An output-sensitive algorithm for computing projections of resultant polytopes

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Outline of the talk

1. Introduction
2. Resultant polytopes
3. The algorithm

The problem

Given polynomials in $K[x, y]$:

$$f_0 = u_2 x + u_1 y + u_0$$
$$f_1 = c_{13} xy + c_{12} x + c_{11} y + c_{10}$$
$$f_2 = c_{23} xy^2 + c_{22} xy + c_{21} x + c_{20}$$

what is the condition that their coefficients $(c_{10}, \ldots, u_2)$ must satisfy so that the system $f_0 = f_1 = f_2 = 0$ has a solution in $(\mathbb{C}^*)^2$?

$$-c_{20} c_{21} u_1^3 c_{12} c_{13}^2 - u_0 c_{21}^2 u_1^2 c_{11} c_{13}^2 + c_{21}^2 u_1^3 c_{10} c_{13}^2 + \ldots \text{another 57 terms} \ldots = 0$$

The problem

When some of the polynomials' coefficients are specialized to constants,

$$f_0 = u_2 x + u_1 y + u_0$$
$$f_1 = 3xy + 11x + 4y + 1$$
$$f_2 = 2xy^2 + 5xy + x + 3$$

what is the condition that the remaining symbolic coefficients must satisfy so that the system $f_0 = f_1 = f_2 = 0$ has a solution in $(\mathbb{C}^*)^2$?

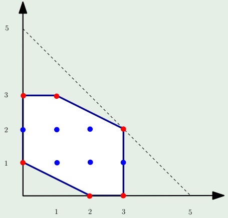
$$80u_0^3 - 136u_1u_0^2 - 180u_2u_0^2 + 288u_1^2u_0 - 674u_2u_1u_0 + 910u_2^2u_0 - 8u_1^3 - 62u_2u_1^2 + 183u_2^2u_1 - 250u_2^3 = 0$$

Preliminaries

The support $A(f)$ of a polynomial f is the set of exponents of its monomials with non-zero coefficient. The Newton polytope $N(f)$ of f is the convex hull of its support.

Example

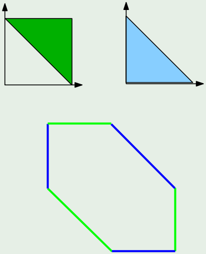
$$f(x,y) = 8y + xy - 24y^2 - 16x^2 + 220x^2y - 34xy^2 - 84x^3y + 6x^2y^2 - 8xy^3 + 8x^3y^2 + 8x^3 + 18y^3$$

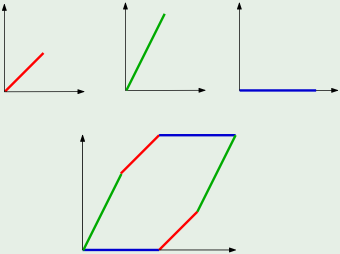


The Newton polytope is the equivalent notion of total degree in sparse elimination theory.

The Minkowski sum of polytopes $P_0, \dots, P_n$ (resp., supports $A_0, \dots, A_n$) is the polytope $P = P_0 + \cdots + P_n := \{p_0 + \cdots + p_n \mid p_i \in P_i\}$
(resp., $P = \mathrm{conv}(A_0 + \cdots + A_n) := \mathrm{conv}\{a_0 + \cdots + a_n \mid a_i \in A_i\}$).

Example



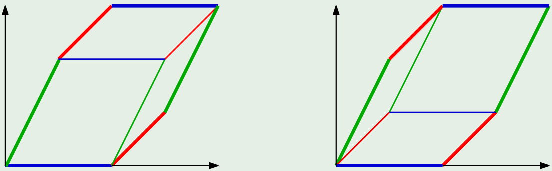


Mixed subdivisions

A fine mixed subdivision of $P = \mathrm{conv}(A_0 + \cdots + A_n) \subset \mathbb{R}^n$, is any family of cells partitioning P and intersecting properly as Minkowski sums of faces $F_i \subset \mathrm{conv}(A_i)$, where $\dim(\sum_i F_i) = \sum_i \dim F_i$.

Example

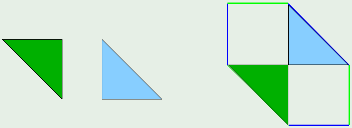
$$f_0 := c_{00} - c_{01}x_1x_2, \; f_1 := c_{10} - c_{11}x_1x_2^2, \; f_2 := c_{20} - c_{21}x_1^2$$

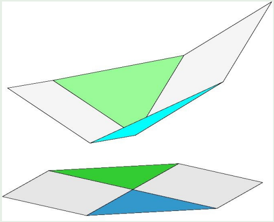


Regular mixed subdivisions

A regular mixed subdivision of $P = \text{conv}(A_0 + \cdots + A_n) \subset \mathbb{R}^n$ is obtained as the projection of the lower (or upper) hull of the Minkowski sum of lifted polytopes $\hat{P}_i := \text{conv}\{(a_i, \omega_i(a_i)) \mid a_i \in A_i\} \subset \mathbb{R}^{n+1}$. Generic ω_i produce regular fine mixed subdivisions.

Example





The sparse (toric) resultant

Definition

The sparse resultant $\mathcal{R}$ of polynomials $f_0, \ldots, f_n$, where

$$f_i = \sum_{a \in A_i} c_{ia} x^a, \quad x^a = (x_1^{a_1}, \ldots, x_n^{a_n}),$$

with symbolic c_{ia} and exponents in $A_i \subset \mathbb{Z}^n$, is the unique irreducible integer polynomial in the c_{ij}'s, which vanishes iff the f_i have a common complex root.

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Example

Let $A_0 = A_1 = \{0, 1\}$. Then $f_0 = c_{01}x + c_{02}$, $f_1 = c_{11}x + c_{12}$.

$$\mathcal{R} = c_{01}c_{12} - c_{02}c_{11} \in \mathbb{Z}[c_{01}, c_{02}, c_{11}, c_{12}]$$

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$$\mathcal{R} = c_{01}c_{12} - c_{02}c_{11} \in \mathbb{Z}[c_{01}, c_{02}, c_{11}, c_{12}]$$

The Newton polytope $N(\mathcal{R})$ of the resultant is called the resultant polytope.

Resultant polytope

Theorem (Sturmfels'94)

There exists a surjection ρ from the regular fine mixed subdivisions of $P = conv(A_0 + \cdots + A_n)$ onto the vertices of $N(\mathcal{R})$. Subdivisions S, S' are equivalent iff $\rho_S = \rho_{S'}$.

Resultant polytope

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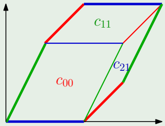
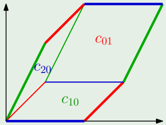
Example

$$f_0 := c_{00} - c_{01}st, \quad f_1 := c_{10} - c_{11}st^2, \quad f_2 := c_{20} - c_{21}s^2$$



$c_{00}^4 c_{11}^2 c_{21}$



$c_{01}^4 c_{10}^2 c_{20}$

$$\rho_S = (4, 0, 0, 2, 0, 1)$$
$$\rho_{S'} = (0, 4, 2, 0, 1, 0)$$

The Cayley Trick

Theorem

Regular mixed subdivisions of $P = conv(A_0 + \cdots + A_n) \subset \mathbb{R}^n$ *correspond bijectively to regular triangulations of* $\mathcal{A} := \bigcup_{i=0}^{n}(A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$*:*
e_i *form an affine basis of* $\mathbb{R}^n$.

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$$\begin{aligned} f_0 &= c_{01}x^2 + c_{02}y^2 + c_{03}x^2y^2 \\ f_1 &= c_{11}x^2 + c_{12}y^2 + c_{14} \end{aligned}$$

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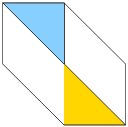

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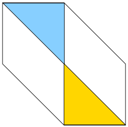
$$\mathcal{A} := \{(2,0,0),(0,2,0),(2,2,0),$$



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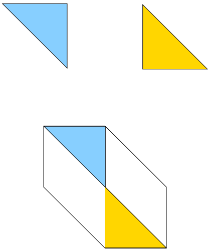

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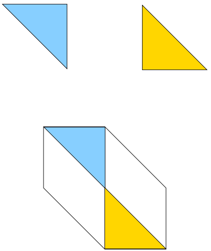
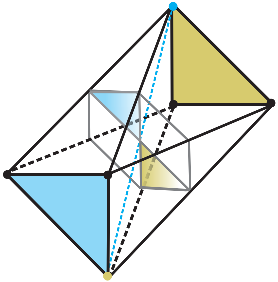
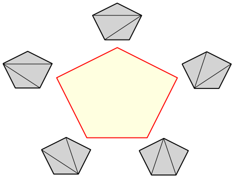




Theorem (Gelfand, Kapranov, Zelevinsky)

For every point set $\mathcal{A} = \{a_1, \dots, a_{|\mathcal{A}|}\} \subset \mathbb{R}^d$, *there exists a polytope* $\Sigma(\mathcal{A}) \subset \mathbb{R}^{|\mathcal{A}|}$, *the* *secondary polytope* *of* $\mathcal{A}$, *s.t.*

1. *Vertices of* $\Sigma(\mathcal{A})$ *correspond bijectively to regular triangulations of* $\mathcal{A}$ *via*
$$T \mapsto \varphi_T : \ (\varphi_T)_i = \sum_{a_i \in \mathrm{Vert}(\sigma):\sigma \in T} \mathrm{Vol}(\sigma), \ i = 1, \dots, |\mathcal{A}|,$$
2. *Edges correspond to flips.*



Secondary polytope of a pentagon.

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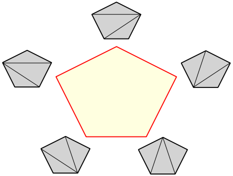
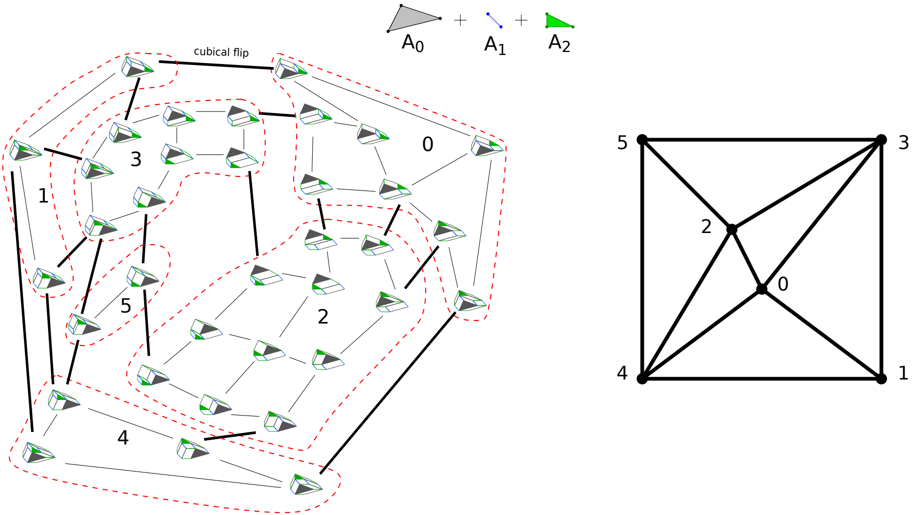


Secondary polytope of a pentagon.

$N(\mathcal{R})$ is a Minkowski summand of $\Sigma(\mathcal{A})$.



(Top) Input polytopes. (Left) Graph of mixed subdivisions of A_0, A_1, A_2 (equiv., triangulations of $\mathcal{A}$): the 6 equivalence classes are dotted. (Right) $N(\mathcal{R}) \subset \mathbb{R}^3$.

The problem

Given $n + 1$ polynomials in n variables with fixed supports $A_i \subset \mathbb{Z}^n$, efficiently compute $\pi(N(\mathcal{R})) = \mathit{\Pi} \subset \mathbb{R}^m$ for some orthogonal projection π.

Idea

- Avoid computing the whole secondary polytope $\Sigma(\mathcal{A})$.
- Incrementally construct $\mathit{\Pi}$ using an oracle that given a direction produces a vertex of $\mathit{\Pi}$, extremal wrt this direction.

Interpolation of resultants

$$f_0 = a_2x^2 + a_1x + a_0,\ \ f_1 = b_1x^2 + b_0$$

$N(\mathcal{R})$: convex hull of $(0,2,0,1,1)$, $(0,0,2,2,0)$, $(2,0,0,0,2)$.
Contains 1 additional lattice point: $(1,0,1,1,1)$.
Possible monomials of $\mathcal{R}$: $a_1^2b_1b_0,\ \ a_0^2b_1^2,\ \ a_2a_0b_1b_0,\ \ a_2^2b_0^2$

Using the 4 monomials and the Horn-Kapranov parameterization of $\mathcal{R}$:

$$a_2 = (2t_1 + t_2)t_3^2t_4,\ \ a_1 = (-2t_1 - 2t_2)t_3t_4,\ a_0 = t_2t_4,\ \ b_1 = -t_1t_3^2t_5,\ \ b_0 = t_1t_5,$$

construct a 4×4 matrix whose kernel vector $(1,1,-2,1)$ contains their coefficients:

$$\mathcal{R} = a_1^2b_1b_0 + a_0^2b_1^2 - 2a_2a_0b_1b_0 + a_2^2b_0^2$$

Implicitization of parametric (hyper)surfaces

$$(y_1, y_2, y_3) = (x_1x_2,\ x_1x_2^2,\ x_1^2).$$

$$f_0 := c_{00} - c_{01}x_1x_2,\ \ f_1 := c_{10} - c_{11}x_1x_2^2,\ \ f_2 := c_{20} - c_{21}x_1^2$$

$$\mathcal{R} = -c_{00}^4c_{11}^2c_{21} + c_{01}^4c_{10}^2c_{20},\ \ N(\mathcal{R}) = ((4,0,0,2,0,1),(0,4,2,0,1,0)) \subset \mathbb{R}^6$$

$$\text{Define } \pi:\ \ \pi(c_{00}, c_{01}, c_{10}, c_{11}, c_{20}, c_{21}) = (y_1, -1, y_2, -1, y_3, -1)$$

Then, $\pi(\mathcal{R}) = -y_1^4 + y_2^2y_3$ is the implicit equation and
$\Pi = \pi(N(\mathcal{R})) = ((4,0,0),(0,2,1))$ is the implicit polytope.

Note that here: $\pi(N(\mathcal{R})) = N(\pi(\mathcal{R}))$

Existing work

Theory

- Michiels & Verschelde [DCG'99] define equivalence classes of mixed subdivisions mapping onto $N(\mathcal{R})$ vertices.
- Michiels & Cools [DCG'00] describe all Minkowski summands of $\Sigma(\mathcal{A})$ including $N(\mathcal{R})$.

Algorithms

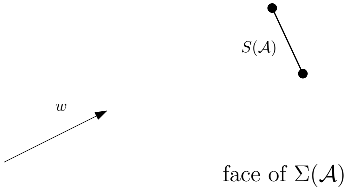
- TOPCOM [Rambau '02] computes all mixed subdivisions. Inefficient for resultant polytope computation.
- Tropical geometry [Sturmfels-Yu '08] leads to algorithms for the polytope of (specialized) resultant based on the `GFan` library [Jensen-Yu '11].

Vertex$\Pi(\mathcal{A}, w)$

Input: $\mathcal{A} \subset \mathbb{Z}^{2n}$, direction $w \in (\mathbb{R}^{|\mathcal{A}|})^{\times}$

Output: vertex $\in \Pi$, extremal wrt w

1. use w as a lifting to construct regular subdivision $S(\mathcal{A})$



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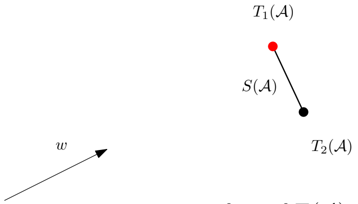

face of $\Sigma(\mathcal{A})$

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Output: vertex $\in \Pi$, extremal wrt w

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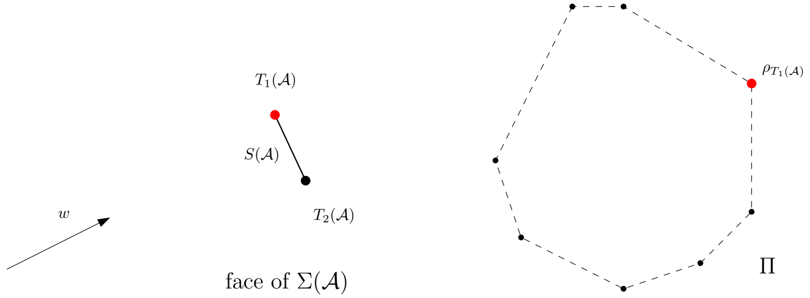
2. refine $S(\mathcal{A})$ into triangulation $T(\mathcal{A})$
3. return $\rho_{T(\mathcal{A})} \in \mathbb{N}^{|\mathcal{A}|}$

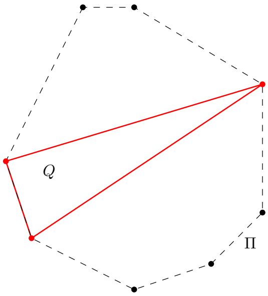


Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = \Pi$

1. initialization step



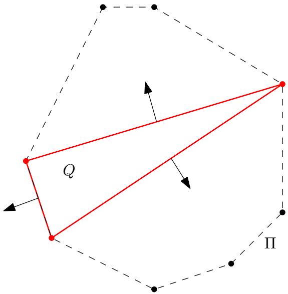
initialization:

- $Q \subset \Pi$
- $\dim(Q) = \dim(\Pi)$

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = \Pi$

1. initialization step
2. all hyperplanes of Q_H are illegal



2 kinds of hyperplanes of Q_H :

- legal if it supports a facet of Π
- illegal otherwise

Incremental Algorithm

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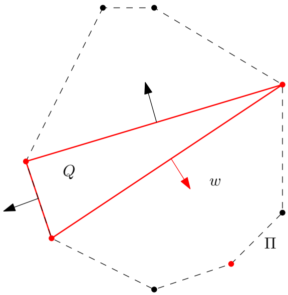
Output: H-rep. Q_H, V-rep. Q_V of $Q = \Pi$

1. initialization step
2. all hyperplanes of Q_H are illegal
3. `while` $\exists$ illegal hyperplane $H \subset Q_H$ with outer normal w do
   - compute $v = \text{Vertex}\Pi(\mathcal{A}, w)$, $Q_V \leftarrow Q_V \cup \{v\}$



Extending an illegal facet

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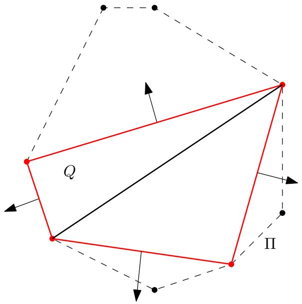

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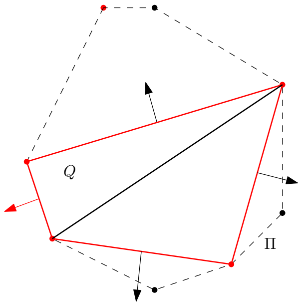

Validating a legal facet

Incremental Algorithm

Input: $\mathcal{A}$

Output: H-rep. Q_H, V-rep. Q_V of $Q = \Pi$

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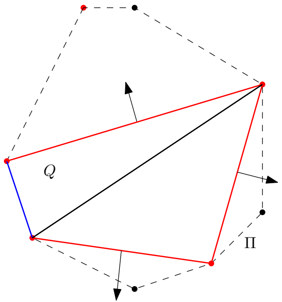

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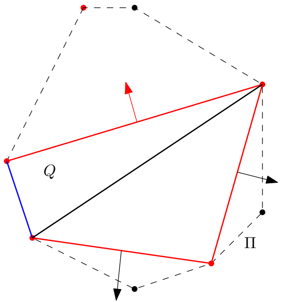

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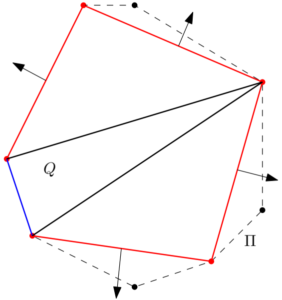

At any step, Q is an inner approximation of Π...

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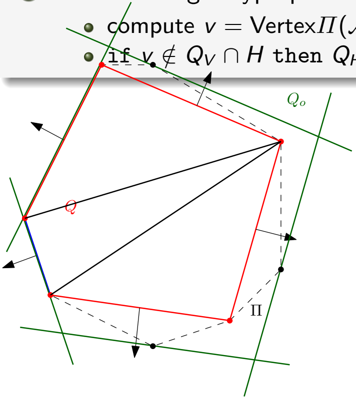

At any step, Q is an inner approximation of Π... from which we can compute an outer approximation Q_o.

Incremental Algorithm

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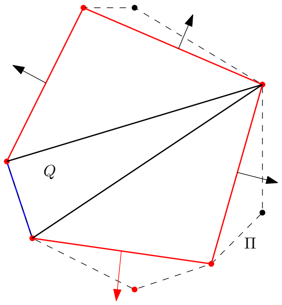

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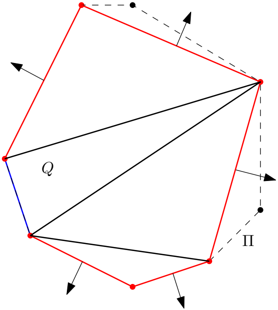

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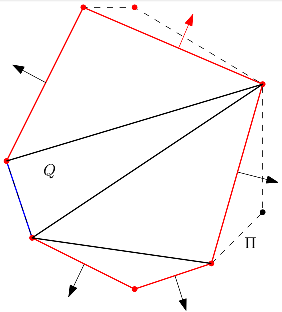

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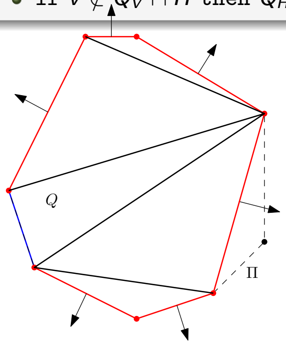

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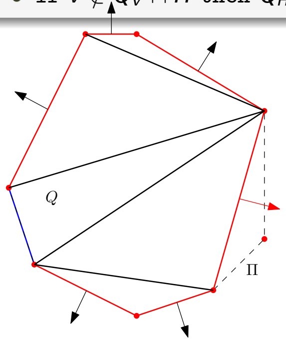

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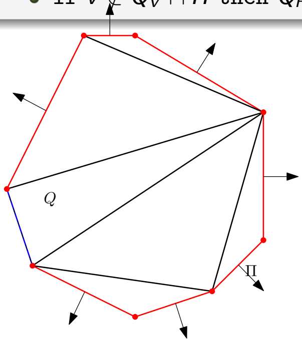

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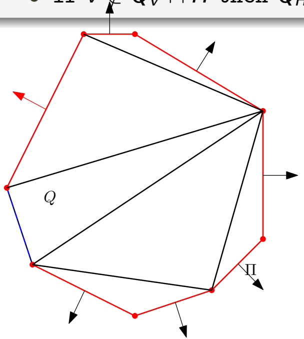

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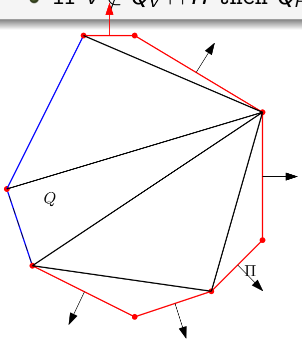

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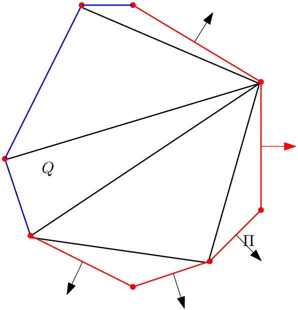

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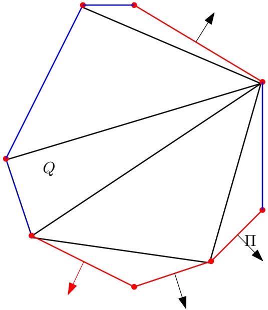

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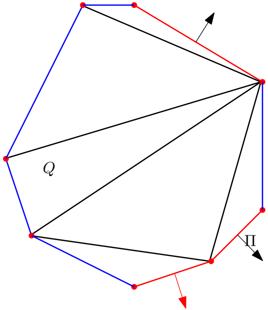

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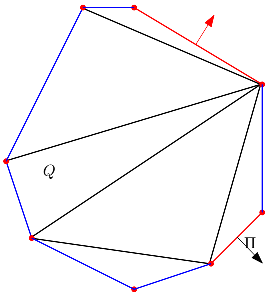

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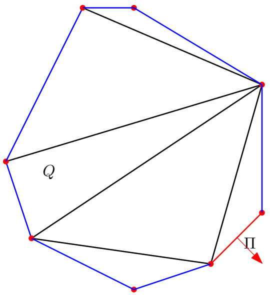

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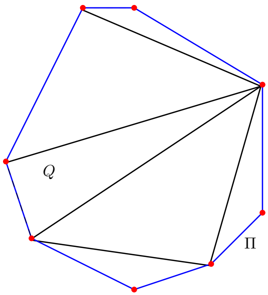


Complexity

Theorem

We compute the Vertex- and Halfspace-representations of Π, as well as a triangulation T of Π, in

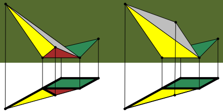
$$O^*(\, m^5\, |vtx(\Pi)| \cdot |T|^2),$$

where $m = \dim \Pi$, *and* $|T|$ *the number of full-dim faces of* T.

Elements of proof

- Most computation is done in dimension $\leq m$.
- At most $\leq \mathrm{vtx}(\Pi) + \mathrm{fct}(\Pi)$ oracle calls
- Beneath-Beyond algorithm for converting V-rep. to H-rep. (bottleneck)

ResPol Implementation



Tools

C++, CGAL, `triangulation` [Boissonnat,Devillers,Hornus], `extreme_points_d` [Gärtner]

Hashing of determinantal predicates

optimizing sequences of similar determinants

$$\widetilde{\mathcal{A}} = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{vmatrix}$$

- Laplace (cofactor) expansion wrt the last row + Hash minors
- If all needed minors computed, orientation$= O(n^2)$, volume$= O(n)$

Future work

- approximate resultant polytopes ($dim(\Pi) \geq 7$)
- preliminary results:

input	m	3	3	4	4	5	5
	$\|\mathcal{A}\|$	200	490	20	30	17	20
exact	$\#$vtx(Π)	98	133	416	1296	1674	5093
	time	2.03	5.87	3.72	25.97	51.54	239.96
approx.	$\#$vtx(Q_{in})	15	11	63	121	–	–
	vol(Q_{in})/vol(Π)	0.96	0.95	0.93	0.94	–	–
	vol(Q_{out})/vol(Π)	1.02	1.03	1.04	1.03	–	–
	time	0.15	0.22	0.37	1.42	$>$ 10hr	$>$ 10hr

- approximate volume computation [Lovsz-Vempala06]

References

The code

- http://respol.sourceforge.net

The paper

- http://arxiv.org/abs/1108.5985v2 (to appear in SOCG'12)

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Thank You !