Nematic phase of spin 1 quantum systems

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Classical model

$\Lambda$: cubic box in $\mathbb{Z}^d$, $\mathcal{E}$: set of nearest-neighbors

Spin configuration $(\vec{\sigma}_x)_{x \in \Lambda}$, $\vec{\sigma}_x \in S^2$

Hamiltonian function: $H(\sigma) = -\sum_{xy \in \mathcal{E}} (J_1 \vec{\sigma}_x \cdot \vec{\sigma}_y + J_2 (\vec{\sigma}_x \cdot \vec{\sigma}_y)^2)$

Gibbs state $\langle A \rangle = \frac{1}{\int d\sigma \, e^{-\beta H(\sigma)}} \int_{(S^2)^\Lambda} d\sigma A(\sigma) e^{-\beta H(\sigma)}$

Main question: low temperature phase diagram in $d = 3$
Results for classical model: $- \sum \left( J_1 \vec{\sigma}_x \cdot \vec{\sigma}_y + J_2 (\vec{\sigma}_x \cdot \vec{\sigma}_y)^2 \right)$

- $J_2 = 0$: Classical Heisenberg model
  - Ferro- and antiferromagnetic order at sufficiently low temperature [Fröhlich, Simon, Spencer ’76]
    $$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \vec{\sigma}_0 \cdot \vec{\sigma}_x \rangle > c, \quad \frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle \vec{\sigma}_0 \cdot \vec{\sigma}_x \rangle > c$$
  - uniformly in the size $|\Lambda|$ of the system

- $J_1 = 0, J_2 > 0$: Spin nematic model
  - $\langle \sigma^i_0 \sigma^i_x \rangle = 0 \quad \forall x \neq 0$
  - Nematic order at sufficiently low temperature [Angelescu, Zagrebnov ’82]
    $$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \left( \langle (\sigma^i_0)^2 (\sigma^i_x)^2 \rangle - \langle (\sigma^i_0)^2 \rangle^2 \right) > c$$

Quantum model

$\Lambda$: cubic box in $\mathbb{Z}^d$, $\mathcal{E}$: set of nearest-neighbors

Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^3$

Spin operators $S^1, S^2, S^3$ on $\mathbb{C}^3$ such that $[S^1, S^2] = iS^3$, etc...

$S^i_x = S^i \otimes \text{Id}_{\Lambda \setminus \{x\}}$

General SU(2)-invar. Hamiltonian: $H = -\sum_{xy \in \mathcal{E}} (J_1 \vec{S}_x \cdot \vec{S}_y + J_2 (\vec{S}_x \cdot \vec{S}_y)^2)$

Gibbs state $\langle A \rangle = \frac{1}{\text{Tr} \ e^{-\beta H}} \text{Tr} A \ e^{-\beta H}$

Main question: low temperature phase diagram in $d = 3$
Phase diagram of the quantum model — conjectured

\[ H = - \sum_{xy \in \mathcal{E}} (J_1 \vec{S}_x \cdot \vec{S}_y + J_2 (\vec{S}_x \cdot \vec{S}_y)^2) \]

[Batista, Ortiz ’04; Tu, Zhang, Xiang ’08; Tóth, Läuchli, Mila, Penc ’12; Fridman, Kosmachev, Klevets ’13]
Rigorous results

**Theorem** [Dyson, Lieb, Simon '78]
Assume that $J_2 = 0$, $J_1 < 0$, and $d \geq 3$. There exists $\beta_0$ and $c > 0$ such that for $\beta > \beta_0$,

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle S^i_0 S^i_x \rangle > c$$

Proved using the method of reflection positivity and infrared bounds

One can check that $-J_2 \sum (\vec{S}_x \cdot \vec{S}_y)^2$ is reflection positive when $J_2 > 0$ so the result extends straightforwardly to small positive $J_2$

**Open problem:** prove antiferromagnetic order in whole quadrant $J_1 < 0$, $J_2 > 0$
Rigorous results

Case $J_1 = 0, J_2 > 0$: Classical model is nematic. The quantum model is not!

**Theorem [DU ’13]**
Assume that $J_1 = 0, J_2 > 0$, and $d \geq 5$. There exists $\beta_0$ and $c > 0$ such that for $\beta > \beta_0$,

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle S_0^i S_x^i \rangle > c$$

Proved using the random loop representation of [Aizenman-Nachtergaele ‘93], and the method of reflection positivity and infrared bounds

Remark: Long-range order is expected for $d \geq 3$
Rigorous results

Finally, a result that seems to confirm the presence of the nematic phase for $0 < J_1 < J_2$

**Theorem [DU ’13]**  
Assume that $0 \leq J_1 \leq \frac{1}{2} J_2$, and $d \geq 5$. There exist $\beta_0$ and $c > 0$ such that for $\beta > \beta_0$,  

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \left( \langle (S_0^i)^2 (S_x^i)^2 \rangle - \langle (S_0^i)^2 \rangle^2 \right) > c$$

Remark: this result is actually proved for $d \geq 3$ when $J_1 \lesssim \frac{1}{2} J_2$
Rigorous results

\[ H = - \sum_{x, y \in \mathcal{E}} \left( J_1 \vec{S}_x \cdot \vec{S}_y + J_2 (\vec{S}_x \cdot \vec{S}_y)^2 \right) \]
Method of proof

- Random loop representation
- Random loop model is reflection positive
- Infrared bound on Duhamel two-point function
- Falk-Bruch inequality + sum rule
Random loop representation

Origins:

- Random-walk representation of [Conlon, Solovej ’91], in order to estimate the free energy of the Heisenberg ferromagnet
- Random loop representation of the spin $\frac{1}{2}$ ferromagnet [Tóth ’93]; improves the estimate of the free energy

(Breaking news! [Giuliani, Seiringer ’13] have obtained optimal result)

- Another loop representation for the antiferromagnet model [Aizenman, Nachtergaele ’94], that allows to relate the quantum spin chain ($d = 1$) to two-dimensional Potts and random cluster models
Random loop representation

Let $\rho$ denote independent Poisson point processes on edges of $\Lambda[0, \beta]$, where:

- **crosses** occur with intensity $u$
- **bars** occur with intensity $1 - u$

$\mathcal{L}(\omega)$: set of loops of realization $\omega$

Relevant probability measure:

$$\frac{1}{Z} 3^{|\mathcal{L}(\omega)|} \rho(d\omega)$$
Illustration (in one-dimension)
Relations between random loops and quantum model

Model $H = - \sum_{xy \in \mathcal{E}} (u \vec{S}_x \cdot \vec{S}_y + (\vec{S}_x \cdot \vec{S}_y)^2)$. That is, $J_1 = u$ and $J_2 = 1$

Arbitrary finite graph $(\Lambda, \mathcal{E})$

**Theorem [DU ’13]**

$$\text{Tr } e^{-\beta H} = \int 3|\mathcal{L}(\omega)| \, d\rho(\omega)$$

$$\langle S^i_0 S^i_x \rangle = \frac{2}{3} \left[ \mathbb{P}(0 \leftrightarrow x, \text{same direction}) - \mathbb{P}(0 \leftrightarrow x, \text{opposite dir.}) \right]$$

$$\langle (S^i_0)^2(S^i_x)^2 \rangle - \langle (S^i_0)^2 \rangle \langle (S^i_x)^2 \rangle = \frac{2}{9} \mathbb{P}(0 \leftrightarrow x)$$
Proofs of the theorems

The random loop representation turns out to be essential:

- A few tricks allow to write the model in a reflection positive fashion
- Rather than looking at spin correlations, one proves the existence of long loops

The method is not purely probabilistic, though:

- The Falk-Bruch inequality [Falk, Bruch ’69; Dyson, Lieb, Simon ’78], that estimates the two-point function with the Duhamel two-point function, is useful
- Estimates are more conveniently done using matrix inequalities
Some insights from the representation

The usual spin correlation function is intriguing:

\[ \langle S^i_0 S^i_x \rangle = \frac{2}{3} \left[ P(0 \leftrightarrow x, \text{same direction}) - P(0 \leftrightarrow x, \text{opposite dir.}) \right] \]

It is tempting to conjecture that it has exponential decay when \( 0 < J_1 < J_2 \). Compatible with the nematic phase

It should be possible to prove this rigorously!!!
Conclusion

- Family of spin 1 Heisenberg models with interesting phase diagram

[Reference: Random loop representations for quantum spin systems, JMP 54, 083301 (2013)]
Conclusion

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- Phase transitions with long-range order and continuous symmetry breaking. Proved using reflection positivity and infrared bounds [Fröhlich, Simon, Spencer ’76; Dyson, Lieb, Simon ’78]

Fascinating representations of quantum spin systems: Random loop models, that generalise [Tóth ’93; Aizenman-Nachtergaele ’94]. Intriguing expressions for quantum spin correlations Random loop representations also play a role in the classification of gapped ground states [Bachmann, Nachtergaele ’13], and in the probability of emptyness formation [Crawford, Ng, Starr ’13].

The joint distribution of the lengths of macroscopic loops is known: Poisson-Dirichlet. Numerical evidence in [Grosskinsky, Lovisolo, U ’12; Nahum, Chalker, Serna, Ortuño, Somoza ’13].


THANK YOU!

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