Vortex motion in models for thin-film ferromagnets

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Domain microstructure and dynamics in magnetic elements
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Overview

Joint work with C. Melcher (Aachen), R. Moser (Bath), D. Spirn (Minnesota)

1. Vortices
2. Energetics of thin-film ferromagnets
3. Thin-film models
4. Motion laws for vortices
5. External fields and currents
Vortices

- Ferromagnets: Magnetization $\mathbf{m}$ is vector field with $|\mathbf{m}| = 1$
- Magnetostatic energy prefers $\mathbf{m}$ to be almost in-plane, tangential at boundary
- Typical states: vortices
- Winding number 1, direction of tangent, polarity up/down

Figure 3: Permalloy film with circular cross-section. Reproduced with permission from Hubert and Schäfer, Magnetic Domains, Springer 1998

Figure 4: A cross-tie wall in a Permalloy film. Reproduced with permission from Hubert and Schäfer, Magnetic Domains, Springer 1998
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Energetics of a ferromagnet

Ferromagnet: $G \subset \mathbb{R}^3$ ferromagnetic body, $\mathbf{m} : G \to \mathbb{R}^3$ its magnetization.

Energy of a magnetization:

$$E(\mathbf{m}) = w^2 \int_G |\nabla \mathbf{m}|^2 + Q \int_G \varphi(\mathbf{m}) + \int_{\mathbb{R}^3} |\nabla U|^2 - 2 \int_G \mathbf{h}_{\text{ext}} \cdot \mathbf{m}$$

Constraints: $|\mathbf{m}| = 1$ in $G$, $\Delta U = \text{div}(\chi_G \mathbf{m})$ in $\mathcal{D}'(\mathbb{R}^3)$.

- $w$: exchange length
- $\varphi$: anisotropy function
- $Q$: quality factor
- $\mathbf{h}_{\text{ext}}$: applied external field

We will use: $Q = 0$ (soft material), $\mathbf{h}_{\text{ext}} = 0$ for most of the talk.
The magnetostatic energy

Magnetostatic energy term is nonlocal (field induced in all of $\mathbb{R}^3$):
$$\int_{\mathbb{R}^3} |\nabla U|^2 \text{ for } \Delta U = \text{div}(\chi_G \mathbf{m})$$

- $\mathbf{h} = -\nabla U$ induced field, $\text{curl} \mathbf{h} = 0$ (static Maxwell equations)
- Fourier picture: $\widehat{\nabla U} = \frac{\xi \otimes \xi}{|\xi|^2} \hat{\mathbf{m}}$
- Distributional divergence has two parts: $\text{div} \mathbf{m}$ inside $G$ and jump part $\mathbf{m} \cdot \nu$ on $\partial G$
- $$\int_{\mathbb{R}^3} |\nabla U|^2 = - \int_G \mathbf{m} \cdot \nabla U = \int_G U \text{div} \mathbf{m} - \int_{\partial G} U \mathbf{m} \cdot \nu$$
Thin film approximation

If \( G = \Omega \times (0, h), \ h \ll 1 \):

- \( \partial_3 \mathbf{m} \to 0 \) if energy not too large; assume
  \[ \mathbf{m} = \mathbf{m}(x_1, x_2) \chi_{(0,h)}(x_3) \]

- Write \( \mathbf{m} = (m, m_3) \), solve \( \Delta U = \text{div}(\chi_G \mathbf{m}) \) then

\[
\int_{\mathbb{R}^3} |\nabla U|^2 \approx h \int_{\Omega} m_3^2 + h^2 |\log h| \int_{\partial \Omega} (m \cdot \nu)^2 + h^2 \| \text{div} \ m \|^2_{H^{-\frac{1}{2}}}
\]

- proofs of asymptotic behavior Gioia-James, Carbou, Kohn-Slastikov, De Simone-Kohn-Müller-Otto, Moser, . . .
A model for boundary vortices

One model problem: \( m \in H^1(\Omega; S^1) \) in plane, consider

\[
E_\varepsilon(m) = \frac{1}{2} \int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2
\]

- As core size \( \varepsilon \to 0 \): Convergence to maps with two boundary singularities at \( a_1, a_2 \in \partial\Omega \)
- Energy expansion \( E_\varepsilon(m_\varepsilon) = \pi |\log \varepsilon| + C_0 + W(a_1, a_2) \) K. 2006
- direct derivation from full micromagnetic energy Ignat-K. 2013
A model for interior vortices

Consider \( m : \Omega \to S^2 \), \( m = \tau \) on \( \partial \Omega \), \( \tau \) tangent field (of degree 1) and energy

\[
E(m) = \frac{1}{2} \int_{\Omega} |\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2
\]

- similar to Ginzburg-Landau functional for \( u : \Omega \to \mathbb{C} \)

\[
\frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2
\]

- analyzed by Hang-Lin based on Bethuel-Brezis-Hélein

- Energy expansion \( E_\varepsilon(m_\varepsilon) = \pi |\log \varepsilon| + C_0 + W(a) \)

- \( W(a) \) depends on position; in disk: \( W(a) = \pi \log \frac{1}{1-|a|^2} \)
Energy expansion for the vortex model

\[ E_\varepsilon(m) = \frac{1}{2} \int_\Omega |\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2 \]

- For small \( \varepsilon \), minimizer \( m_\varepsilon(z) = (m(z), \pm \sqrt{1 - |m(z)|^2}) \) with 
  \( m(z) = e^{i\psi(z)} \rho\left( \frac{z-a}{\varepsilon} \right) \frac{z-a}{|z-a|} \)
- \( m_*(z) = e^{i\psi(z)} \frac{z-a}{|z-a|} \), \( \psi \) harmonic
- renormalized energy: \( W(a) = \lim_{r \to 0} \frac{1}{2} \int_{\Omega \setminus B_r(a)} |\nabla m_\ast|^2 - \pi |\log r| \)
- If \( J(m) \approx \pi \delta_a \) then energy excess
  \[ D_\varepsilon(m; a) = E_\varepsilon(m) - W(a) - \pi |\log \varepsilon| - \gamma \geq 0 \]
- coercivity: \( D_\varepsilon \) controls convergence of \( m_\varepsilon \to m_* \) away from \( a \)
Dynamics: Landau-Lifshitz-Gilbert equation

Evolution equation:

\[ \alpha \partial_t \mathbf{m} + \mathbf{m} \times \partial_t \mathbf{m} = -\text{proj}_{T_m S^2} \nabla E(\mathbf{m}) \]

- damped precessional motion of \( \mathbf{m} \) around \( \mathcal{H}_{\text{eff}} = -\nabla E(\mathbf{m}) \)
- even without precession, problems with long-time existence and uniqueness (bubbling off of harmonic spheres in harmonic map heat flow)
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Dynamics for the boundary vortex model

LLG for boundary vortex model $m : \Omega \to S^1$, 

$$E_\varepsilon(m) = \frac{1}{2} \int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon} \int_{\partial \Omega} (m \cdot \nu)^2$$

reduces to gradient flow (no precession in $S^1$) Kohn-Slastikov

$$\alpha_\varepsilon \partial_t m = -\text{proj}_{T_m S^2} \nabla_{L^2} E_\varepsilon(m)$$

• can see vortex motion for $\alpha_\varepsilon = \frac{1}{|\log \varepsilon|}$
• vortices move with gradient flow of renormalized energy: 
  $$\dot{a}_j = -\frac{2}{\pi} \nabla_{a_j} W(a_1, a_2)$$
• proof using Sandier-Serfaty method of $\Gamma$-convergence of gradient flows: K. 2007 (steepest descent for PDE $\rightsquigarrow$ steepest descent for ODE)
Dynamics for interior vortices

Huber 1982: vortex should observe a Thiele equation (vortex center as collective coordinate)

\[ \pi(\dot{a} + 2qa^\perp) = -\nabla W(a) \]

if damping \( \alpha \approx \frac{1}{|\log \varepsilon|} \) (formal calculations), \( q = \pm 1 \) polarity

Theorem (K.-Melcher-Moser-Spirn, ARMA 2011)

Given well-prepared initial data, the LLG equations for the model problem have smooth solutions for all time. Vorticity \( \omega \) and energy density \( \alpha_\varepsilon e_\varepsilon \) converge as \( \varepsilon \to 0 \) to delta masses supported at the vortex centers. The vortex centers follow the Thiele-Huber ODE.
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Towards a proof: fundamental quantities

Set $f_\varepsilon(m) = \Delta m + |\nabla m|^2 m - \frac{1}{\varepsilon^2} (m_3 e_3 - m_3^2 m)$ then

$$\alpha_\varepsilon \partial_t m + m \times \partial_t m = f_\varepsilon$$

Can consider

- "spin": where is $m_3 = \pm 1$?
- energy density $e_\varepsilon(m) = \frac{1}{2} |\nabla m|^2 + \frac{1}{2\varepsilon^2} m_3^2 \approx \pi |\log \varepsilon| \delta_a$
- magnetic vorticity $\omega(m) = \langle m, (\partial_x m \times \partial_y m) \rangle \approx 2\pi \delta_a$
- in-plane Jacobian $J(m) = \partial_1 m_1 \partial_2 m_2 - \partial_2 m_1 \partial_1 m_2 \approx \pi \delta_a$
Evolution of fundamental quantities

\[ \alpha_\varepsilon \partial_t \mathbf{m} + \mathbf{m} \times \partial_t \mathbf{m} = \mathbf{f}_\varepsilon \]

Energy density (multiply with \( \partial_t \mathbf{m} \))

\[ \partial_t e_\varepsilon(\mathbf{m}) + \alpha_\varepsilon |\partial_t \mathbf{m}|^2 = \text{div} \left\langle \partial_t \mathbf{m}, \nabla \mathbf{m} \right\rangle \]

Vorticity (multiply with \( \nabla \mathbf{m} \) and take the curl):

\[ \partial_t \omega(\mathbf{m}) = \text{curl div}(\nabla \mathbf{m} \otimes \nabla \mathbf{m}) - \alpha_\varepsilon \text{curl} \left\langle \partial_t \mathbf{m}, \nabla \mathbf{m} \right\rangle \]

- \( \text{div}(\nabla \mathbf{m} \otimes \nabla \mathbf{m}) \) related to \( \nabla \mathbf{W} \) (error bounded by \( D_\varepsilon \))
- Convergence of \( \left\langle \partial_t \mathbf{m}, \nabla \mathbf{m} \right\rangle \)?
- Can avoid this term by cancellation using good test functions that behave like \( x \) and \( x^\perp \) near the vortex (proof of KMMS11)
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Energy density (multiply with \( \partial_t \mathbf{m} \))

\[ \partial_t e_\varepsilon(m) + \alpha_\varepsilon |\partial_t \mathbf{m}|^2 = \text{div} \left( \langle \partial_t \mathbf{m}, \nabla \mathbf{m} \rangle \right) \]

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Remarks

- Problem: bubbles where $\omega \approx 4\pi \delta_b$ (full cover), but $J \approx \pi \delta_{b+} - \pi \delta_{b-} \approx 0$ in $W^{-1,1}$ (dipole)
- typical bubble energy is $4\pi$ (not divergent as $\varepsilon \to 0$)
- can rule out bubbling if excess energy $D_\varepsilon(m_\varepsilon(t); a(t))$ is $\ll 4\pi$ for all times
- can avoid cancellation approach using convergence results for $\langle \partial_t m, \nabla m \rangle$ K.-Melcher-Moser 2011
- evolution identity for spin is used to show that $\text{div} \langle m^\perp, \nabla m \rangle \to 0$ in spacetime
Convergence results (KMM 2011)

There is a curve $a(t) \in H^1$ such that $\alpha_\varepsilon e_\varepsilon (m_\varepsilon (t)) \to \pi \delta_{a(t)}$ and $\omega (m_\varepsilon (t)) \to 2\pi \delta_{a(t)}$. For any $\eta \in C^1(\Omega)$,

$$\pi (\eta (a(t_1)) - \eta (a(t_2))) = \lim_{\varepsilon \to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_{t_1}^{t_2} \int_{\Omega} \nabla \eta \cdot \langle \partial_t m_\varepsilon, \nabla m_\varepsilon \rangle$$

$$\pi \int_{t_1}^{t_2} |\dot{a}|^2 \leq \liminf_{\varepsilon \to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_{t_1}^{t_2} \int_{\Omega} |\partial_t m_\varepsilon|^2$$

$$\pi \text{id}_{2 \times 2} \int_{t_1}^{t_2} \eta (a(t)) = \lim_{\varepsilon \to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_{t_1}^{t_2} \int_{\Omega} \eta \nabla m_\varepsilon \otimes \nabla m_\varepsilon$$

Main idea for proof: test energy identity with time-dependent $\xi(t, x) = \dot{a}(t) \cdot (x - a(t)) \chi(x - a(t))$ for a cutoff function $\chi$

Quantitative convergence rate and equipartition K.-Spirn 2010
Proof via evolution identity for vorticity

Multiply equation with $\nabla m$ and take the curl:

$$\partial_t \omega(m) = \text{curl} \left( \text{div}(\nabla m \otimes \nabla m) - \langle \alpha \varepsilon \partial_t m, \nabla m \rangle \right)$$

Integrate against a test function $\phi$ with $\phi(x) = x^\perp$ near the vortex and pass to the limit. Formally (up to small error terms):

- $\int_{\Omega} \partial_t \omega(m) \phi \to 2\pi \dot{a}^\perp$
- $\int_{\Omega} \nabla \nabla^\perp \phi : (\nabla m \otimes \nabla m) \to -\pi \nabla^\perp \phi \cdot \nabla a W = -\pi \nabla a W$
- Using $\psi(x) = x$ so $\nabla \psi = \nabla^\perp \phi$:
  $$\int_{t_1}^{t_2} \int_{\Omega} \alpha \varepsilon \nabla \psi \cdot (\partial_t m, \nabla m) \to -\pi(a(t_2) - a(t_1)); \text{ differentiate}$$
- Combine this to obtain motion law!
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- Combine this to obtain motion law!
Control of the error terms

- Let $a(t) =$ location of concentration of $\omega(m(t))$
- Let $\hat{a}(t) =$ solution of ODE
- Growth of equation error $|\hat{a} - \dot{a}|$ can be bounded by $D_\varepsilon$
- Growth of excess $\frac{d}{dt} D_\varepsilon$ can be controlled (via ODE and control of kinetic energy) by $|\hat{a} - \dot{a}|$
- Gronwall: If $D_\varepsilon(0) \to 0$ and $a(0) = \hat{a}(0)$ then $D_\varepsilon(t) \to 0$ and $\hat{a}(t) = a(t)$ for all times
Moving vortices out of equilibrium

Can we affect vortex motion by forces?

- **external (spin-polarized) current**: replace $\partial_t$ by $\partial_t + \gamma v \cdot \nabla$ (possibly different $\gamma$'s for gradient and Schrödinger term)
  - for small $v$ does not lead away from well-preparedness
  - need additional convergence result K.-Melcher-Moser 2011

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2\pi \int_{t_1}^{t_2} v^\perp \cdot \dot{a} = - \lim_{\varepsilon \to 0} \int_{t_1}^{t_2} \int_\Omega \langle m_\varepsilon \times (v \cdot \nabla) m_\varepsilon, \partial_t m_\varepsilon \rangle
\]

- **external magnetic field**: add $- \int_\Omega h(t) \cdot m$ to energy
  - changes renormalized energy $W$
  - for small fields still have coercivity
  - K.-Melcher-Moser 2012
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Some remarks about the problem with applied field
K.-Melcher-Moser 2012

- Energy
  \[ E_{\varepsilon}(h; m) = \int_{\Omega} \frac{1}{2} |\nabla m|^2 + \frac{m^2}{\varepsilon^2} - h \cdot m \]

- optimal "h-harmonic" limit map
  \[ \tilde{m}_*(z; a) = m_*(z; a)e^{i\phi} = e^{i\psi_\phi(h,a)} \frac{z - a}{|z - a|} \]
  where \( \phi(h, a) \) minimizes
  \[ \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 - h \cdot (e^{i\phi} m_*(z; a)) \]

no applied field: \( m_*(\cdot, 0) \) in a disk

numerics by J. Steiner
Some remarks about the problem with applied field
K.-Melcher-Moser 2012

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  \[ E_\varepsilon(h; m) = \int_\Omega \frac{1}{2} |\nabla m|^2 + \frac{m_3^2}{\varepsilon^2} - h \cdot m \]

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applied field \( h = (0, -40), \tilde{m}_*(\cdot, 0) \)

numerics by J. Steiner
Some remarks about the problem with applied field

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- optimal “\(h\)-harmonic” limit map

\[ \tilde{m}_*(z; a) = m_*(z; a) e^{i\phi} = e^{i\psi+\phi(h,a)} \frac{z - a}{|z - a|} \]

where \(\phi(h, a)\) minimizes \(\int_\Omega \frac{1}{2} |\nabla \phi|^2 - h \cdot (e^{i\phi} m_*(z; a))\)

applied field \(h = (0, -40), \tilde{m}_*(\cdot, a_*)\); \(a_*\) chosen as minimizer

numerics by J. Steiner
Renormalized energy for the problem with field

- New renormalized energy

\[ W(h, a) = \lim_{r \to 0} \int_{\Omega \setminus B_r(a)} \frac{1}{2} |\nabla \tilde{m}_*|^2 - h \cdot \tilde{m}_* - \pi \log \frac{1}{r} \]

- Noether theorem: if \( \Phi: \Omega \to \mathbb{R}^2 \) is constant near \( a \) then

\[ \Phi(a) \cdot \nabla_a W(h, a) = \int_{\Omega} \nabla \Phi : \left( \left( \frac{1}{2} |\nabla \tilde{m}_*|^2 - h \cdot m_* \right) \text{id} - \nabla \tilde{m}_* \otimes \nabla \tilde{m}_* \right) \]

- Coercivity for small energy excess and small field can be shown using Taylor expansion of exp

- Smallness from smallest Dirichlet eigenvalue
Alternative PDE approach

Previously

- control $D_\varepsilon$ using evolution identities
- $D_\varepsilon \to 0$ implies strong convergence

Can we show this more directly? K.-Melcher-Moser-Spirn 2012

- if $D_\varepsilon = O(1)$ we have weak convergence $\nabla m_\ast \otimes \nabla m_\ast \rightharpoonup \mu(t)$ for some matrix valued measure
- show $\mu(t) = \nabla m_\ast \otimes \nabla m_\ast + \sum_k A_k(t) \delta_{a_k(t)}$ using the PDE and partial regularity style estimates
- $A_k$ do not affect the motion law
- bubbling may occur and change vortex polarity (changes motion law):

$$\pi(\dot{a} + 2q(t)\dot{a}^\perp) = -\nabla W(a), \quad q(t) \in \{\pm 1\}$$
Alternative PDE approach

Previously

- control $D_\varepsilon$ using evolution identities
- $D_\varepsilon \to 0$ implies strong convergence

Can we show this more directly? K.-Melcher-Moser-Spirn 2012

- if $D_\varepsilon = O(1)$ we have weak convergence $\nabla m \otimes \nabla m \rightharpoonup \mu(t)$ for some matrix valued measure
- show $\mu(t) = \nabla m_* \otimes \nabla m_* + \sum_k A_k(t) \delta_{a_k(t)}$ using the PDE and partial regularity style estimates
- $A_k$ do not affect the motion law
- bubbling may occur and change vortex polarity (changes motion law):

$$\pi(\dot{a} + 2q(t)\dot{a}^\perp) = -\nabla W(a), \quad q(t) \in \{\pm 1\}$$
Open problems and ongoing research

- Uniqueness and coercivity for large fields? Related to elliptic sine-Gordon equation
  \[-\Delta u + \lambda \sin(u + \theta(x)) = 0\]

- Include magnetostatic interaction
- Large fields / currents can create bubbles. How?
- Bubbling in PDE approach is not controlled
- Do not know how energy dissipates (vortex-free: damped wave equation Miot 2010); nonlinear version?
- Finite \(\varepsilon\)? Explicit estimates for GLS Jerrard-Spirn, GLH K.-Spirn 2011
- Realistic derivation of motion law from full micromagnetic energy