Data decomposition of complex scenes for motion estimation and imaging with synthetic aperture radar

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Waves and imaging in complex media
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Spotlight-mode Synthetic Aperture Radar (SAR) Imaging

Radar system emits probing signals $f(t)$ and records echoes $D(s, t)$ (slow time $s$ of the SAR platform displacement, fast time $t$ of the probing signal)

X-band Regime

- $|\vec{r}_p(s) - \vec{\rho}^\mathcal{I}| \approx 10$ km
- Central wavelength $\lambda_0 = 3$cm
- Bandwidth $B = 622$ MHz
- $V = 70$ m/s

\[\mathcal{I}(\hat{\rho}^\mathcal{I}) = \int_{-S^{(a)}}^{S^{(a)}} ds \int dt \frac{D(s, t)f(t - \tau(s, \vec{\rho}^\mathcal{I}))}{\sqrt{S^{(a)}}} \]

\[= \int_{\omega_0 - \pi B}^{\omega_0 + \pi B} d\omega \frac{1}{2\pi} \int_{-S^{(a)}}^{S^{(a)}} ds \hat{f}(\omega)\hat{D}(s, \omega)e^{-i\omega\tau(s, \vec{\rho}^\mathcal{I})}\]

where $\tau(s, \vec{\rho}^\mathcal{I}) = 2|\vec{r}_p(s) - \vec{\rho}^\mathcal{I}|/c$ is the roundtrip travel time
Autofocus

- In practice, resolution always lost due to imperfect knowledge of the flight path
- Due to high frequency regime, a very small error in travel time can corrupt image
- Large bandwidth systems have potential for cm resolution but not without autofocus
- Previous Work:
  - Phase Gradient Autofocus (PGA): Jakowatz 1993
  - Contrast Metric Methods (Morrison 2007, etc)
Similarly, motion of targets with sufficiently large reflectivity leads to poor image resolution

- Slow target motion yields similar results as platform perturbations (displacement, blurring)
- Fast target motion smears image entirely
Overview

We study motion estimation and autofocus problems for

1. Single scatterer scene
2. Multiple scatter scenes that have same velocity
3. Complex scenes with many stationary targets and a few moving targets with independent velocities
Outline

Phase-Space Approach to Motion Estimation and Autofocus
  Motivation and Model Framework
  Motion Estimation
  Autofocus

Decomposing SAR Data into Moving and Stationary Data
  Annihilation-based data filtering
  Robust PCA
  Numerical: Comparison of Decomposition Methods
Outline

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Phase-Space Methods

Motivation for our approach

- High frequency regime results in highly oscillatory integrals
- In phase-space, small phase shifts can be determined robustly
- Use Wigner transform and ambiguity function because position of peaks relate to target motion and platform perturbations
- Use properly segmented “sub-apertures” of our data

Earlier work on phase space methods by Barbarosa (1992), Munson (2002) and others.
Data Model

- Suppose a single point target $\mathbf{\rho}(s)$ moves in imaging plane with velocity $\mathbf{u}(s)$
  - Assume target location is known at some time $s$
  - Assume velocity $\mathbf{u}(s)$ is constant over small sub-aperture
- Suppose measured flight trajectory is $\mathbf{r}_p(s)$ while actual trajectory is $\mathbf{r}_p(s) + \mathbf{\mu}(s)$
- Assume single scattering (Born) approximation of solution to wave equation
- Regime: High frequency, relatively low bandwidth, long distance ($\lambda_0 \ll L$)
- Under these assumptions, the data model is:

$$D_r(s, t) = \frac{(\omega_o/c)^2}{(4\pi|\mathbf{r}_p(s) - \mathbf{\rho}(s)|)^2} f(t - (\tau(s, \mathbf{\rho}(s)) - \tau(s, \mathbf{\rho}_o)))$$  \hspace{1cm} (1)

where we offset the travel time by a reference point $\mathbf{\rho}_o$ in the imaging plane. The model of the range compressed data in the frequency domain is

$$\hat{D}_r(s, \omega) \approx \frac{\omega_o^2}{c^2} \left| \hat{f}_B(0) \right|^2 \frac{1_{[\omega_o, \pi B]}(\omega)}{(4\pi|\mathbf{r}_p(s) - \mathbf{\rho}(s)|)^2} \exp \{ i\omega [\tau(s, \mathbf{\rho}(s)) - \tau(s, \mathbf{\rho}_o)] \}. \hspace{1cm} (2)$$

where

$$\tau(s, \mathbf{\rho}(s)) = \frac{2|\mathbf{r}_p(s) + \mathbf{\mu}(s) - \mathbf{\rho}(s)|}{c}$$
Wigner Transform and Ambiguity Function of Data

- The Wigner transform of the data of a point target moving with velocity $|\vec{u}| \leq V$ is

$$
\mathcal{W}(s, \Omega, \omega, T) = \int_{-\tilde{\Omega}}^{\tilde{\Omega}} d\omega \int_{-\tilde{S}}^{\tilde{S}} d\tilde{s} \hat{D}_r \left( s + \frac{\tilde{s}}{2}, \omega + \frac{\tilde{\omega}}{2} \right) \hat{D}_r \left( s - \frac{\tilde{s}}{2}, \omega - \frac{\tilde{\omega}}{2} \right) e^{is\Omega - i\tilde{\omega} T}
$$

where $\tilde{\Omega} = 2\pi B - 2|\omega - \omega_o|$ and $\tilde{S} = \frac{a}{2V}$.

- The ambiguity function of the data is

$$
\mathcal{A}(s, \Omega, \tilde{s}, T) = \int_{\omega_o - \pi B}^{\omega_o + \pi B} d\omega \int_{-\tilde{S}}^{\tilde{S}} d\tilde{s} \hat{D}_r \left( s + \tilde{s} + \frac{\tilde{s}}{2}, \omega \right) \hat{D}_r \left( s + \tilde{s} - \frac{\tilde{s}}{2}, \omega \right) e^{i\tilde{s}\Omega - i\omega T}
$$

where $\tilde{S} = \frac{a}{2V}$. 
Phase-Space Transforms for Motion Estimation ($\vec{\mu}(s) \equiv 0$)

Result\textsuperscript{1}: Under conditions on the size of the aperture that restrict the size of the Fresnel number $a^2/(\lambda_0 L)$ so we can linearize phases

\[
\mathcal{W}(s, \Omega, \omega_0, T) \sim \text{sinc}\{\pi B \left[ T - \Delta \tau(s) \right]\} \text{sinc} \left\{ \frac{4\pi a}{\lambda_0} \left[ \frac{\Omega c}{2\omega_0 V} - \Phi(s) \right] \right\}
\]

\[
\mathcal{A} \left( s, \Omega, \frac{a}{2V}, T \right) \sim \text{sinc} \left\{ \frac{\pi Ba}{c} \left[ \frac{c T}{a} + \Phi(s) \right] \right\} \text{sinc} \left[ \frac{a\Omega}{2V} + \frac{\pi a^2 \Phi^\perp(s)}{\lambda_0 |\vec{r}'_p(s) - \vec{\rho}(s)|} \right],
\]

where $\vec{m}(s)$ and $\vec{t}(s)$ are the unit vectors

\[
\vec{m}(s) = \frac{\vec{r}'_p(s) - \vec{\rho}}{|\vec{r}'_p(s) - \vec{\rho}|}, \quad \vec{t}(s) = \frac{\vec{r}'_p(s)}{V}
\]

and $\mathbb{P}(s) = I - \vec{m}(s)\vec{m}(s)^T$ is the projection matrix orthogonal to $\vec{m}(s)$

\[
\Delta \tau(s) = \tau(s, \vec{\rho}(s)) - \tau(s, \vec{\rho}_o) \quad \text{and} \quad \Phi(s) = \frac{\vec{u}}{V} \cdot \vec{m}(s) - \vec{t}(s) \cdot (\vec{m}(s) - \vec{m}_o(s)).
\]

\[
\Phi^\perp(s) = \left[ \mathbb{P}(s) \left( \frac{\vec{t}(s) - \frac{\vec{u}}{V}}{V} \right) \right]^2 - |\vec{r}_p(s) - \vec{\rho}(s)| \left[ \frac{|\mathbb{P}_o(s)\vec{t}(s)|^2}{|\vec{r}_p(s) - \vec{\rho}_o|} - \frac{\vec{t}'(s)}{V} \cdot (\vec{m}(s) - \vec{m}_o(s)) \right].
\]

Complementary Phase Information for Motion Estimation

**Wigner Transform**
- By selecting the peaks in the Wigner transform \((\Omega^W(s), \mathcal{T}^W(s))\) we can extract the estimate:

\[
\frac{\vec{u}}{V} \cdot \vec{m}(s) = \frac{c \Omega^W(s)}{2\omega_0 V} + \vec{t}(s) \cdot \vec{m}(s) + O\left(\frac{\lambda_o}{a}\right)
\]

- Orthogonal estimate can be derived from peaks of Wigner transform, but requires a numerical differentiation

**Ambiguity Function**
- By selecting the peaks in the ambiguity function \((\Omega^A(s), \mathcal{T}^A(s))\) we can extract an estimate of the velocity in an orthogonal direction \(|\mathcal{P}(s)\left(\vec{t}(s) - \frac{\vec{u}}{V}\right)|^2\) and of \(\frac{\vec{u}}{V} \cdot \vec{m}(s)\) but with worse resolution than Wigner estimate.

The Wigner transform estimate of \(\frac{\vec{u}}{V} \cdot \vec{m}(s)\) and the ambiguity function estimate of \(\left|\mathcal{P}(s)\left(\vec{t}(s) - \frac{\vec{u}}{V}\right)\right|^2\) are complementary estimates that can be combined to form an estimate of \(\vec{u}(s)\).
Motion Estimation on Single Sub-aperture of Single Target

(a) Uncorrected Image

(b) Corrected Image

(c) Estimated Motion

(d) Error

Cross Range (meters)

Range (meters)

Image of 28 m/s moving target

Image with Motion Estimation from Data

Motion Estimation Path in Imaging Plane

Pointwise Error in Motion Estimation

Distance from True Trajectory (meters)

Slow Time

Pointwise Error in Motion Estimation

Distance from True Trajectory (meters)

Slow Time

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Autofocus

\[ I(\rho^I) = \int_{-S}^{S} ds \int_{\omega_o-\pi B}^{\omega_o+\pi B} d\omega \frac{\hat{D}_r(s,\omega)}{2\pi} e^{-\frac{2i\omega}{c} (|\bar{r}_p(s)-\rho^I|-|\bar{r}_p(s)-\rho_o|)} \]

Through similar approximation of phases by a second degree polynomial in \( s \) and the same constraints as before:

\[
I(\rho^I) \sim \int_{\omega_o-\pi B}^{\omega_o+\pi B} d\omega \exp \left\{ \frac{2i\omega}{c} \left( |\bar{r}_p - \rho| - |\bar{r}_p - \rho^I| + \varphi_0 \right) \right\} \int_{-S}^{S} ds \exp \left\{ \frac{2i\omega sV}{c} \left[ \vec{t} \cdot (\vec{m} - \vec{m}_I) + \varphi_1 \right] + i\omega_o (sV)^2 \right\} \]

The focusing of \( I(\rho^I) \) is determined by the phases

\[
\varphi_o = \vec{m} \cdot \vec{\mu}, \quad \varphi_1 = \vec{m} \cdot \frac{\vec{\mu}'}{V} + \vec{t} \cdot \frac{\vec{P} \vec{\mu}}{|\bar{r}_p - \rho|}, \quad \varphi_2 = \vec{m} \cdot \frac{\vec{\mu}''}{V^2} + \frac{2 \left( \vec{t} + \frac{\vec{\mu}'}{V} \right) \cdot \vec{P} \vec{\mu}'}{|\bar{r}_p - \rho|}.
\]

Autofocus process consists in applying the correction \( \vec{\mu}^{AF}(s) = \left[ \varphi_o + sV \varphi_1 + \frac{(sV)^2}{2} \varphi_2 \right] \vec{m} \) to the SAR platform trajectory and forming the image

\[
I^{AF}(\rho^I) = \int_{-S}^{S} ds \int_{\omega_o-\pi B}^{\omega_o+\pi B} d\omega \frac{\hat{D}_r(s,\omega)}{2\pi} e^{-\frac{2i\omega}{c} (|\bar{r}_p(s)+\vec{\mu}^{AF}(s)-\rho^I|-|\bar{r}_p(s)-\rho_o|)}.
\]
Phase-Space Transforms for Autofocus ($\vec{u}(s) \equiv 0$)

**Result:** Under conditions on the size of the aperture that restrict the size of the Fresnel number $a^2/(\lambda_0 L)$

$$\mathcal{W}(s = 0, \Omega, \omega_o, T) \sim \text{sinc} \left\{ \pi B \left[ T + \delta T^W \right] \right\} \text{sinc} \left\{ \frac{4\pi ac(\Omega + \delta \Omega^W)}{2\lambda_o V \omega_o} \right\}$$

$$\mathcal{A} \left( s = 0, \Omega, \frac{a}{2V}, T \right) \sim \text{sinc} \left\{ \pi B (T + \delta T^A) \right\} \text{sinc} \left[ \frac{a(\Omega + \delta \Omega^A)}{2V} \right]$$

where

$$\delta T^W = \frac{2\vec{\mu} \cdot \vec{m}}{c}, \quad \frac{c \delta \Omega^W}{2V \omega_o} = \vec{m} \cdot \frac{\vec{\mu}'}{V} + \vec{t} \cdot \frac{\mathbb{P} \vec{\mu}}{|\vec{r}_p - \vec{\rho}|}.$$  

and

$$\delta T^A = -\frac{a}{V} \frac{\delta \Omega^W}{\omega_o} \quad \text{and} \quad \frac{\delta \Omega^A}{\omega_o} = \frac{Va}{c} \left[ \frac{2\left( \frac{\vec{t}}{V} - \frac{\vec{u}}{V} \right)}{|\vec{r}_p - \vec{\rho}|} \cdot \mathbb{P} \vec{\mu}' + \frac{\vec{\mu}''}{V^2} \cdot \vec{m} \right].$$
Complementary Phase Information for Autofocus

**Wigner Transform**

- By selecting the peaks in the Wigner transform \((\Omega^W(s), T^W(s))\) we can extract the estimate:

\[
\varphi_o(s) = -\frac{c}{2} T^W(s) + O\left(\frac{c}{B}\right), \quad \varphi_1(s) = -\frac{\lambda_o}{4\pi V} \Omega^W(s) + O\left(\frac{\lambda_o}{a}\right).
\]

**Ambiguity Function**

- By selecting the peaks in the ambiguity function \((\Omega^A(s), T^A(s))\) we can extract the estimate:

\[
\varphi_1(s) = \frac{c}{2a} T^A(s) + O\left(\frac{c}{aB}\right), \quad \varphi_2(s) = -\frac{\lambda_o}{2\pi a V} \Omega^A(s) + O\left(\frac{\lambda_o}{a^2}\right).
\]

Similarly, we get a redundant estimate of \(\varphi_1\), with worse resolution, because

\[
\frac{c}{aB} \sim \frac{\lambda_o \omega_o}{aB} \gg \frac{\lambda_o}{a}.
\]

The ambiguity function is useful for the estimation of \(\varphi_2\), and thus complements the Wigner transform in the autofocus process.
Extending Single Scatterer Model

We can extend this approach to situations where the data consists of

- **Motion Estimation:** Cluster of targets moving together
- **Autofocus:** Cluster of stationary targets

by computing the centroid of Wigner transform and ambiguity function instead of picking the peak

![Graphs](a) \(W(\Omega, T)\) 1 scatt  
(b) \(A(\Omega, T)\) 1 scatt  
(c) \(W(\Omega, T)\) 81 scatts  
(d) \(A(\Omega, T)\) 81 scatts

But what about more general scenes with several stationary targets and possibly multiple targets moving in different directions that you wish to track?
Autofocus for a Single Scatterer Image

- Single Sub-Aperture (100 m, 1 degree)

- 1 km Aperture (10 degrees)
Autofocus over 1 km aperture (10°) with Complex Scene

(a) Unfocused

(b) Autofocused
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Can we separate stationary target data from moving target data?

Phase-space method works for autofocus of stationary targets and motion estimation where all targets move with same velocity. In general we have more complex scenes. Ideally, we could decompose the data into stationary plus moving.

This motivates using a data pre-processing step to separate data:

- Geometric travel-time transformation and data filtering
- Robust PCA: Low Rank + Sparse Decomposition

After separation, we can apply existing algorithms individually for motion estimation and autofocus.
Multiple Scatterer Model

Assuming Born (single scattering) approximation, we get superposition of traces

\[
D_r(s, t) = \sum_{m=1}^{M} R_m f(t - (\tau(s, \tilde{\rho}_m(s)) - \tau(s, \tilde{\rho}_o))) + \sum_{n=1}^{N} R_n f(t - (\tau(s, \tilde{\rho}_n) - \tau(s, \tilde{\rho}_o)))
\]

where \(\tilde{\rho}_m(s)\) and \(\tilde{\rho}_n\) denote the \(m\)-th moving target and the \(n\)-th stationary target respectively. \(R_m, R_n\) absorb the reflectivity and other amplitude terms. Assume preliminary autofocus has already eliminated platform perturbations. Probing signal has the form:

\[
f(t) = \cos(\omega_o t) e^{-B^2 t^2 / 2}
\]
Geometric travel-time transformation and data filtering

**Idea:** Since the stationary targets can be imaged reasonably well, we can use the preliminary images in our data processing step.

1. Estimate locations of stationary targets from preliminary image $\{\hat{\bm{\rho}}_\ell\}$

2. Apply a geometrical transformation to $D_r(s, t)$, using travel times from the estimated platform location to one target at a time in the image,

$$
D_r^{\hat{\bm{\rho}}_\ell}(s, t) = D_r \left( s, t + \left( \tau(s, \hat{\bm{\rho}}_\ell) - \tau(s, \bar{\bm{\rho}}_o) \right) \right)
$$

$$
= \sum_{m=1}^{M} f(t - (\tau(s, \bar{\bm{\rho}}_m(s)) - \tau(s, \hat{\bm{\rho}}_\ell))) + \sum_{n \neq \ell}^{N} f(t - (\tau(s, \bar{\bm{\rho}}_n) - \tau(s, \hat{\bm{\rho}}_\ell)))
$$

$$
+ f \left( t - (\tau(s, \hat{\bm{\rho}}_\ell) - \tau(s, \hat{\bm{\rho}}_\ell)) \right)
$$

When $\hat{\bm{\rho}}_\ell = \bar{\bm{\rho}}_\ell$, this transformation removes the slow time dependence of the trace of the echo from that particular target.
3. Remove this trace from the rest of the data using a filter. One technique, which has been successful in seismic imaging\(^2\), amounts in this case to taking an approximation to the derivative in slow time of the transformed data

\[
[QD_{r}^{\hat{\rho}}](s, t) = D_{r}^{\hat{\rho}}(s, t) - \frac{1}{|I(h)|} \int_{I(h)} D_{r}^{\hat{\rho}}(s + h, t) dh
\]

where \(I(h)\) is a small interval of length \(|I(h)|\).

4. Repeat for each estimated stationary target location.
Geometric travel-time transformation and data filtering

Figure: Filtering of a stationary scatterer. Left: Original trace. Middle: Transformed trace. Right: Filtered trace by slow time derivative. Notice that the stationary trace has been eliminated while the moving target trace remains.
Travel-time transformation and Filtering Separation

Figure: Before and after filtering of the data from a 10 stationary scatterer scene with a single moving target with velocity 28 m/s. Top/Bottom=Before/After. Right column is in dB scale.
Robust Principle Component Analysis (Robust PCA)

**Goal:** Decompose $D_r(s, t) = D^s_r(s, t) + D^m_r(s, t) = \text{stationary data} + \text{moving data}$.

**Idea:** Think of $D_r(s, t)$ as a matrix $M$. Decompose into low rank $L$ plus sparse $S$

- Traces from stationary targets $\approx$ columns in $M$.
  Structure $\approx$ low rank part $L$ of $M$.

- The remainder, $S = M - L \approx$ trace from moving target.
  Structure $\approx$ full rank, but sparse.

The RPCA method\(^3\) does such decompositions of matrices, into a low rank part $L$ and a sparse part $S$, by solving the following convex optimization problem:

$$\begin{align*}
\text{minimize} & \quad ||L||_* + \lambda ||S||_1 \\
\text{subject to} & \quad L + S = M.
\end{align*}$$

Here $|| \cdot ||_*$ is the nuclear norm, i.e. the sum of the singular values, and $|| \cdot ||_1$ is the matrix 1-norm. The Lagrange multiplier $\lambda$ has the optimal value of $1/\sqrt{\dim(M)}$.

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\(^3\)Candes, E.J. and Li, X. and Ma, Y. and Wright, J., *Robust principal component analysis?*, Journal of ACM 58(1), 1-37, 2009
Robust PCA: Main Result (of Candes et al)

Let the SVD of $L_0 \in \mathbb{R}^{n_1 \times n_2}$ be:

$$L_0 = U \Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

where $r$ is the rank of $L_0$. The incoherence conditions (with parameter $\mu$) used for the theorem are

$$\max_i ||U^* e_i||^2 \leq \frac{\mu r}{n_1}, \quad \max_i ||V^* e_i||^2 \leq \frac{\mu r}{n_2}, \quad ||UV^*||_{\infty} \leq \sqrt{\frac{\mu r}{n_1 n_2}}$$

Below $n(1) = \max(n_1, n_2)$ and $n(2) = \min(n_1, n_2)$

**Theorem**

Suppose $L_0$ is $n_1 \times n_2$, obeys 3, and that the support set of $S_0$ is uniformly distributed among all sets of cardinality $m$. Then there is a numerical constant $c$ such that with probability at least $1 - cn^{-10}_{(1)}$ (over the choice of support of $S_0$), Principal Component Pursuit with $\lambda = 1/\sqrt{n(1)}$ is exact, i.e. $\hat{L} = L_0$ and $\hat{S} = S_0$, provided that

$$\text{rank}(L_0) \leq \rho_r n(2) \mu^{-1} (\log n(1))^{-2} \quad \text{and} \quad m \leq \rho_s n_1 n_2$$

Above, $\rho_r$ and $\rho_s$ are positive numerical constants.
Decomposing SAR Data into Moving and Stationary Data

Robust PCA

(a) Original frames
(b) Low-rank $\hat{L}$
(c) Sparse $\hat{S}$
Robust PCA: Requirements

Essentially, to recover $L$ and $S$ from $M = L + S$ via this optimization problem, it requires

1. rank($L$) sufficiently small
2. $L$ not sparse
3. rank($S$) sufficiently large
4. $S$ sufficiently sparse

To satisfy these requirements, we need to understand how

1. the number and velocity of moving targets affect the rank and sparsity of $S$, and
2. the density of the stationary scatterers affects the rank and sparsity of $L$

Success will require windowing of $D_r(s, t)$, i.e. applying Robust PCA to $n_s \times n_f$ subsets $(s, t) \in [s_k, \ldots, s_k + n_s \Delta s] \times [t_k, \ldots, t_k + n_f \Delta t]$
Example 1: $D_r^s(s, t)$ too sparse
Example 2: $D_r^S(s, t)$ not sufficiently low rank
Angle between rows of SAR data

Consider a single scatterer SAR scene, we model the data as

\[ D_r(s, t) = f(t - \Delta \tau(s, \vec{\rho}(s))) = \cos(\omega_0(t - \Delta \tau(s, \vec{\rho}(s))))e^{-B^2(t-\Delta\vec{\rho}(s))^2/2} \]

where \( \Delta \tau(s, \vec{\rho}(s)) = \tau(s, \vec{\rho}(s)) - \tau(s, \vec{\rho}_0) \).

First step to understanding rank, examine the angle between two rows of SAR data:

\[
\cos \angle(D_r(s_1, \cdot), D_r(s_2, \cdot)) = \frac{\int_{-\infty}^{\infty} dt \ D_r(s_1, t)D_r(s_2, t)}{\sqrt{\int_{-\infty}^{\infty} dt \ (D_r(s_1, t))^2} \sqrt{\int_{-\infty}^{\infty} dt \ (D_r(s_2, t))^2}}
\]

Figure: \( |D_r(s, t)| \) windowed around a scatterer at \( \vec{\rho}_o = (0, 0) \) (left) and at \( (-10, -10) \) (right).
Angle between rows of SAR data

Linearizing the travel time

\[
\Delta \tau(s, \vec{\rho}_k) \approx \frac{2}{c} \left( \Delta \vec{\rho}_k^m + s \frac{V \Delta \vec{\rho}_k^t}{L} + \frac{(\Delta \vec{\rho}_k^t)^2}{2L} \right)
\]

where

\[
\Delta \vec{\rho}_k^m = (\vec{\rho}_k - \vec{\rho}_o) \cdot \vec{m}(0) \quad \text{(range offset)}
\]

\[
\Delta \vec{\rho}_k^t = \vec{t}(0) \cdot \vec{P}(0) (\vec{\rho}_k - \vec{\rho}_o) \quad \text{(cross-range offset)}
\]

We see that

\[
\cos \angle(D_r(s_1, \cdot), D_r(s_2, \cdot)) \approx \cos(\omega_o \alpha(s_2 - s_1)) e^{-\frac{(B \alpha(s_2-s_1))^2}{4}}
\]

where

\[
\alpha = -\frac{2 V \Delta \vec{\rho}_k^t}{c L}
\]

This tells us that for \(|s_2 - s_1| < \frac{6}{B \alpha}\), the angle between the rows of the data matrix are small, though they oscillate quickly. This supports the hypothesis that \(D_r(s, t)\) should be low rank. In particular, if \(\alpha = 0\) then \(\cos \angle(D_r(s_1, \cdot), D_r(s_2, \cdot)) \approx 1\) and each row is linearly dependent. In this case, \(D_r(s, t)\) is rank 1.
Covariance matrix

Using this result, if we instead compute the entire covariance matrix

\[ M(s_1, s_2) = \int_{-\infty}^{\infty} dt \ D_r(s_1, t)D_r(s_2, t) \approx \frac{\sqrt{\pi}}{2B} \cos(\omega_o \alpha (s_2 - s_1)) e^{-\frac{(B\alpha(s_2-s_1))^2}{4}} \]

Since rank(A)=rank(AA^T), we can study the rank of the matrix M whose entries are M(s_1, s_2). This matrix is symmetric and Toeplitz.

Figure: Covariance matrix of a one scatterer scene with scatterer \( \vec{\rho}_k = (0, 15) \).
Eigenvalues of Toeplitz matrices

Using a theorem by Szego, we know that for real symmetric Toeplitz matrices, which have the form \((c_k)\) for \(k = -n, \ldots, -1, 0, 1, \ldots, n\) and \(c_{-k} = c_k\),

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n+1} F \left( \lambda_k^{(n)} \right)}{n + 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[Q(\theta)] \, d\theta
\]

where \(F(\lambda)\) is any Riemann-integrable function defined on the interval \(m \leq \lambda \leq M\) for finite \(m\) and \(M\).

\[
Q(\theta) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}
\]

is called the symbol of the matrix \((c_k)\). A special case of this theorem is the following:

\[
\lim_{n \to \infty} \frac{N(n; a, b)}{n + 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} I[a \leq Q(\theta) \leq b] \, d\theta
\]

where \(I[\cdot]\) is the indicator function and \(N(n; a, b)\) is the number of eigenvalues between \(a\) and \(b\). This asymptotic result allows us to compute functions of the eigenvalues by simply computing integrals of the matrix’s symbol.
Computing Rank

Applying this to the SAR data, let $s_2 - s_1 = k\Delta s$ be discretized slow-time, we have

$$c_k = \frac{\sqrt{\pi}}{2B} e^{-\frac{(B\alpha k\Delta s)^2}{4}} \cos(\omega_0 \alpha k\Delta s)$$  \hspace{1cm} (4)

The symbol is then

$$Q(\theta) = \frac{\pi}{2B^2|\alpha|\Delta s} \left( e^{-\frac{(\theta-\omega_0 \alpha \Delta s)^2}{B^2 \alpha^2 \Delta s^2}} + e^{-\frac{(\theta+\omega_0 \alpha \Delta s)^2}{B^2 \alpha^2 \Delta s^2}} \right)$$  \hspace{1cm} (5)

Define rank of $M$ then as

$$\text{rank}[M] = N(n; \epsilon \bar{\lambda}, \infty) , \quad \bar{\lambda} \equiv \max Q(\theta) = \frac{\pi}{2B^2 \alpha \Delta s}$$

where $\bar{\lambda}$ is an upper bound on the largest eigenvalue of $M$.

$$\text{rank}[M] \approx (n + 1) \frac{1}{2\pi} \int_{-\pi}^{\pi} I \left[ Q(\theta) \geq \epsilon \bar{\lambda} \right] d\theta$$

$$= (n + 1) \frac{2B|\alpha|\Delta s \sqrt{\log 1/\epsilon}}{\pi} = (n + 1) \frac{2B \frac{2V \Delta s}{L} \frac{\sqrt{\log 1/\epsilon}}{\pi}}{c} |\Delta \hat{\rho}_k^t|$$
Simulation Results

Figure: Left: Comparison of the analytical, computed, and MATLAB rank for a single scatterer scene for various cross-range off-sets of the scatterer. Right: Convergence of the average number of eigenvalues over a threshold, i.e. the rank normalized by the size of the data matrix. It is plotted for various offsets. It requires a large aperture to see close agreement between the asymptotic rank and the rank for our simulated data.
Rank of Moving Target Data

$$\text{rank}[M] \approx (n + 1) \frac{2B|\alpha|\Delta s \sqrt{\log 1/\epsilon}}{\pi},$$

$$\alpha = \frac{2\bar{u} \cdot \bar{m}(0)}{c} - \frac{2V \Delta \bar{\rho}^t_k}{cL} + \frac{2\bar{u} \cdot P(0) \Delta \bar{\rho}_k}{cL}$$

**Figure:** Comparison of the rank for a moving target with various velocities. Here we see good agreement between the asymptotic result and the computed results using the data. $M$ is $296 \times 296$ matrix and so for fast enough targets the matrix becomes full rank.
Two Scatterer Scene

The covariance matrix for a two scatterer \((\vec{\rho}_{k1}, \vec{\rho}_{k2})\) scene is

\[
M(s_1, s_2) \approx \frac{\sqrt{\pi}}{2B} \left[ \cos(\omega_o \alpha_1(s_2 - s_1))e^{-\frac{(B\alpha_1(s_2-s_1))^2}{4}} + \cos(\omega_o \alpha_2(s_2 - s_1))e^{-\frac{(B\alpha_2(s_2-s_1))^2}{4}} \right.
\]

\[
+ \cos(\omega_o (\alpha_1 s_1 - \alpha_2 s_2 + \Delta \beta))e^{-\frac{(B(\alpha_1 s_1 - \alpha_2 s_2 + \Delta \beta))^2}{4}} + \cos(\omega_o (\alpha_1 s_2 - \alpha_2 s_1 + \Delta \beta))e^{-\frac{(B(\alpha_1 s_2 - \alpha_2 s_1 + \Delta \beta))^2}{4}} \right]
\]

where

\[
\alpha_j = -\frac{2V\Delta \vec{\rho}_{k_j}^t}{cL}, \quad \beta_j = -\frac{2}{c} \left( \Delta \vec{\rho}_{m_{k_j}}^t + \frac{(\Delta \vec{\rho}_{k_j}^t)^2}{2L} \right), \quad \Delta \beta = \beta_1 - \beta_2
\]

No longer a Toeplitz matrix unless scatterers are well separated in range:

\[
|\vec{\rho}_{m_{k1}}^t - \vec{\rho}_{m_{k2}}^t| > \frac{c}{B} + \frac{2a}{L} |\vec{\rho}_{k1}^t - \vec{\rho}_{k2}^t|
\]
Decomposing SAR Data into Moving and Stationary Data

Robust PCA

Figure: Left: scatterers located at (5, 5) and (−5, −5). Right: scatterers located at (0.25, 5) and (−0.25, −5).
Well-separated scatterer scenes

Assuming the scene has well separated scatterers in range, the symbol is

\[
Q(\theta) = \frac{\pi}{2B^2|\alpha_1|\Delta s} \left( e^{-\frac{(\theta - \omega_0 \alpha_1 \Delta s)^2}{B^2 \alpha_1^2 \Delta s^2}} + e^{-\frac{(\theta + \omega_0 \alpha_1 \Delta s)^2}{B^2 \alpha_1^2 \Delta s^2}} \right) + \frac{\pi}{2B^2|\alpha_2|\Delta s} \left( e^{-\frac{(\theta - \omega_0 \alpha_2 \Delta s)^2}{B^2 \alpha_2^2 \Delta s^2}} + e^{-\frac{(\theta + \omega_0 \alpha_2 \Delta s)^2}{B^2 \alpha_2^2 \Delta s^2}} \right)
\]

and thus

\[
\text{rank}[M] \approx (n + 1) \left[ \frac{2B|\alpha_1|\Delta s \sqrt{\log(1/\epsilon)}}{\pi} + \frac{2B|\alpha_2|\Delta s \sqrt{\log(|\alpha_1|/(|\alpha_2|\epsilon))}}{\pi} \\
- \max \left( 0, \frac{1}{\pi} \left( \omega_0 \Delta s (|\alpha_1| - |\alpha_2|) + B\Delta s \left( |\alpha_1| \sqrt{\log(1/\epsilon)} + |\alpha_2| \sqrt{\log(|\alpha_1|/(\epsilon|\alpha_2|))} \right) \right) \right]
\]

Figure: Rank of SAR data of two scatterer scene. One scatterer at location \((-1.25, 10)\) and other scatterer varied between \((1.25, 0.01)\) to \((1.25, 50)\) linearly in the y direction.
General scenes

- Szego’s theorem has been generalized to consider other Hermitian matrices. However, more general results do not seem to apply to scenes with scatterers clustered in range.
- However, Robust PCA can work on these scenes.
- Empirically, clustered scenes have lower rank than scenes with well-separated targets in range.
Windowing

Recall: We need to window SAR data in order for Robust PCA to work.

1. Preliminary image formation can be used to compute approximate locations and number of stationary scatterers
2. Analysis results can help us estimate rank of windowed data in order to choose window sizes on which to apply Robust PCA.
3. Calibration is needed.
## Windowing

To match the criteria from the theorem of Candes et al, we need to constrain the rank of the stationary scatterer data and the sparsity of the moving target data, i.e.,

\[
\text{rank}(D_r^s) < \frac{\rho_r \min(n_s, n_f)}{\mu \left(\log(\max(n_s, n_f))\right)^2}
\]

\[
|D_r^m| < \rho_s n_s n_f
\]

It also makes sense to constrain the rank of \(D_r^m\) and the sparsity of \(D_r^s\):

\[
\text{rank}(D_r^m) > \frac{\bar{\rho}_r \min(n_s, n_f)}{\mu \left(\log(\max(n_s, n_f))\right)^2}
\]

\[
|D_r^s| > \bar{\rho}_s n_s n_f
\]

where \(0 < \rho_r < \bar{\rho}_r\) and \(0 < \rho_s < \bar{\rho}_s < 1\) are constants that require calibration. Here \(|\cdot|\) denotes number of non-zero entries. These lead to bounds on \(n_s, n_f\) but they still involve calibration constants. Also rank and sparsity of \(D_r^s\) depends on the velocity of the moving target which is a priori unknown.
Comparison of Methods: Decomposition of 10 scatterers

Figure: Left: Robust PCA; Right: Annihilation filter. Top: Original data
Comparison of Methods: Motion Estimation

**Figure:** Comparing the motion estimation of a moving target in a complex scene with 10 stationary scatterers uniformly placed at random in a $50 \times 50$ $m^2$ area using the Robust PCA method and the annihilation based filtering approach.
Comparison of Methods: Decomposition of 30 scatterers

Figure: Comparison of the filtering done with the Robust PCA decomposition and the annihilation based filter of a stationary scene with 30 scatterers uniformly placed at random in a $50 \times 50m^2$ area.
Comparison of Methods: Motion Estimation

Figure: Comparing the velocity and motion estimation of a moving target in a complex scene with 30 stationary scatterers uniformly placed at random in a 50 × 50 m² area using the Robust PCA method and the annihilation based filtering approach.
Comparison of Methods

- **Annihilation-based Filtering**
  1. Inaccurate location estimates of stationary scatterers. Sensitivity analysis for this is work in progress.
  2. Numerous applications of filter for scenes with numerous strong stationary scatterers degrade moving target signal. This is due to numerical differentiation of a function with limited sampling in slow time.
  3. Slow moving targets

- **Robust PCA**
  1. Requires calibration to choose windows
  2. May struggle to work on dense scenes
  3. Slow moving targets
Summary

- Peaks (centroids) in phase-space of calibrated sub-apertures of the echoes correspond to motion of interest in data (target motion or platform perturbations).

- Wigner transform and ambiguity function give complementary information. Issues: Computational complexity. Tradeoffs between aperture size, resolution and SNR.

- Two approaches to pre-processing of data to separate moving targets from stationary targets in order to decouple motion estimation of moving targets from imaging of stationary scenes.

- Future work: SAR motion estimation with an array of sensors on platform instead of a single antenna.