Correlation-Based Imaging in Random Media

Kinetic Models

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Inverse scattering in random media
Kirchhoff migration: weakly scattering media
Kirchhoff migration: strongly scattering case
Another strongly scattering case
Kirchhoff migration: weakly scattering media
Transport: weakly scattering media
Kirchhoff: strongly scattering media
Transport: strongly scattering media
Outline

1. Waves in Random media and kinetic models

2. Some numerical and experimental validations of kinetic models

3. Statistical stability

4. Optimal Stability

5. Resolution Theories in Coherent and Incoherent imaging
Scalar wave equation

The pressure $p(t, x)$ solves following closed form wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2(x) \Delta p, \quad c^2(x) = \frac{1}{\rho_0 \kappa(x)}.$$ 

The total energy is conserved and given by

$$\mathcal{E}_H(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \kappa(x) \left( \frac{\partial p}{\partial t} \right)^2 (t, x) + \frac{|\nabla p|^2(t, x)}{\rho_0} \right) dx = \mathcal{E}_H(0).$$

Kinetic models predict the spatial distribution of conserved energy.

We are interested in the high frequency regime, where initial conditions have the form $p(0, x) = p\left(\frac{x}{\varepsilon}\right)$. 
Let $\varepsilon$ be a small parameter modeling typical wavelength.

The Wigner transform of two propagating vector fields is defined by:

$$W_\varepsilon[u, v](x, k) = \int_{\mathbb{R}^d} e^{i y \cdot k} u(x - \varepsilon \frac{y}{2}) v^*(x + \varepsilon \frac{y}{2}) \frac{dy}{(2\pi)^d}.$$  

It is the inverse Fourier transform of the product:

$$W_\varepsilon[u, v](x, k) = \mathcal{F}^{-1}\left(u(x + \varepsilon \frac{y}{2})v^*(x - \varepsilon \frac{y}{2})\right).$$

We verify that

$$\int_{\mathbb{R}^d} W[u, v](x, k) dk = (uv^*)(x)$$

The Wigner transform captures **field-field correlations**.
Weak-Coupling Regime

In the weak coupling regime, the random fluctuations of the media are modeled by

\[(c^\varphi_\varepsilon)^2(x) = c_0^2 - \sqrt{\varepsilon} V^\varphi(\frac{x}{\varepsilon}), \quad \varphi = 1, 2,\]

\[c_0^2 = \frac{1}{\kappa_0 \rho_0}, \quad V^\varphi(x) = \frac{c_0^2}{\kappa_0} \kappa^\varphi_1(x),\]

where \(c_0\) is the average background speed and \(\kappa^\varphi_1\) and \(V^\varphi\) are random fluctuations in the compressibility and sound speed, respectively. We assume that \(V^\varphi(x), \varphi = 1, 2,\) are statistically homogeneous mean-zero random fields with correlation functions and power spectra given by:

\[c_0^4 R^{\varphi \psi}(x) = \langle V^\varphi(y)V^\psi(y + x) \rangle, \quad 1 \leq \varphi, \psi \leq 2,\]

\[(2\pi)^d c_0^4 \hat{R}^{\varphi \psi}(p) \delta(p + q) = \langle \hat{V}^\varphi(p)\hat{V}^\psi(q) \rangle.\]
Kinetic theory in weak coupling regime

Let $W_\varepsilon(t, x, k)$ be the Wigner transform of two fields $u_1$ and $u_2$ propagating in two media $\varphi = 1, 2$. The limit Wigner distribution is decomposed as: $W(t, x, k) = a_+(t, x, k)b_+b^* + a_-(t, x, k)b_-b^*$. The radiative transfer equation for $a_+$ is (with $\omega_\pm = \pm c_0|k|$) [Ryzhik Papanicolaou Keller W.M. 1996; B. Verastegui M.M.S. 2004]:

$$\frac{\partial a_+}{\partial t} + c_0\hat{k} \cdot \nabla a_+ + (\Sigma(k) + i\Pi(k))a_+$$

$$= \frac{\pi \omega_+^2(k)}{2(2\pi)^d} \int_{\mathbb{R}^d} \hat{R}^{12}(k - q)a_+(q)\delta(\omega_+(q) - \omega_+(k))dq,$$

$$\Sigma(k) = \frac{\pi \omega_+^2(k)}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{R}^{11} + \hat{R}^{22}}{2}(k - q)\delta(\omega_+(q) - \omega_+(k))dq,$$

$$i\Pi(k) = \frac{i\pi \sum_{j=\pm} \text{p.v.}}{4(2\pi)^d} \int_{\mathbb{R}^d} (\hat{R}^{11} - \hat{R}^{22})(k - q)\frac{\omega_j(k)\omega_+(q)}{\omega_j(q) - \omega_+(k)}dq.$$
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Numerical validation

The domain size is roughly $20,000 \times 10,000 = 200M$ nodes
**Effect of void inclusions**

Correction generated by an inclusion of radius $R = 50$ with random fluctuations suppressed. **Left:** 5% RMS. **Right:** 8% RMS.

[B. Pinaud, Wave Motion 2006; M.M.S. 2007; M3AS, 2011]
Discrete Scatterers and frequency domain

The sound speed fluctuations take the form $\sqrt{\epsilon} V_{\epsilon}(\frac{x}{\epsilon})$, where $V_{\epsilon}(x) = \epsilon^{-(\gamma+2\beta)d} \sum_{j} \tau_j V\left(\frac{x - x_{\epsilon j}}{\epsilon^\beta}\right)$ with $\beta > 0$ and $\gamma < 1$. The scatterers are modeled as a Poisson point process: the points $x_j(\omega)$ form a Poisson point process of density $\nu_{\epsilon} = \epsilon^{\gamma d} n_0$.

At the wave level, the scatterers are sufficiently localized so that we can use a Foldy-Lax model. At the transport level, we observe that the power spectrum $\tilde{R}_{\epsilon}$ converges to $\tilde{R}_0 = L^{2d} \mathbb{E}\{\tau^2\} n_0$. 
Comparison Wave Energy - Transport

Small detector (left) and Large detector (right). Error bars = 1 standard deviation (50 realizations). [B. Ren, SIAM App. Math. 08]
Duke experimental set-up
Correlation after spatial phase shift

Back-propagated signal Duke Experiments

Duke numerical set-up

All subsequent reconstructions based on frequency average on the interval $[9GHz, 11GHz]$. Best fit for mean free path is 42cm (91cm for band $[6.5GHz, 8.5GHz]$, showing behavior of mfp in $k^{-3}$).
Reconstruction of four (small) rods outside the random medium from differential measurements.
Reconstruction of voids inside random medium

Reconstruction of four holes created by the removal of three rods. Reconstruction based on differential measurements.
[B., Carin, Liu, Ren, Inverse Problems 2007]
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Stability and Imaging

Mathematically rigorous derivations show that the ensemble average of the energy density solves the (deterministic) radiative transfer equation in dimension $d \geq 2$ [Spohn 1977; Erdös Yau, 1998]:

$$\mathbb{E}\{\mathcal{E}_\varepsilon(t, x)\} \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^d} a_+(t, x, k) dk.$$ 

Does $\mathcal{E}_\varepsilon(t, x)$ converge and is the limit independent of the realization of the random medium?

Statistical Stability is crucial to address inverse problems.

We need to quantify the statistical instability of the field-field measurements as incomplete knowledge of the medium translates into noise.
Stability result

For the paraxial approximation to wave propagation we have the result:

**Theorem** [B. Papanicolaou Ryzhik Nonlinearity 2003; S&D 2003].

$W_\varepsilon$ converges in probability and weakly in $L^2(\mathbb{R}^{2d})$ to the (deterministic) solution $\overline{W}$ of the transport equation

$$\frac{\partial \overline{W}}{\partial z} + \frac{1}{\kappa} \mathbf{k} \cdot \nabla_x \overline{W} = \kappa \mathcal{L} \overline{W},$$

with $L^2(\mathbb{R}^{2d})$ initial data $W_0(x, k)$ and operator $\mathcal{L}$ defined by

$$\mathcal{L} \lambda = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \hat{R}(\frac{|p|^2 - |k|^2}{2}, p - k)(\lambda(p) - \lambda(k)),$$

where $\hat{R}(\omega, p)$ is the Fourier transform of the correlation function of $V$.

For any test function $\lambda \in L^2(\mathbb{R}^{2d})$, the process $\langle W_\varepsilon(z), \lambda \rangle$ converges to $\langle \overline{W}(z), \lambda \rangle$ in probability as $\varepsilon \to 0$. 
Radiative transfer and diffusion regimes

Consider more generally the following Schrödinger equation

$$i\varepsilon^{1+\delta}\frac{\partial \psi_\varepsilon}{\partial z} + \frac{\varepsilon^2}{2}\Delta \psi_\varepsilon - \frac{\varepsilon}{2}\beta \frac{\delta - \varepsilon}{\alpha} V\left(\frac{z}{\varepsilon^\alpha}, \frac{x}{\varepsilon^\beta}\right) \psi_\varepsilon = 0.$$ 

Theorem [B. Komorowsky Ryzhik K.R.M. 2011] The Wigner distribution converges in probability and weakly to the deterministic solution of

(i) the radiative transfer equation when $\alpha < \beta = 1$ and $\delta = 0$:

$$\frac{\partial \hat{W}}{\partial z} + k \cdot \nabla_x \hat{W} = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \delta \left(\frac{|p|^2 - |k|^2}{2}\right) \hat{R}_0(p - k)(W(p) - W(k))$$

(ii) the diffusion equation when $\alpha < \beta = 1$ and $\delta > 0$:

$$\frac{\partial \hat{W}}{\partial z} - \nabla_x D \nabla_x \hat{W} = 0, \quad D = k \otimes L^{-1}k,$$
(iii) the Fokker-Planck equation when $\alpha < \beta < 1$ and $\delta = 0$.

$$\frac{\partial \overline{W}}{\partial z} + \mathbf{k} \cdot \nabla_x \overline{W} = \frac{1}{2} \nabla_{\mathbf{k}} \cdot \left( \int_{\mathbb{R}^d} \delta(\mathbf{k} \cdot \mathbf{p}) \hat{R}_0(p) \mathbf{p} \otimes \mathbf{p} \frac{dp}{(2\pi)^d} \right) \nabla_{\mathbf{k}} W$$

The Fokker Planck equation is valid the wavelength is much shorter than the correlation length of the medium. We may see the dynamics as wave packets propagating in a random Hamiltonian.

The Fokker-Planck may be seen as an approximation to radiative transfer when scattering is highly peaked forward so that the direction of the wavepackets follows Brownian dynamics.

[B. Komorowsky Ryzhik, Review in Kinetics and Related Models 2011]
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Kinetic model for the scintillation function

A natural object in the study of the statistical stability of $W_\varepsilon$ is the following covariance (scintillation) function:

$$J_\varepsilon(t, x, k, y, p) = \mathbb{E}\{W_\varepsilon(t, x, k)W_\varepsilon(t, y, p)\} - \mathbb{E}\{W_\varepsilon(t, x, k)\}\mathbb{E}\{W_\varepsilon(t, y, p)\}.$$  

Define domains of measurements:

$$\varphi_{\varepsilon,s_1,s_2}(x, k) = \frac{1}{\varepsilon d(s_1 + s_2)} \varphi\left(\frac{x}{\varepsilon s_1}, \frac{k - k_1}{\varepsilon s_2}\right).$$

By using the Chebyshev inequality, we obtain the following estimate on the probability that $W_\varepsilon$ deviate from its ensemble average $a_\varepsilon$:

$$\mathbb{P}\left(\left|\langle W_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\rangle - \langle a_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\rangle\right| \geq \delta\right) \leq \frac{1}{\delta^2} \langle J_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2} \otimes \varphi_{\varepsilon,s_1,s_2}\rangle.$$  

We are therefore interested in estimating the above right-hand side.
Fluctuations in Itô-Schrödinger regime

**Theorem.** [B. Pinaud KRM 08] Let $\psi_\varepsilon(x, 0)$ be a sequence of functions uniformly bounded in $L^2(\mathbb{R}^d)$, compact at infinity, and $\varepsilon$-oscillatory. Let $a_\varepsilon(0, x, k)$ be the corresponding sequence of Wigner transforms. We assume that

$$\| F_x a_\varepsilon(0, u, k) \|_{L^1(\mathbb{R}^{2d})} \lesssim \varepsilon^{-\alpha d} \quad \text{and} \quad \| F_k a_\varepsilon(0, x, \xi) \|_{L^1(\mathbb{R}^{2d})} \lesssim \varepsilon^{-\beta d}.$$ 

This means the source concentrates at the scale $\varepsilon^\alpha$ in space and $\varepsilon^\beta$ in wavenumbers. Physically, $\alpha + \beta = 1$. Then we find that

$$\langle J_\varepsilon(t), \varphi, s_1, s_2 \otimes \varphi, s_1, s_2 \rangle \lesssim g_\varepsilon$$

$$g_\varepsilon = \varepsilon^{d(1-\alpha)-2d(s_1+s_2)} \left[ \varepsilon^{2(1-\alpha)-s_1-s_1 \vee s_2+(\alpha-\beta) \vee 0} \right]$$

$$\vee \varepsilon^{1-\beta+((\alpha-\beta) \vee 0) \wedge ((d-1)(1-\alpha-\beta)+\alpha)},$$

when $d \geq 3$ (with a modified expression when $d = 2$).
Single scattering scintillations

For \( \varphi \) a (large) detector array characteristic, we wish to estimate

$$I_{\varepsilon} := \int J_{11\varepsilon}(t, x, k, y, q)\varphi(x, k)\varphi(y, q)dxdydq.$$  

Long range correlations (slow decay of \( R(x) \)) are modeled by singular behavior of power spectrum \( \hat{R}(\xi) \) at \( \xi = 0 \):

$$\hat{R}(\xi) = \frac{\hat{S}(\xi)}{|\xi|^\delta}, \quad \delta = d - p, \quad R(x) \sim \frac{\kappa}{|x|^p}, \quad x \to \infty.$$  

The size of \( I_{\varepsilon} \) depends on \( \alpha, \beta = 1 - \alpha, \) and \( \delta \) [B. Langmore Pinaud 09]:

$$0 \leq I_{\varepsilon} \leq C_{\varepsilon}(d-\delta)(1-\alpha)+\alpha\vee(1-\alpha), \quad d \geq 1(!).$$

When \( \delta = 0 \), the maximum scintillation is of order \( \varepsilon \) and \( \alpha = 1 \).

When \( \delta \to d \) (very long range \( p \to 0 \)), the maximum scintillation approaches \( \varepsilon^{\frac{1}{2}} \) and \( \alpha = \beta = \frac{1}{2} \).

Similar models exist for double scattering [B. Pinaud CPDE 12].
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Imaging functionals from Coherent signal

We want to reconstruct a scatterer modeled as an Initial Condition (secondary source). Random Environment is Noise.

A wave-based, Kirchhoff-like functional to image the scatterer is

$$I^W(x) = \mathcal{F}_{k \rightarrow x}^{-1} \left( \cos |k| \mathcal{F}(\chi_D p(T, \cdot))(k) \right).$$

Consider the correlation (kinetic quantity)

$$W_S(T, x, k) = \frac{1}{(2\pi)^n} \int |\varepsilon y| \lesssim \frac{1}{N_c} p(t, x - \frac{\varepsilon y}{2}) p^*(t, x + \frac{\varepsilon y}{2}) dy.$$ 

Here, $\varepsilon = \frac{\lambda}{L}$ with $\lambda$ wavelength of I.C. and $N_c$ selects $p - p$ correlations. A correlation-based imaging functional (like CINT) is

$$I^C(x) = \int_D W_S(T, x + c_0 T \vec{k}, k) dk.$$ 

In the absence of random fluctuations, both functionals reconstruct the object with resolution $\frac{\lambda}{2}$ (or $N_c \frac{\lambda}{2}$ for correlations when $N_c \gg 1$).
Calculations based on low-scattering approximations of kinetic equations for wave models [B. Pinaud Ryzhik ≥2013] show that:

Fluctuations mainly caused by single scattering generated by source; the variances of the two imaging functionals are given by:

\[
\text{Var}^W = \sigma_0^2 \frac{L}{l_c N_l^4}, \quad \text{Var}^C = \sigma_0^2 \frac{L}{l_c N_l^4 N_c^2} = \text{Var}^W \frac{1}{N_c^2}
\]

Here \(\sigma_0\) is disorder, \(l_c\) is medium correlation length, \(\lambda = N_l l_c = L\varepsilon\) is wavelength, \(L\) is domain size, and \(N_c\) is \(p-p\) correlation selection. Thus,

\[
SNR^2_{W,N_l} = N_l^4 \frac{l_c}{\sigma_0^2 L}, \quad SNR^2_{C,N_l,N_c} = N_c^2 N_l^4 \frac{l_c}{\sigma_0^2 L}, \quad \text{Resolution} = \frac{\lambda}{2 N_l N_c}.
\]

Increasing \(N_c\) stabilizes \(I^C\), but less efficiently than increasing \(N_l\) for the same effect on resolution. Stabilization always comes at the expense of resolution. Results consistent with [Ammari-Bretin-Garnier-Jugnon] for random travel times.
Wave and Correlation Models; Resolution

Resolution of $\frac{\lambda}{2}$ accessible only for sufficiently small disorder. If all frequencies available, choose the smallest one with low disorder such that $N_l^4 \frac{l_c}{\sigma_0^2 L} = \text{Comfortable SNR}^2$. Then Resolution $= N_l \frac{1}{2} \lambda$.

If lower frequencies not available ($N_l$ cannot increase), then correlations increase SNR$^2$ by $N_c^2$ for loss of resolution of $N_c$.

There are many settings where $N_l$ cannot increase:

(i) coherent signal not available (object to image behind blocker);
(ii) we want statistical properties $\hat{R}(x, k)$ at fixed $k = \lambda^{-1}$
(iii) low frequencies are absorbed because of resonances (as in plasmas)

Then the incoherent part must be used and modeled and kinetic equations offer simplest models. Moreover, statistical stabilization becomes crucial for inverse problems and imaging.
Resolution for incoherent measurements

Resolution is then more difficult to define. Let $\varepsilon$ be probing wavelength (now fixed; we no longer play with $N_l$). Let $\varepsilon^a$, $a \in [0, 1]$ be resolution. An inverse problem always increases high frequencies, for instance as $N^\delta$. Statistical stability provides stability of the form

$$\text{Stability Correlations} = \varepsilon^{\gamma(1-a)} \ll 1 \quad \text{when} \quad a > 0.$$ 

Noise amplification provides noise in the reconstruction

$$\text{Noise in Reconstruction} = \varepsilon^{\gamma(1-a)} e^{-\delta a}, \quad N = \varepsilon^{-a}.$$ 

Then choose $a$, and hence Resolution $\varepsilon^a$, such that

$$\frac{1}{\text{Noise}} = \varepsilon^{-\gamma + (\gamma + \delta) a} = \text{Comfortable SNR} \quad (=\varepsilon^{-\beta}, \quad a = \frac{\gamma - \beta}{\gamma + \delta}).$$

Most Inverse Transport Problems are severely ill-posed with $N^\delta \to e^N = e^{\varepsilon^{-a}}$.

$$\text{CSNR} = \varepsilon^{\gamma(1-a)} e^{-\varepsilon^{-a}} \quad \text{so that} \quad \varepsilon^a \sim \frac{1}{\ln \text{CSNR} + \gamma} = O(1).$$
Conclusions

Kinetic models for field-field correlations in random media are homogenization models. They work well in many numerical and experimental settings and can be used for parameter identifications.

Limitations are caused by scintillation (statistical instabilities). Scintillation is a complex functional of the initial conditions, the array of detectors and the long range properties of the randomness.

Models for statistical instabilities can be used to devise resolution theories. In coherent imaging, it is preferable to probe with smaller wavelength (increase $N_l$) rather than clean up correlations (increase $N_c$).

In IPs based on correlation measurements with fixed wavelength, we need to clean up correlations (by spatial and angular averaging) and may then define resolution as highest reconstructed frequency for given CSNR. This strongly depends on IP and PI at hand.