Research Statement

I am interested in analysis, calculus of variations and geometric measure theory, with applications to partial differential equations (PDE) and mathematical physics.

1 Numerical analysis and PDE

In my B.S. program in Vietnam National University at Ho Chi Minh City, I learned analysis and its application in partial differential equations. In particular, I was interested in the ill-posed problems in classical mechanics, where a small error of data may introduce a large error of solutions. The aim is to find a regularization process, in which from any approximate datum we construct a numerical solution approximating the expected (but *unknown*) solution.

A typical example is the backward heat problem, where we are given the final temperature $u(T)$ and we have to recover the previous temperature $u(t)$ $(0 \le t < T)$. This problem was considered in my B.S. thesis. Another example is the problem of finding the heat source, given the final temperature. More precisely, we want to find a pair functions (u, f) satisfies the heat equation

$$\begin{cases} \partial_t u - \Delta_x u = \varphi(t) f(x), \ (t,x) \in (0,T) \times \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \, u|_\Gamma = 0, u(x,T) = g(x), \end{cases}$$

where Ω is an open bounded subset in $\mathbb{R}^d$ and Γ is an open subset of $\partial\Omega$, and the data (φ, g) is given approximately. By applying Fourier transform and an interpolation inequality of entire functions, we were able to construct an approximate solution from the approximate data. This joint work with Dang Duc Trong, Alain Pham and Phan Thanh Nam was published in [23] (it was previously announced in [22]).

2 Financial mathematics

I am also interested in financial mathematics. After graduating B.S., I spent one year to work in Prudential Vietnam, an insurance company. My job was of pricing products and analyzing the actual data to forecast the risks and profitability.

This was a chance for me to learn more about the probability and mathematical finances (to pass two exams: Probability and Financial Mathematics, among the total five exams to become an ASA-Associate of the Society of Actuaries). Moreover, I found the skill of modeling problems very useful; for example I used Prophet to model and manage risks.

In this time, we found by chance a beautiful property of power means. Given two positive number x, y and a number $t \in (0,1)$, we can define the power means

$$M(p) := (tx^p + (1-t)y^p)^{1/p}.$$

This function relates to the elasticity of substitution in economy. Olivier de La Grandville and Robert M. Solow [5] conjectured that the function $p \mapsto M(p)$ has one and only one inflection point; moreover, between its limiting values, $M(p)$ is in a first phase convex and then turns concave. Those authors considered the property as a key tool to prove the benefits of a high elasticity of substitution, which relates to the celebrated 1956 contribution to economic growth theory by Robert M. Solow (which gained him the 1987 Nobel Prize in Economics), but they could not give a proof of the conjecture. Phan Thanh Nam and I found a mathematical proof [19] of this conjecture. The application of this result in economics is explained in details in a recent article of O. de La Grandville [6].

3 Weak solutions of rate-independent systems

In my Ph.D. at Università di Pisa, I considered evolutions of rate-independent systems. I will describe some problems I have been working on.

We consider the system including an initial position x_0 belongs to some normed vector space X, an energy functional $\mathscr{E} : [0,T] \times X \to [0,+\infty)$, which is usually non-convex and non-smooth, and a dissipation effect $\psi : X \to [0,+\infty)$, which is supposed to be positively 1-homogeneous, i.e. $\psi(\lambda v) = \lambda\psi(v)$ for all $\lambda > 0$. The evolution $x(t)$ of this system can be illustrated by the following sub-differential inclusion

$$0 \in \nabla_x \mathscr{E}(t, x(t)) + \partial\psi(\dot{x}(t)) \text{ for a.e. } \ t \in (0,T) \tag{1}$$

In general, if the energy functional $\mathscr{E}$ is not convex, then the equation (1) may have no strong solution [20]. Hence, the question on how to define suitable weak solutions to (1) naturally arises. The first answer was introduced in [14] by Mielke and Theil for shape-memory alloys via the concept of **energetic solution**, and then was developed in many specific rate-independent systems as well as in the general case, see [15, 8, 4, 10]. It is well-known that in the simple model, where the evolution $x(t)$ is a point in a Hilbert space X, ψ is the distance, and $\mathscr{E}$ satisfies some reasonable smoothness conditions, the system has an energetic solution $x(t)$ which satisfies the global stability condition (S)

$$\mathscr{E}(t, x(t)) \le \mathscr{E}(t, x) + \psi(x - x(t)) \ \text{ for all } t \in [0,T] \text{ and for all } x \in X$$

and the energy - dissipation balance (E)

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, x(t))\, dt - \mathrm{Diss}(x(\cdot); [t_1, t_2]) \text{ for all } 0 \le t_1 \le t_2 \le T,$$

here

$$\mathrm{Diss}(x(\cdot); [a,b]) := \sup\{\sum_{n=1}^{N} \psi(x(t_n) - x(t_{n-1})) | \text{ for all } N \text{ and } a = t_0 \le t_1 \le \cdots \le t_N = b\}.$$

Energetic solution is applied widely in many mathematical frameworks as well as in physical models. However, in some situations the requirement on the global stability is so strong that the energetic solutions would jump sooner than strong solutions (see [11, Example 2.3]). Therefore, the models that deal with *local minimization* are preferred.

In general, any solution using the local minimization (for short we call it *weak solution*) should satisfy at least the following two conditions.

(Local stability) For all $t \in (0,T)$, if $x(\cdot)$ is continuous at t, then

$$|\nabla_x \mathscr{E}(t, x(t))| \le 1. \tag{2}$$

(Energy-Dissipation upper bound) For all $0 \le t_1 \le t_2 \le T$,

$$\mathscr{E}(t_2, x(t_2)) - \mathscr{E}(t_1, x(t_1)) \le \int_{t_1}^{t_2} \partial_t \mathscr{E}(s, x(s))\, ds - \mathrm{Diss}(x; [t_1, t_2]). \tag{3}$$

Remark. (i) The above definition of weak solutions contains energetic solutions [15], BV solutions [13], ε-neighborhood solutions [16], parametrized solutions [3], and ε-stable solutions [7].

(ii) The condition (2) is a consequence of the following stronger condition

(Stronger local stability) For all $t \in (0, T)$, if $x(\cdot)$ is continuous at t, then $z = x(t)$ is a local minimizer of the functional $z \mapsto \mathscr{E}(t, z) + |z - x(t)|$. (4)

In below I will discuss about two approaches of weak solutions: BV solutions constructed by vanishing viscosity and BV solutions constructed by epsilon-neighborhood method.

4 BV solutions constructed by vanishing viscosity

One way to deal with the local minimization is to add a viscosity into the dissipation function. This idea was first mentioned in [3], and then was developed in [13] for finite-dimensional normed spaces. More precisely, we consider the system with the viscous term

$$0 \in \nabla_x \mathscr{E}(t, x(t)) + \partial \psi_\varepsilon(\dot{x}(t)) \text{ for a.e. } t \in (0, T) \tag{5}$$

where the viscous term ψ_ε is convex and super-linear, and Ψ_ε converges to ψ in an appropriate sense as $\varepsilon \to 0$. For example, $\psi(v) = |v|$ and

$$\psi_\varepsilon(v) = |v| + \frac{\varepsilon}{2}|v|^2.$$

The existence of an absolutely continuous solution u^ε to (5) was proved by Colli and Visintin [2, 1]. Moreover, Mielke, Rossi and Savaré [13] showed that the limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ solves problem (1) and it satisfies the local stability in (2) and the new energy-dissipation balance

$$\mathscr{E}(t_2, u(t_2)) - \mathscr{E}(t_1, u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(t, u(t))\, dt - \mathrm{Diss}_{new}(u(\cdot); [t_1, t_2]) \text{ for all } 0 \le t_1 \le t_2 \le T, \tag{6}$$

where

$$\mathrm{Diss}_{\mathrm{new}}(u(\cdot); [a, b]) := \mathrm{Diss}(u(\cdot); [a, b]) + \sum_{t \in J} (\Delta_{\mathrm{new}}(t) - \Delta(t)),$$

with J being the jump set of $u(t)$, $\Delta(t) := |u(t^+) - u(t^-)|$ being the jump steps, and

$$\Delta_{\mathrm{new}}(t) = \inf \left\{ \int_0^1 |\dot{v}(r)| \cdot \max\{1, |\nabla_x \mathscr{E}(t, v(r))|\}\, dr, v \in \mathrm{AC}([0,1]), v(0) = u(t^-), v(1) = u(t^+) \right\}.$$

Notice that $\mathrm{Diss}_{new}(u(\cdot); [a, b]) \ge \mathrm{Diss}(u(\cdot); [a, b])$. Hence, BV solutions also satisfy the energy-dissipation upper bound (3).

There are examples [11] in which the jumps of BV solutions are physically more plausible than those of energetic solutions. However, the definition of BV solution still has some drawbacks. First, we see that to construct BV solutions, one must add the viscous term into the dissipation. And it was pointed out by Stefanelli in [20] that the different viscous terms may produce different solutions. In other words, there are examples showing that BV solutions depend on viscosity (see also in [18]). Secondly, in some situation, BV solutions are stable in very weak sense, i.e. they satisfy the local stability (2), but not the strong local stability (4). Moreover, in [16] we have examples showing that with some specific viscosity, BV solutions jumps later than we expect and in these cases, local minimality is violated.

5 BV solutions constructed by epsilon-neighborhood

The idea is to use time-discretization and restrict the minimizer $x_n = x(t_n)$ of the discretized problem in some small neighborhood of order ε

$$x_n \in \operatorname{argmin}\{\mathscr{E}(t_n, x) + \psi(x - x_{n-1}), \|x - x_{n-1}\| \leq \varepsilon\}. \tag{7}$$

By the time-discretization, we have a sequence $\{x^\varepsilon(t)\}$ of solutions of (7). Then, letting $\varepsilon \to 0$, the limit will be the solution of (1) corresponds to local minimizer.

This approach was discussed in [9] (Section 6) for one-dimensional case (when ε is chosen proportional to the square root of the time-step). The main result of my work [16] is to show that the solutions constructed by this method in fact satisfies the local stability and the new energy-dissipation balance.

Theorem 1 (BV solutions constructed by epsilon-neighborhood method)**.** *Let $\mathscr{E} : [0,T] \times \mathbb{R}^d \to [0, +\infty]$ be of class C^1 and satisfy*

$$|\partial_t \mathscr{E}(s,x)| \leq \lambda\, \mathscr{E}(s,x) \text{ for all } (s,x) \in [0,T] \times \mathbb{R}^d. \tag{8}$$

Let an initial datum $x_0 \in \mathbb{R}^d$ be such that x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. Then the following statements hold true.

(i) (Discrete solution) For any $\varepsilon > 0$ and $\tau > 0$, there exists a discretized solution $t \mapsto x^{\varepsilon,\tau}(\cdot)$ as described above.

(ii) (Epsilon-neighborhood solution) For any $\varepsilon > 0$ fixed, there exists a subsequence $\tau_n \to 0$ such that $x^{\varepsilon,\tau_n}(\cdot)$ converges pointwise to some limit $x^\varepsilon(\cdot)$. The function $x^\varepsilon(\cdot)$ satisfies

- *(Epsilon local stability) If $x^\varepsilon(\cdot)$ is right-continuous at t, namely $\lim_{t' \to t^+} x^\varepsilon(t') = x^\varepsilon(t)$, then $x^\varepsilon(t)$ satisfies the epsilon local stability*

$$\mathscr{E}(t, x^\varepsilon(t)) \leq \mathscr{E}(t,x) + |x - x^\varepsilon(t)| \quad \text{for all } |x - x^\varepsilon(t)| \leq \varepsilon. \tag{eps-LS}$$

- *(Energy-dissipation inequalities) We have $\mathrm{Diss}(x^\varepsilon; [0,T]) \leq C$ (independent of ε), $\partial_t \mathscr{E}(\cdot, x^\varepsilon(\cdot)) \in L^1(0,T)$ and for all $0 \leq s \leq t \leq T$,*

$$-\mathrm{Diss}_{new}(x^\varepsilon; [s,t]) \leq \mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t \mathscr{E}(r, x^\varepsilon(r))\, dr \leq -\mathrm{Diss}(x^\varepsilon; [s,t]).$$

(iii) (BV solution constructed by epsilon-neighborhood method) There exists a subsequence $\varepsilon_n \to 0$ such that x^{ε_n} converges pointwise to some BV function u. The function u satisfies

- *(Weak local stability) If $t \mapsto u(t)$ is continuous at t, then*

$$|\nabla_x \mathscr{E}(t, u(t))| \leq 1.$$

- *(New energy-dissipation balance) For all $0 \leq s \leq t \leq T$, one has*

$$\mathscr{E}(t, u(t)) - \mathscr{E}(s, u(s)) = \int_s^t \partial_t \mathscr{E}(r, u(r))\, dr - \mathrm{Diss}_{new}(u; [s,t]).$$

Intuitively, BV solutions constructed by epsilon-neighborhood coincide to strong solutions up to strong solutions exist. Moreover, if we choose the neighborhood to be the closed ball of radius ε, the corresponding solutions will follow the path of steepest descent in the energy (plus dissipation) landscape.

Problem 1. Find a stronger version of the local stability for BV solutions constructed by epsilon-neighborhood method.

Problem 2. The existence of BV solutions constructed by epsilon-neighborhood method for capillary drops.

6 Regularity

Although as a-priori, energetic solutions are BV functions, in some certain situations a better smoothness of the solutions is expected. When the energy functional is convex, the regularity was already considered by Mielke, Rossi and Thomas [10, 12, 21]. However, if the energy functional is not convex, there are very few results on the regularity of the energetic solutions.

The first question one may ask is about the jump set of the solutions. Since each solution has bounded variation, the number of jumps is at most countable. But can we know something more?

Problem 3. Under which conditions, the jump set is not dense? Or is finite? Or is empty?

Notice that, without the convexity assumption, these questions are not obvious to answer. In [17], we show that for some smooth energy functionals, the corresponding energetic solutions may have *dense jumps*. However, in one dimension we have in [17] some conditions to guarantee that the jump set of an energetic solution is finite.

Theorem 2. *In one dimension, if $\mathscr{E}$ is C^2, ψ is the usual distance, and the set $\{(t,x) : \partial_x \mathscr{E}(t,x) = \pm 1, \partial_{xx}\mathscr{E}(t,x) = 0\}$ is empty, then any weak solution (in particular, energetic solution and BV solution) has finitely many jumps.*

As a consequence, in one dimension, if $\mathscr{E}$ is C^2, ψ is the usual distance, and for all t, the distance between two elements of the set $\{x : \partial_x \mathscr{E}(t,x) = \pm 1\}$ is always greater than some constant $C > 0$ (C does not depend on t), then any weak solution has finitely many jumps.

The next question one may ask is about the smoothness of the solutions. In [17] we prove that any increasing function is an energetic solution of some smooth energy functional. Hence, in general an energetic solution may be not SBV. In [17] we provide some conditions so that energetic solutions, or even more generally, weak solutions to (1) is SBV. More precisely, we call $x : [0, T] \to \mathbb{R}$ a weak solution to (1) if $x(\cdot)$ has bounded variation and it satisfies the local stability (2) and the energy-dissipation inequality (3). Then we have (see [17])

Theorem 3. *Assume that $x(\cdot)$ is a BV function satisfying (2) and (3). If $\mathscr{E}(t, x)$ is of class C^3 and the set*

$$\{(t, x) \in (0, T) \times \mathbb{R} \,|\, \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x) = \partial_{xxx} \mathscr{E}(t, x) = 0\}$$

is countable, then $x(\cdot)$ is of SBV class.

The SBV regularity can be generalized in multi-dimensions as following.

Theorem 4. *Assume that $x : [0, T] \to \mathbb{R}^d$ is a BV function satisfying (2) and (3). If $\mathscr{E}(t, x)$ is of class $C^3(\mathbb{R}^{d+1})$ and the set*

$$\{(t, x) \in (0, T) \times \mathbb{R}^d \,|\, |\nabla_x \mathscr{E}(t, x)| = 1, F(t, x) = (\nabla_x \mathscr{E}(t, x)) \cdot (\nabla_x F(t, x)) = 0\}$$

is countable, then $x(\cdot)$ is of SBV class, here the function $F(t, x)$ is defined by

$$F(t, x) := (\nabla_x \mathscr{E}(t, x)) \cdot H(t, x) \cdot (\nabla_x \mathscr{E}(t, x))^T$$

and the Hessian matrix $H(t, x)$ is defined by

$$[H(t, x)]_{ij} := (\partial_{x_i} \partial_{x_j} \mathscr{E})(t, x).$$

The conditions in Theorem 4 are acceptable when $d = 2$ and $d = 3$. However, in higher dimensions, these conditions turn out to be too strong.

Problem 4. Find a weaker condition to guarantee the SBV regularity for energetic solutions - BV solutions - weak solutions in finite-dimension and infinite-dimension.

Another problem that I am interested is about the differentiability of solutions to (1). It has been showed in [17] that if $x(\cdot)$ has finitely many jumps, then $x(\cdot)$ can be expected to be piecewise differential. By technical reasons, in the following we shall replace condition (2) by the stronger condition (4).

Theorem 5. *Assume that $x(\cdot)$ is a BV function satisfying (3) and (4) and $x(\cdot)$ has only finitely many jump points. If $\mathscr{E}(t, x)$ is of class C^3 and the set*

$$\{(t, x) \in (0, T) \times \mathbb{R} \,|\, \partial_x \mathscr{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathscr{E}(t, x) = \partial_{xxx} \mathscr{E}(t, x) = 0\}$$

has only finitely many elements, then $x(\cdot)$ is differentiable except a set $I = I_1 \cup I_2$, where I_1 is a set of isolated points t such that either $x'_-(t) = 0$ or $x'_+(t) = 0$, and I_2 is a set of isolated points t such that $x'_+(t)$, $x'_-(t)$ exist and satisfy

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

Moreover, piecewise C^1 result is obtained under some stronger conditions on $\mathscr{E}$, see Theorem 3 in [17] for details. However, the techniques used in [17] seem rather specific for one dimension. The similar results for finite and infinite dimensions should be the subject of a future work.

Problem 5. Differentiability of weak (or energetic or BV) solutions in finite-dimension? And in infinite-dimension?

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