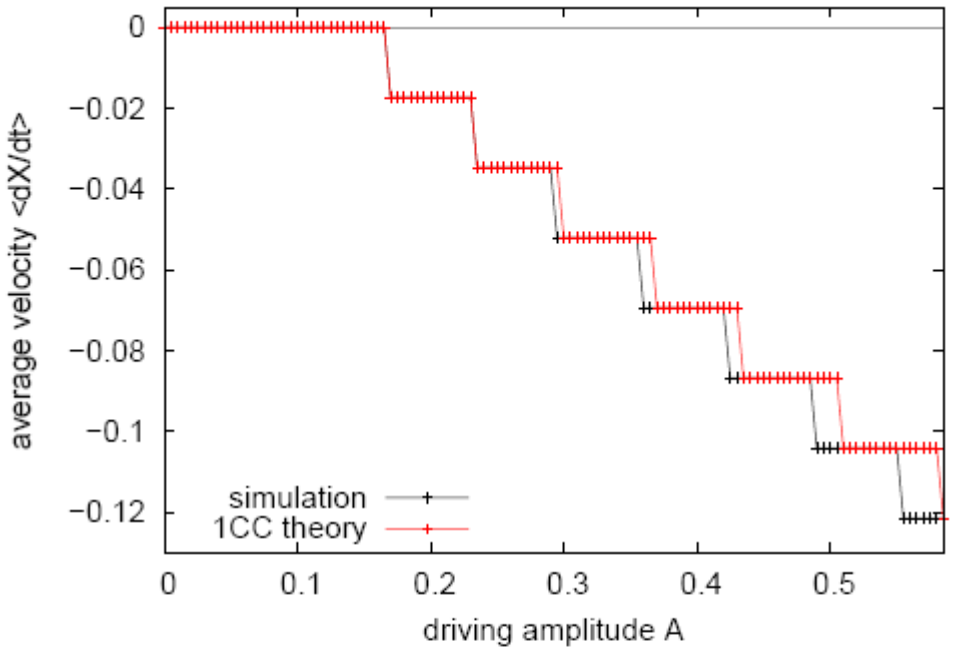


FIG. 6: (colour online) Average velocity of the kink vs. driving amplitude. Comparison between 1CC theory (red) and simulation (black). Parameters: $L = 21.8$, $w = 0.14$, $g_1 = 15.9$, $\omega = 0.005$.

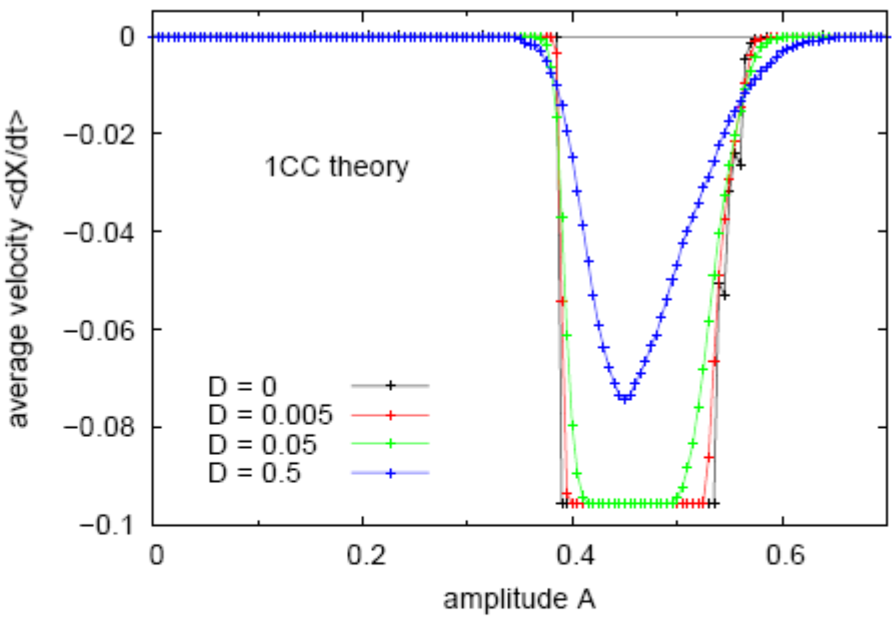
Good agreement with the simulations, except for large A

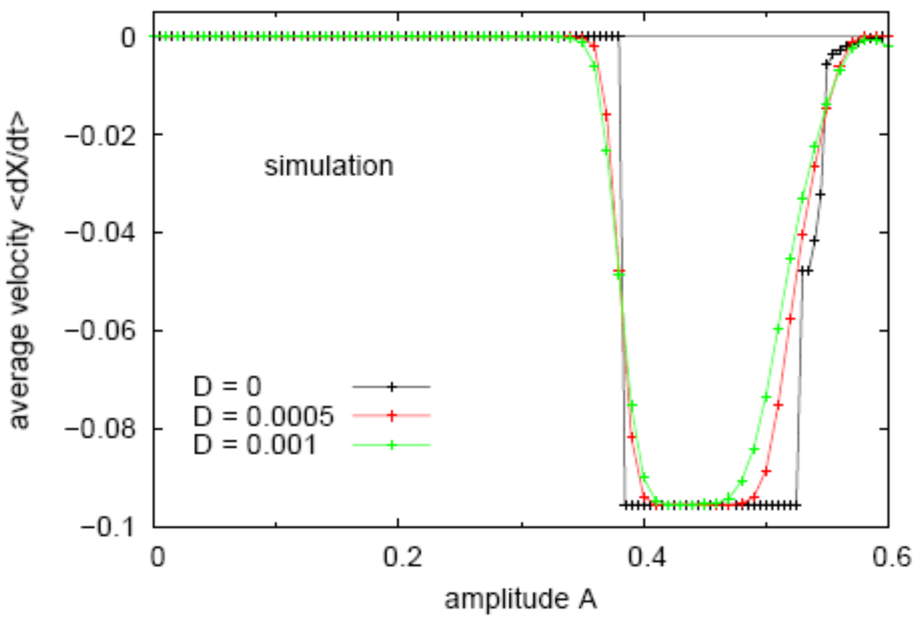
Stehr, Müller, Mertens, and Bishop, PRE 79, 036601 (2009)

Include thermal noise: $\phi_{tt} - \phi_{xx} + \sin\phi = -\beta\phi_t + f(t) + g(x) + \eta(x,t)$

1 stochastic ODE $\gamma^3 M_0 \ddot{X} + \gamma\beta M_0 \dot{X} = -qf(t) - \frac{\partial U}{\partial X} + F_{st}(t)$

with the stochastic force $F_{st}(t) = \sqrt{\gamma \cdot M_0 \cdot D} \cdot \xi(t)$





good qualitative agreement, but effect of noise in the theory much weaker than in the simulation !

Possible reason for the discrepancy: approximation in the calculation of the variance of the stochastic force

$$\begin{aligned}
\langle F_{\mathrm{st}}(t) \cdot F_{\mathrm{st}}(t')\rangle &= \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}x'\, \frac{\partial \Phi(x,t)}{\partial X} \cdot \frac{\partial \Phi(x',t')}{\partial X} \cdot \eta(x,t) \cdot \eta(x',t') \right\rangle \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}x'\, \frac{\partial \Phi(x,t)}{\partial X} \cdot \frac{\partial \Phi(x',t')}{\partial X} \cdot \langle \eta(x,t) \cdot \eta(x',t') \rangle \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}x'\, \frac{\partial \Phi(x,t)}{\partial X} \cdot \frac{\partial \Phi(x',t')}{\partial X} \cdot D \cdot \delta(x-x') \cdot \delta(t-t') \\
&= D \cdot \delta(t-t') \cdot \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\partial \Phi(x,t)}{\partial X} \right)^2 \\
&= \gamma \cdot M_0 \cdot D \cdot \delta(t-t')
\end{aligned}$$

How could this be improved?

Alternative:
Improve CC-theory, look at shape of moving soliton in the simulation (without noise):

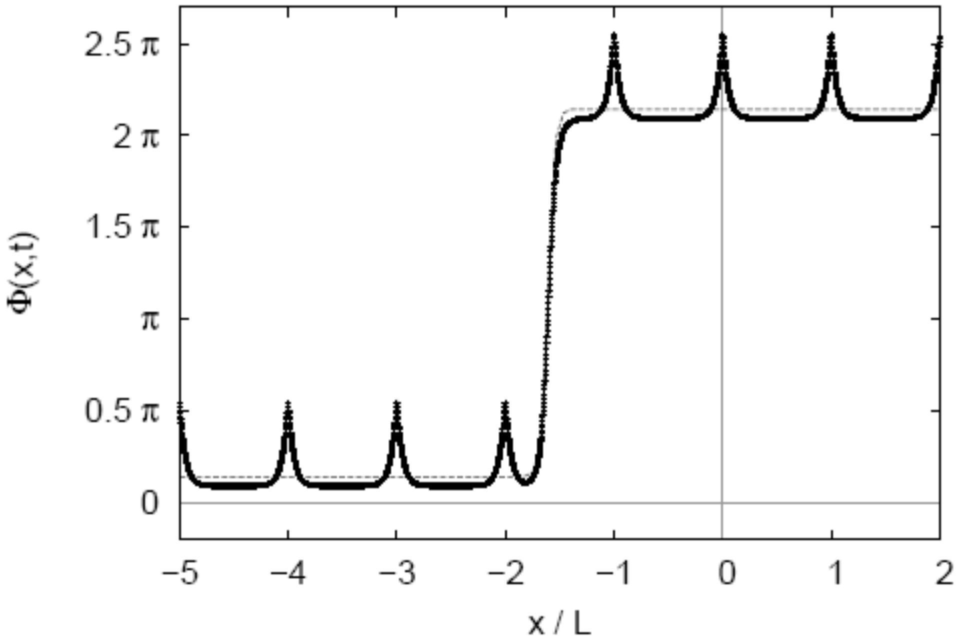


FIG. 3: The kink 300 time units after the initialisation with Eq. 4. The thin line is a fit with Eq. 5. The parameters are the same as in Fig. 1.

Driving amplitude 0.4, driving frequency 0.025

Two new features:

Soliton moves up and down

Spikes, due to inhomogeneities, do not move with the soliton

Theory with 3 CCs, soliton position , width and offset,
spikes not taken into account, because they do not move with the soliton

$$\phi(x,t) = 4 \arctan \exp\left(\frac{x - X(t)}{l(t)}\right) + \varphi(t)$$

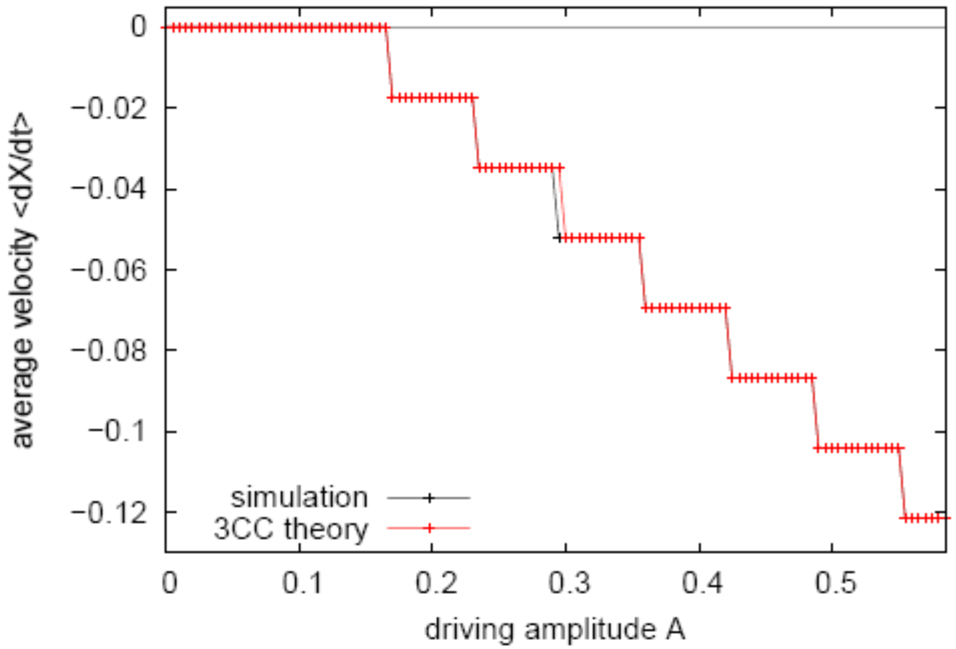


FIG. 9: (colour online) Average velocity of the kink depending on the driving amplitude, comparison between 3CC theory (red) and simulation (black). Parameters are the same as in Fig. 6.

Perfect agreement with the simulations.
For A > 0.6, kink-antikink pairs appear in the simulations, caused by the spikes.

Stehr, Müller, Mertens, Bishop
PRE 79, 036601 (2009)

Include noise: 3 stochastic ODEs, not yet solved

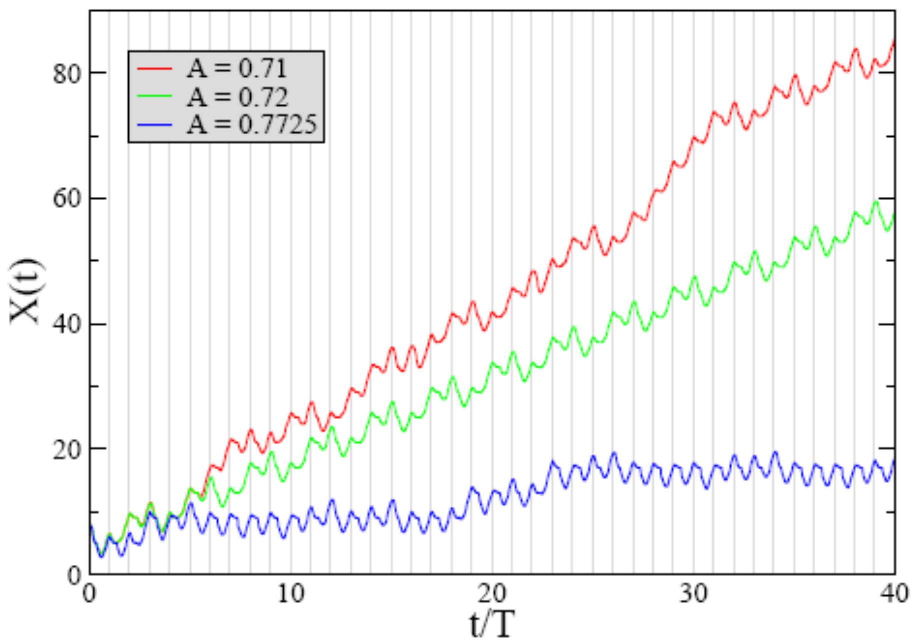
Is effect of noise increased by one to two orders of magnitude?

Then we would have agreement with the simulations.

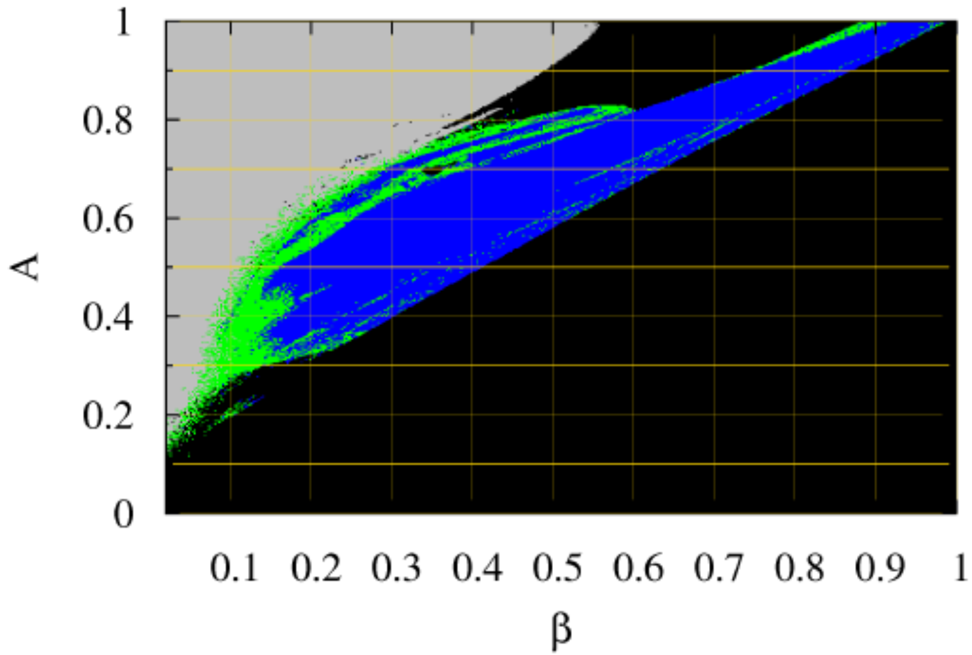
Trend is okay, because deformable soliton is more sensitive to noise.

Remark:

Inhomogeneous sine-Gordon ratchet systems exhibit **deterministic chaos**, **but ratchet effect not destroyed**



transport in spite of chaos;
only weak intermittency, in contrast to particle ratchets



Black: no ratchet effect
Blue: regular, periodic trajectories
Green: chaotic trajectories
Grey: no well-defined trajectories

Müller, Mertens, and Bishop, PRE 79, 016207 (2009)

6. Conclusions

Main effects of thermal noise:

a) Ratchet effect robust under temperature

c) Effect even enhanced (in homogeneous sine-Gordon ratchets)

e) New window appears in a region of the driving amplitude, where no window existed in the deterministic case, and optimal temperature exists (in inhomogeneous ratchets)

d) New window appears for a very narrow frequency range, similarity with Stochastic Resonance

Remark: Deterministic chaos in inhomogeneous sine-Gordon ratchets does not destroy the transport