Some recent contributions to (theoretical and numerical) random homogenization

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based on a series of joint works with
X. Blanc (CEA), PL. Lions (Collège de France)
A. Anantharaman, R. Costaouec, F. Legoll and F. Thomines (ENPC and INRIA)
Real composite material

Courtesy M. Thomas and EADS
Common denominator of many of the works presented:

- go beyond the idealistic setting of periodic materials
- do not treat fully general random materials (fine theoretically, but too expensive to treat practically);
- consider materials that are, in a sense to be made precise, random *perturbations* of periodic materials;
- and adapt the modelling and the numerical approach.

Even more broadly,

**Make mechanics of random materials practical**

Work on the simplest possible equation $- \text{div} \left( a(\varepsilon) \nabla u^\varepsilon \right) = f$. 
Homogenization theory

\[-\text{div} \left( a \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) = f, \quad a \text{ periodic}\]

\[-\text{div} \left( a \left( \frac{x}{\varepsilon}, \omega \right) \nabla u_\varepsilon \right) = f \quad a \text{ stationary}\]

\[-\text{div} \left( a(\Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right)) \nabla u_\varepsilon \right) = f \quad a \text{ periodic, } \nabla \Phi \text{ stationary}\]

\[-\text{div} \left( a(\Phi^{-1} \left( \frac{x}{\varepsilon}, \omega, \omega' \right)) \nabla u_\varepsilon \right) = f \quad a \text{ stationary, } \nabla \Phi \text{ stationary}\]

\[-\text{div} \left( a \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) = f \quad a \text{ belongs to a general, abstract, algebra}\]
Homogenization 1.0.1: the periodic setting

\[-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \mathcal{D},\]

with \( A_{\text{per}} \) symmetric and \( \mathbb{Z}^d \)-periodic: \( A_{\text{per}}(x + k) = A_{\text{per}}(x) \) for any \( k \in \mathbb{Z}^d \).
Homogenization 1.0.1: the periodic setting

\[-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad D, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial D,\]

with \(A_{\text{per}}\) symmetric and \(\mathbb{Z}^d\)-periodic: \(A_{\text{per}}(x + k) = A_{\text{per}}(x)\) for any \(k \in \mathbb{Z}^d\).

When \(\varepsilon \to 0\), \(u^\varepsilon\) converges to \(u^*\) solution to

\[-\text{div} \left[ A^* \nabla u^* \right] = f \quad \text{in} \quad D, \quad u^* = 0 \quad \text{on} \quad \partial D.\]

The effective matrix \(A^*\) is given by

\[[A^*]_{ij} = \int_Q (e_i + \nabla w_{e_i}(y))^T A_{\text{per}}(y) e_j \, dy, \quad Q = \text{unit cube} = (0, 1)^d\]

with, for any \(p \in \mathbb{R}^d\), \(w_p\) solves the so-called corrector problem:

\[-\text{div} \left[ A_{\text{per}}(y) (p + \nabla w_p) \right] = 0 \quad \text{in} \quad \mathbb{R}^d, \quad w_p \text{ is } \mathbb{Z}^d\text{-periodic.}\]

Note that \(u_p(y) = p \cdot y + w_p(y)\) satisfies \(\langle \nabla u_p \rangle = p\).
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Note that \( u_p(y) = p \cdot y + w_p(y) \) satisfies \( \langle \nabla u_p \rangle = p \).

→ Solve \( d \) PDEs (for \( p = e_i, \ 1 \leq i \leq d \)) on the bounded domain \( Q \): easy!
We consider statistically homogeneous random materials:

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}\]

The tensor \( A(x, \omega) \) is such that

- \( A(x, \omega) \) and \( A(x + k, \omega) \) share the same probability distribution, for any \( k \in \mathbb{Z}^d \). For a given realization of the randomness, properties may be different. But, on average, they are identical: the material is statistically homogeneous.

- \( x \mapsto \mathbb{E}(A(x, \cdot)) \) is a periodic function.

- Ergodicity property: space average \( \sim \) average over realizations:

\[
\frac{1}{|Q_N|} \int_{Q_N} A(x, \omega) \, dx \xrightarrow{N \to \infty} \mathbb{E} \left[ \int_Q A(x, \cdot) \, dx \right]
\]

with \( Q = (0, 1)^d \) and \( Q_N = (-N, N)^d \).
Stochastic homogenization

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \mathcal{D}.\]

$u^\varepsilon(\cdot, \omega)$ converges to $u^*$ solution to

\[-\text{div} [A^* \nabla u^*] = f \quad \text{in} \quad \mathcal{D}, \quad u^* = 0 \quad \text{on} \quad \partial \mathcal{D},\]
\[ -\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \mathcal{D}. \]

\( u^\varepsilon(\cdot, \omega) \) converges to \( u^* \) solution to

\[ -\text{div} \left[ A^* \nabla u^* \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^* = 0 \quad \text{on} \quad \partial \mathcal{D}, \]

where the effective matrix \( A^* \) is given by

\[ [A^*]_{ij} = \mathbb{E} \left( \int_{Q} (e_i + \nabla w_{e_i}(y, \cdot))^T A(y, \cdot) e_j \, dy \right), \]

where \( w_p \) solves

\[
\begin{cases}
-\text{div} \left[ A(y, \omega) (p + \nabla w_p(y, \omega)) \right] = 0 & \quad \text{in} \quad \mathbb{R}^d, \quad p \in \mathbb{R}^d, \\
\nabla w_p \text{ is statist. homog., } & \quad \mathbb{E} \left( \int_{Q} \nabla w_p(y, \cdot) \, dy \right) = 0.
\end{cases}
\]

The corrector problem is set on \( \mathbb{R}^d \). Theoretically, the RVE is infinite.
A variant of the classical homogenization setting
A variant of classical stochastic homogenization

Classical stochastic homogenization:

$$-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^{\varepsilon}(x, \omega) \right] = f(x) \quad \text{in} \ D, \quad u^{\varepsilon} = 0 \quad \text{on} \ \partial D$$

where the matrix $A$ is stationary.

We consider here a variant:

$$-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^{\varepsilon}(x, \omega) \right] = f(x) \quad \text{in} \ D, \quad u^{\varepsilon} = 0 \quad \text{on} \ \partial D$$

for a periodic matrix $A_{\text{per}}$ and a random diffeomorphism $\Phi$, with $\nabla \Phi$ stationary.

In general, $A_{\text{per}} \circ \Phi^{-1}$ is NOT stationary.

Random diffeomorphism

\[ \Phi(.,\omega) \]

A 'real' material \( \equiv \) a random deformation of a reference periodic material

Up to (more or less) random glasses, the material is periodic!
Deformed structure

The periodic structure corresponds to identical fibers set on a $\mathbb{Z}^2$ lattice.
Discrete stationary setting

Let \((\tau_k)_{k \in \mathbb{Z}^d}\) be a group action that preserves the measure \(\mathbb{P}\) and is ergodic:

\[
\forall k \in \mathbb{Z}^d, \quad \forall B \in \mathcal{F}, \quad \mathbb{P}(\tau_k B) = \mathbb{P}(B)
\]

\[
\forall B \in \mathcal{F}, \quad (\tau_k B = B \text{ for any } k \in \mathbb{Z}^d) \implies \mathbb{P}(A) = 0 \text{ or } 1.
\]

A function \(F\) is said stationary if,

\[
\forall k \in \mathbb{Z}^d, \quad F(x + k, \omega) = F(x, \tau_k \omega) \quad \text{a.e., a.s.}
\]

Only discrete shifts are allowed.

Ergodic theorem:

\[
F \left( \frac{x}{\varepsilon}, \omega \right) \underset{\varepsilon \to 0}{\ast} \mathbb{E} \left( \int_Q F(x, \cdot) \, dx \right) \text{ in } L^\infty(\mathbb{R}^d), \text{ a.s.}
\]
Stochastic deformations

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon(x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\varepsilon = 0 \text{ on } \partial \mathcal{D}\]

Assumptions:

- the map \( \Phi(\cdot, \omega) \) is almost surely a diffeomorphism from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), with

  \[
  \text{EssInf}_{\omega \in \Omega, x \in \mathbb{R}^d} (\det(\nabla \Phi(x, \omega))) = \nu > 0, \\
  \text{EssSup}_{\omega \in \Omega, x \in \mathbb{R}^d} |\nabla \Phi(x, \omega)| = M < +\infty,
  \]

- \( \nabla \Phi(x, \omega) \) is stationary:

  \[
  \forall k \in \mathbb{Z}^d, \quad \nabla \Phi(x + k, \omega) = \nabla \Phi(x, \tau_k \omega)
  \]

Why do we need \( \nabla \Phi \) to be stationary?
Consider \( b(x, \omega) = b_{\text{per}}(\Phi^{-1}(x, \omega)) \). Let us compute its average,

\[
\frac{1}{R} \int_0^R b(x, \omega) \, dx = \frac{1}{R} \int_0^R b_{\text{per}}(\Phi^{-1}(x, \omega)) \, dx
\]

With the change of variables \( y = \Phi^{-1}(x, \omega) \), we have

\[
\int_0^R b(x, \omega) \, dx \cdot \frac{1}{R} = \frac{1}{\Phi^{-1}(R, \omega) - \Phi^{-1}(0, \omega)} \int_{\Phi^{-1}(0, \omega)}^{\Phi^{-1}(R, \omega)} b_{\text{per}}(y) \Phi'(y, \omega) \, dy
\]

Our assumptions on \( \Phi \) imply that \( \Phi^{-1}(R, \omega) - \Phi^{-1}(0, \omega) \geq M^{-1} R \) a.s. Thus, if \( \Phi'(y, \omega) \) is stationary, this converges to

\[
\left[ \mathbb{E} \int_0^1 \Phi'(y, \omega) \, dy \right]^{-1} \mathbb{E} \int_0^1 b_{\text{per}}(y) \Phi'(y, \omega) \, dy.
\]
Homogenization result (X. Blanc, CLB, P.-L. Lions, 2006)

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \right) \nabla u^\varepsilon(x, \omega) \right] = f(x) \text{ in } D, \quad u^\varepsilon = 0 \text{ on } \partial D \]

\( u^\varepsilon(\cdot, \omega) \) converges (weakly in \( H^1 \) and strongly in \( L^2 \)) to \( u_* \) almost surely, with

\[-\text{div} \left[ A^* \nabla u_*(x) \right] = f(x) \text{ in } D, \quad u_* = 0 \text{ on } \partial D \]

with the homogenized matrix

\[
A^*_{ij} = \det \left[ \mathbb{E} \left[ \int_Q \nabla \Phi(y, \cdot) dy \right] \right]^{-1} \mathbb{E} \left[ \int_{\Phi(Q, \cdot)} \left( e_i + \nabla w_{e_i}(y, \cdot) \right)^T A_{\text{per}} \left( \Phi^{-1}(y, \cdot) \right) e_j dy \right]
\]
Homogenization result (X. Blanc, CLB, P.-L. Lions, 2006)

\[-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} \left( \frac{x}{\epsilon}, \omega \right) \right) \nabla u^\epsilon (x, \omega) \right] = f(x) \text{ in } \mathcal{D}, \quad u^\epsilon = 0 \text{ on } \partial \mathcal{D}\]

\(u^\epsilon (\cdot, \omega)\) converges (weakly in \(H^1\) and strongly in \(L^2\)) to \(u_*\) almost surely, with

\[-\text{div} \left[ A^* \nabla u_* (x) \right] = f(x) \text{ in } \mathcal{D}, \quad u_* = 0 \text{ on } \partial \mathcal{D}\]

with the homogenized matrix

\[A^*_{ij} = \det \left[ \mathbb{E} \left[ \int_Q \nabla \Phi (y, \cdot) dy \right] \right]^{-1} \mathbb{E} \left[ \int_{\Phi(Q, \cdot)} \left( e_i + \nabla w_{ei} (y, \cdot) \right)^T A_{\text{per}} \left( \Phi^{-1} (y, \cdot) \right) e_j dy \right]\]

where, for all \(p \in \mathbb{R}^d, w_p\) is the corrector defined by

\[
\begin{cases}
-\text{div} \left[ A_{\text{per}} \left( \Phi^{-1} (x, \omega) \right) (p + \nabla w_p (x, \omega)) \right] = 0 \text{ in } \mathbb{R}^d, \\
\[w_p (x, \omega) = \tilde{w}_p (\Phi^{-1} (x, \omega), \omega), \quad \nabla \tilde{w}_p \text{ is stationary,}
\[\mathbb{E} \left( \int_{\Phi(Q, \cdot)} \nabla w_p (y, \cdot) dy \right) = 0.\]

\end{cases}
\]
Rate of convergence

\[ \frac{u^\varepsilon(x, \omega) - u^*(x)}{\sqrt{\varepsilon}} \rightarrow ? \quad \varepsilon \rightarrow 0 \]

Analysis in the 1D case by F. Legoll and F. Thomines.
Related results

\[- \frac{d}{dx} \left[ a \left( \frac{x}{\varepsilon}, \omega \right) \frac{d u^\varepsilon}{dx} (x, \omega) \right] = f(x) \text{ in } D = (0, 1), \quad u^\varepsilon = 0 \text{ on } \partial D\]

Let

\[R(\tau) = \text{Cov} \left( \frac{1}{a(x, \cdot)}, \frac{1}{a(x + \tau, \cdot)} \right)\]

- case \(R(\tau) \sim 1/\tau^\alpha\) with \(\alpha > 1\):

- case \(R(\tau) \sim 1/\tau^\alpha\) with \(0 < \alpha < 1\):
Going further
Assume the diffeomorphism $\Phi$ is close to the identity:

$$\Phi(x, \omega) = x + \eta \Psi(x, \omega) + O(\eta^2),$$

for $\eta$ small. Then

$$w_p(x, \omega) = w_p^0(x) + \eta w_p^1(x, \omega) + O(\eta^2),$$

with

$$-\text{div} \left( A_{\text{per}} (p + \nabla w_p^0) \right) = 0, \quad w_p^0 \text{ is } \mathbb{Z}^d\text{-periodic},$$

and

$$\begin{cases} 
-\text{div} \left[ A_{\text{per}} (\nabla w_p^1 - \nabla \Psi \nabla w_p^0) + (\nabla \Psi^T - (\text{div} \Psi) \text{Id}) A_{\text{per}} (p + \nabla w_p^0) \right] = 0, \\
\mathbb{E} \left( \int_Q \nabla w_p^1 \right) = \mathbb{E} \left( \int_Q (\nabla \Psi - (\text{div} \Psi) \text{Id}) \nabla w_p^0 \right), \quad \nabla w_p^1 \text{ stationary.}
\end{cases}$$

However, to compute $A^*$, only the expectation of $w_p^1$ is needed!

$$A_{ij}^* \sim \mathbb{E} \left[ \int (e_i + \nabla w_{ei}(y, \cdot))^T A_{\text{per}} (\Phi^{-1}(y, \cdot)) e_j dy \right].$$
Taking the expectation and setting $\bar{w}_p^1 = \mathbb{E}(w_p^1)$, \\

$$
\left\{
\begin{align*}
-\text{div} \left[ A_{\text{per}} \nabla \bar{w}_p^1 \right] &= \text{RHS} \left( A_{\text{per}}, \mathbb{E}(\nabla \Psi), \nabla w_0^0 \right), \\
\int_Q \nabla \bar{w}_p^1 &= \int_Q \left( \mathbb{E}(\nabla \Psi) - \mathbb{E}(\text{div} \ \Psi) \text{Id} \right) \nabla w_0^0, \quad \nabla \bar{w}_p^1 \text{ periodic}.
\end{align*}
\right.
$$

Eventually, \\

$$
A^* = A^0 + \eta A^1 + O(\eta^2),
$$

with \\

$$
A^0_{ij} = \int_Q (e_i + \nabla w^0_{e_i})^T A_{\text{per}} e_j
$$

$$
A^1_{ij} = \int_Q \text{fct} \left[ \mathbb{E}(\nabla \Psi), A^0, \nabla w^0, A_{\text{per}} \right] + \int_Q \left( \nabla \bar{w}_e^1 - \mathbb{E}(\nabla \Psi) \nabla w^0_{e_i} \right)^T A_{\text{per}} e_j.
$$

Two periodic computations instead of an expensive stochastic one.
In practice, the corrector problem is solved on $Q_N = (-N, N)^d$, using e.g. FEM.

$\rightarrow$ We do not compute $A^*$, but some $A_{N,h}^*(\omega)$.

Yet, a similar result holds (R. Costaouec, CLB, F. Legoll, CRAS 2010):

$$A_{N,h}^*(\omega) = A_0^h + \eta A_1^{N,h}(\omega) + O(\eta^2)$$

where

- $A_0^h$ is the Finite Element approximation of $A^0$, obtained by periodic homogenization of $A_{\text{per}}$.

- $A_1^h := \mathbb{E} \left[ A_{N,h}^1 \right]$ is independent of $N$, and is easy to compute.
Numerical illustration (2D)

\[ \Phi_\eta(x, \omega) = x + \eta \Psi(x, \omega) \quad \text{with} \quad \Psi(x, \omega) = \begin{pmatrix} \psi_A(x_1, \omega) \\ \psi_B(x_2, \omega) \end{pmatrix} \]

with

\[ \psi_A(x, \omega) = \sum_{k \in \mathbb{Z}} 1_{[k, k+1)}(x) \left( \sum_{q=0}^{k-1} 2A_q(\omega) + A_k(\omega) \int_k^x \sin^2(2\pi t) dt \right) \]

where \((A_k)_{k \in \mathbb{Z}}\) and \((B_k)_{k \in \mathbb{Z}}\) are all i.i.d. uniform random variables.

Take as periodic reference structure

\[ A_{\text{per}}(x) = a_{\text{per}}(x) \text{ Id} \]

with

\[ a_{\text{per}}(x) = \beta + (\alpha - \beta) \sin^2(\pi x) \sin^2(\pi y) \in C^\infty(\mathbb{R}^2) \]
Deformed structure in $Q_N$, with $N = 5$ and $\eta = 0.05$
Error at order $\eta^2$

Relative error: \[ \frac{A_{N,h}^*(\omega) - A_h^0 - \eta A_{N,h}^1(\omega)}{\eta^2} \] for $h = 1/3$, $N = 20$
Numerical illustration (R. Costaouec, CLB, F. Legoll, CRAS 2010)

\[ A^0_h + \eta A^1_h \]

\[ A^*_N,h = \frac{1}{3}, \quad N = 20 \]

\[ A^*_N,h(\omega) = A^0_h + \eta A^1_N,h(\omega) + O(\eta^2) \quad \text{and} \quad A^1_h := \mathbb{E} \left[ A^1_N,h \right] \]
Further extension: more randomness!

Define a random “perturbation” in a different topology and provide an efficient approach to compute the homogenized coefficients

Joint work with A. Anantharaman.

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In the previous setting, we have considered \( a_{\text{per}}(\Phi^{-1}(x, \omega)) \) with

\[
\Phi_{\eta}(x, \omega) = x + \eta \Psi(x, \omega) + \ldots,
\]

and \( \eta \) a small scalar. We can alternately consider

\[
\Phi_{\eta}(x, \omega) = x + b_{\eta}(x, \omega) \Psi(x, \omega) + \ldots,
\]

with \( b_{\eta} \) small in some appropriate (random) norm.

To keep things simple, let us assume

\[
a(x, \omega) = a_{\text{per}}(x) + b_{\eta}(x, \omega) c_{\text{per}}(x).
\]

The most interesting case is typically \( b_{\eta}(x, \omega) \) a Bernoulli random variable.
Random $\eta$ (Anantharaman/LB)
Random \( \eta \) (Anantharaman/LB)
Random $\eta$ (Anantharaman/LB)

Law of the material:

$$\delta_a + \eta (\delta_c - \delta_a)$$

on each cell. Cells are independent from one another. Product. Expand at first order in $\eta$:

$$\prod_{k=1}^{N} \delta_a(\text{cell } k) + \eta N \left[ \delta_c(\text{cell } k = 1) \prod_{k=2}^{N} \delta_a(\text{cell } k) - \prod_{k=1}^{N} \delta_a(\text{cell } k) \right].$$

Think of a jellium model in Physics, or, otherwise stated, of a model for defects.

Next remark that in $-\text{div} \left(A(x, \omega)(p + \nabla w(x, \omega))\right) = 0$, the only source of randomness is in $A$. Otherwise stated, $w$ is a deterministic function of $A$. Thus, formally

$$A_{i,j}^* = \int \int (e_i + \nabla w_{e_i}(A, y))^T A(y)e_j \, dy \, \rho(A) \, dA.$$

$$A^* = A_0 + \eta A_1 + \ldots$$

No proof of the expansion. $A_0$, $A_1$ (and $A_2,...$) finite.
Random $\eta$ (Anantharaman/LB)
Inclusions – \( \eta = 0.4 \)
What if the material is “fully” stochastic?

We go back to the standard setting:

\[-\text{div} \left[ A \left( \frac{x}{\varepsilon}, \omega \right) \nabla u^\varepsilon \right] = f \quad \text{in} \quad \mathcal{D}, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \mathcal{D}\]

with $A$ stationary.


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with $A$ stationary. Then the effective matrix $A^*$ is given by

$$[A^*]_{ij} = \mathbb{E} \left( \int_Q (e_i + \nabla w_{e_i}(y, \cdot))^{T} A(y, \cdot) e_j \, dy \right),$$

where $w_p$ solves

$$\begin{cases} 
-\text{div} \left[ A(y, \omega) (p + \nabla w_p(y, \omega)) \right] = 0 & \text{in} \quad \mathbb{R}^d, \\
\mathbb{E} \left( \int_Q \nabla w_p(y, \cdot) \, dy \right) = 0, \quad \nabla w_p \text{ stationary}
\end{cases}$$

This corrector problem is set on $\mathbb{R}^d$. Some approximation is in order to get to a tractable problem.
Standard discretization

- Solve the corrector problem on a truncated domain:

\[
\begin{cases}
- \text{div} \left[ A \left( y, \omega \right) \left( p + \nabla w_p^N (y, \omega) \right) \right] = 0 \quad \text{in} \quad \mathbb{R}^d,
\\
w_p^N \quad \text{is } Q_N\text{-periodic,} \quad Q_N = (-N, N)^d
\end{cases}
\]

- This yields an approximate effective matrix

\[
[A_N^*]_{ij} (\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla w^N_{e_i} (y, \cdot) \right)^T A (y, \omega) e_j \, dy
\]
Standard discretization

- Solve the corrector problem on a truncated domain:

\[
\begin{cases}
-\text{div} \left[ A(y, \omega) \left( p + \nabla w_p^N (y, \omega) \right) \right] = 0 \quad \text{in} \quad \mathbb{R}^d, \\
w_p^N \quad \text{is} \quad Q_N\text{-periodic}, \quad Q_N = (\neg N, N)^d
\end{cases}
\]

- This yields an approximate effective matrix

\[
[A_N^*]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla w_{e_i}^N (y, \cdot) \right)^T A(y, \omega) e_j \, dy
\]

Due to numerical truncation, \( A_N^* \) is random!

When \( N \to \infty \), we have \( A_N^* \to A^* \) a.s. (Bourgeat/Piatnitski, 2004).
Reducing the statistical error

\[ [A^*_N]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} (e_i + \nabla w^N_{e_i}(y, \omega))^T A(y, \omega) e_j \, dy \]

At fixed \( N \),

\[
\text{(small) systematic error} + \text{(large) statistical error}
\]

Can we reduce the statistical error? Can we compute more accurately \( \mathbb{E}[A^*_N] \)?

[Related works by Gloria and Otto]
An academic random material

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \text{Id}_2, \quad a_k \text{ independent identically distributed} \]

\[ a_k = \alpha \text{ or } \beta \text{ with equal probability.} \]
Monte Carlo approximation

Consider $2M$ realizations $A^m(y, \omega)$, compute for each of these
the corrector $w^N_p, m$, solution to

$$-\text{div} \left[ A^m(y, \omega) \left( p + \nabla w^N_p, m(y, \omega) \right) \right] = 0, \quad w^N_p, m \text{ is } Q_N\text{-periodic},$$

and the approximate homogenized matrix

$$[A^*_N, m]_{ij}(\omega) = \frac{1}{|Q_N|} \int_{Q_N} \left( e_i + \nabla w^N_p, m(y, \cdot) \right)^T A^m(y, \omega) e_j \, dy.$$

Approximate $\mathbb{E}(A^*_N)$ by

$$I_{2M} = \frac{1}{2M} \sum_{m=1}^{2M} A^*_N, m(\omega).$$

Classical confidence interval:

$$\left| \mathbb{E}([A^*_N]_{ij}) - [I_{2M}]_{ij} \right| \leq 1.96 \sqrt{\text{Var}([A^*_N]_{ij})} \sqrt{2M}$$

The accuracy of $I_{2M}$ is directly linked with the variance of $A^*_N$. 
In practice, on a 2D example . . .

$I_{2M} \approx \mathbb{E}([A_N^*]_{11})$ (along with confidence intervals) for a given number $2M$ of realizations, and several sizes for $Q_N$.

For moderate $N$, the statistical error $\gg$ systematic error

Our aim: compute $\mathbb{E}(A_N^*)$ more efficiently, for any given $N$. 
Antithetic variables

Goal: compute $\mathbb{E}(f(U))$, with $U$ a random variable uniformly distributed in $[0, 1]$.

- Basic Monte Carlo method: using $2M$ independent realizations of $U(\omega)$,

$$I_{2M}(\omega) = \frac{1}{2M} \sum_{m=1}^{2M} f(U_m(\omega))$$

- Alternative approximation:

$$\overline{I}_{2M}(\omega) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( f(U_m(\omega)) + f(1 - U_m(\omega)) \right)$$

$1 - U(\omega)$ has the same law as $U(\omega)$:

$I_{2M}$ and $\overline{I}_{2M}$ both converge to $\mathbb{E}(f(U))$

At fixed $M$,

- both estimators have the same cost (same number of evaluations of $f$)
- accuracy?
When does it work?

\[ I_{2M}(\omega) = \frac{1}{2M} \sum_{m=1}^{2M} f(U_m(\omega)) \]

\[ \overline{I}_{2M}(\omega) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{2} \left( f(U_m(\omega)) + f(1 - U_m(\omega)) \right) \]

Let's compare the variance:

\[ \text{Var} \overline{I}_{2M} = \text{Var} I_{2M} + \frac{1}{2M} \text{Cov}(f(U), f(1 - U)) \]

\( \overline{I}_{2M} \) is a better estimator than \( I_{2M} \) \iff \( \text{Cov}(f(U), f(1 - U)) \leq 0 \)

Lemma: assume that \( f : [0, 1] \mapsto \mathbb{R} \) is non-decreasing. Then \( \text{Cov}(f(U), f(1 - U)) \leq 0 \).
Back to the homogenization context

We will apply the exact same idea to homogenization, with

\[
\text{input} \equiv U(\omega) \sim A(x, \omega) \mid x \in \mathbb{Q}_N, \quad \text{output} \equiv f(U) \sim A_N^*(\omega)
\]

Any time a random structure is considered, we will also make the computations with the antithetic structure.

Example: each time we see

\[
A(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ a_k(\omega) \ \text{Id}, \quad a_k(\omega) \ \text{are i.i.d.}
\]

we also do the computations with the antithetic field:

\[
B(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ b_k(\omega) \ \text{Id}
\]

where \(b_k(\omega)\) is antithetic to \(a_k(\omega)\).

At each point \(x\), we replace the local microstructure by the antithetic microstructure.
Antithetic materials

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ a_k(\omega) \ \text{Id} \quad \rightarrow \quad B(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) \ b_k(\omega) \ \text{Id} \]

If \( a_k = \alpha \) or \( \beta \) with equal probability, then set \( b_k(\omega) = \beta \) whenever \( a_k(\omega) = \alpha \).

If \( a_k \) is uniformly distributed in \([\alpha, \beta]\), then set \( b_k(\omega) = \alpha + \beta - a_k(\omega) \).
Numerical experiments

\[ A(x, \omega) = \sum_{k \in \mathbb{Z}^2} 1_{Q+k}(x) a_k(\omega) \text{Id}_2, \quad a_k \text{ i.i.d., } a_k \sim \mathcal{U}[\alpha, \beta] \]

\[ I_{2M} \text{ and } \overline{I}_{2M} \text{ (and confidence interval), } \alpha = 3, \beta = 20. \]

Accuracy gain \( \geq \sqrt{6} \) (e.g. CPU time gain of 6 for equal accuracy).

Approach efficient even if \( N \) is not large!
Overview of the results

- The method has been tested in several 2D situations:
  - various input fields,
  - various outputs
- It has been proved to reduce the variance in 1D, for 2D weakly random settings, and for some truly random 2D problems.

Behind the scene:  \(-\text{div} \left[ a(x, \omega) \nabla u(x, \omega) \right] = f(x, \omega)\)
MsFEM approaches for weakly stochastic materials

Keep $\varepsilon$ at its small fixed value

MsFEM approach: introduced by Efendiev, Hou and Wu (subsequent large literature: Aarnes, Allaire & Brizzi, Dostert, Ginting, Chen et al . . . )
Joint work with F. Legoll and F. Thomines.
M2AN, submitted
Asymptotic Analysis, in press.
MsFEM approach in the deterministic setting

\[- \text{div}(A^\varepsilon(x) \nabla u^\varepsilon(x)) = f(x) \quad \text{in} \quad D, \quad u^\varepsilon = 0 \quad \text{on} \quad \partial D.\]

where $A^\varepsilon$ is not necessarily periodic.

Variational formulation: find $u^\varepsilon$ such that

\[ \forall v \in H^1_0(D), \quad A^\varepsilon(u^\varepsilon, v) = b(v), \]

where

\[ A^\varepsilon(u, v) = \int_D (\nabla v)^T A^\varepsilon \nabla u \quad \text{and} \quad b(v) = \int_D f v \, dx. \]

Idea: introduce an approximation with suitably chosen basis functions.

We introduce a classical P1 discretization of the domain $D$, with $L$ nodes, and denote $\phi_i^0$ the basis functions.
Basis functions

One possible definition (many have been proposed: Allaire, Hou, Efendiev):

\[
\begin{aligned}
-\text{div}(A^\varepsilon(x) \nabla \phi^\varepsilon_i, K) &= 0 \quad \text{in } K \\
\phi^\varepsilon_i, K &= \phi^0_i|_K \quad \text{on } \partial K.
\end{aligned}
\]

These problems, indexed by \( K \), are all independent from one another.

Introduce the finite dimensional space

\[
\mathcal{W}_h := \text{span} \{ \phi_i^\varepsilon, \ i = 1, \ldots, L \},
\]

where \( \phi_i^\varepsilon \) is such that \( \phi_i^\varepsilon|_K = \phi_i^\varepsilon, K \) for all \( K \), and proceed with a standard Galerkin approximation using \( \mathcal{W}_h \):

Find \( u_h^\varepsilon \in \mathcal{W}_h \) such that, \( \forall v \in \mathcal{W}_h \),

\[
A^\varepsilon(u_h^\varepsilon, v) = b(v).
\]

This only involves a limited number of degrees of freedom.
A three step method
A three step method

Coarse mesh with a P1 Finite Element basis functions $\phi_i^0$. 

- p. 49
A three step method

Coarse mesh with a P1 Finite Element basis functions $\phi_i^0$.

MsFEM basis

\[
\begin{aligned}
  -\text{div}(A^\varepsilon(x) \nabla \phi_i^{\varepsilon,K}) &= 0 & \text{in } K \\
  \phi_i^{\varepsilon,K} &= \phi_i^0 \big|_K & \text{on } \partial K
\end{aligned}
\]

and glue them together: $\phi_i^{\varepsilon}$ such that $\phi_i^{\varepsilon \big|_K} = \phi_i^{\varepsilon,K}$ for all $K$. The MsFEM functions are computed independently (in parallel) over each $K$. 

- p. 49
A three step method

- Coarse mesh with a P1 Finite Element basis functions $\phi_i^0$.

- MsFEM basis

\[
\begin{aligned}
\begin{cases}
- \text{div}(A^\varepsilon(x) \nabla \phi_i^{\varepsilon,K}) = 0 & \text{in } K \\
\phi_i^{\varepsilon,K} = \phi_i^0|_K & \text{on } \partial K
\end{cases}
\end{aligned}
\]

and glue them together: $\phi_i^{\varepsilon}$ such that $\phi_i^{\varepsilon|_K} = \phi_i^{\varepsilon,K}$ for all $K$. The MsFEM functions are computed independently (in parallel) over each $K$.

- Solve the macro problem with MsFEM basis functions $\phi_i^{\varepsilon}$.
The MsFEM method is accurate even with a coarse mesh, because the basis functions encode the specific fast oscillations of the problem.

\[-\text{div}(A^\varepsilon(x)\nabla \phi_i^{\varepsilon, K}) = 0 \quad \text{in } K \quad \text{with} \quad \phi_i^{\varepsilon, K} = \phi_i^0|_K \quad \text{on } \partial K\]

\[w := \phi_i^{\varepsilon, K} - \phi_i^0|_K\]

\[-\text{div}(A^\varepsilon(x)(\nabla \phi_i^0 + \nabla w) = 0 \quad \text{+ Dirichlet BC: } \sim \text{ corrector problem}\]
Consider the stochastic problem
\[
\begin{cases}
-\text{div}(A^\varepsilon(x, \omega) \nabla u^\varepsilon(x, \omega)) = f(x) \text{ in } \mathcal{D}, \\
u^\varepsilon = 0 \text{ on } \partial \mathcal{D},
\end{cases}
\]
and assume that we wish to build an estimate of the mean \( \mathbb{E}(u^\varepsilon(x, \cdot)) \) using a Monte-Carlo simulation method.

Then, for each realization of \( A^{\varepsilon,m}(x, \omega) \),
- first construct a (random) MsFEM basis \( \phi_{i}^{\varepsilon,m}(x, \omega) \)
- and next solve the macroscale problem to compute \( u^{\varepsilon,m}(x, \omega) \).

Eventually, \( \mathbb{E}(u^\varepsilon(x, \cdot)) \approx \frac{1}{M} \sum_{m=1}^{M} u^{\varepsilon,m}(x, \omega) \).

This is extremely expensive.
Weakly stochastic setting

\[ A^\varepsilon(x, \omega) \equiv A^\varepsilon_\eta(x, \omega) = A^\varepsilon_0(x) + \eta A^\varepsilon_1(x, \omega), \]

where \( A^\varepsilon_0 \) is a deterministic matrix, and \( \eta \) is a small parameter, uniquely determined by
\[ \left\| \frac{A^\varepsilon_1}{A^\varepsilon_0} \right\|_{L^\infty} = 1. \]

Alternative MsFEM method:

1. compute the MsFEM basis functions only once, with the deterministic part of the matrix \( A^\varepsilon_\eta \):
\[ -\text{div}(A^\varepsilon_0(x) \nabla \phi^\varepsilon_i, K) = 0 \quad \text{in } K, \quad \phi^\varepsilon_i, K = \phi^0_i |_K \quad \text{on } \partial K. \]

Deterministic approximation space
\[ \mathcal{W}_h := \text{span} \{ \phi^\varepsilon_i, i = 1, \ldots, L \}, \]

2. next perform Monte-Carlo realizations only for the macro scale problem:
\[ \forall v \in \mathcal{W}_h, \quad \int_D (\nabla v(x))^T A^\varepsilon,m(x, \omega) \nabla u^m(x, \omega) dx = \int_D f v. \]
Numerical illustration \((\varepsilon = 0.025, \mathcal{D} = [0, 1]^2, f = 1)\)

\[
A_\varepsilon(x, y, \omega) = A_0^\varepsilon(x, y) + \eta \sum_{(k,\ell) \in \mathbb{Z}^2} 1_{(k,k+1]}(\frac{x}{\varepsilon}) 1_{(\ell,\ell+1]}(\frac{y}{\varepsilon}) X_{k,\ell}(\omega) A_0^\varepsilon(x, y)
\]

where \((X_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}\) are i.i.d. scalar random variables, \(X_{k,\ell} \sim \mathcal{U}[0, 1]\), and \(A_0^\varepsilon(x, y)\) is a classical test case of the literature.

- We build the deterministic basis functions by locally solving
  \[-\text{div}(A^\varepsilon(x) \nabla \phi_i^\varepsilon) = 0.\]

- For each realization \(A_\varepsilon^m(x, y, \omega)\), solve for \(u^m(x, \omega) = \sum_i U_i^m(\omega) \phi_i^\varepsilon(x)\).

Compare three functions:

- \(u_{\text{ref}}\): reference solution
- \(u_M\): approximation by the general MsFEM approach
- \(u_{w-S}\): approximation by the weakly-stochastic MsFEM approach
Errors ($u_M$: general MsFEM; $u_{w-S}$: weakly-stochastic MsFEM)

Relative error in % (here, with $H^1$ norm; similar conclusion with $L^2$ norm, although errors are 10 times smaller):

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$e(u_M, u_{\text{ref}})$</th>
<th>$e(u_{w-S}, u_{\text{ref}})$</th>
<th>$e(u_{w-S}, u_M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.12 ± 0.19</td>
<td>17.37 ± 0.78</td>
<td>15.51 ± 0.87</td>
</tr>
<tr>
<td>0.1</td>
<td>7.17 ± 0.02</td>
<td>7.62 ± 0.07</td>
<td>2.56 ± 0.10</td>
</tr>
<tr>
<td>0.01</td>
<td>7.15 ± 0.002</td>
<td>7.28 ± 0.007</td>
<td>1.39 ± 0.002</td>
</tr>
</tbody>
</table>

- when $\eta$ is small (here, $\eta \leq 0.1$), $u_{w-S}$ is an approximation of $u_{\text{ref}}$ as accurate as $u_M$, and is obtained for a much smaller cost (the MsFEM basis has only been computed once!)

- as expected, when $\eta$ is not small (say $\eta \approx 1$), the accuracy of the solution $u_{w-S}$ computed with the alternative approach proposed here decreases.

CLB, F. Legoll, F. Thomines,

“General” but “explicit” deterministic problems

Joint work with X. Blanc, PL. Lions.

What is the most general property that allows homogenization while keeping formulae explicit and staying deterministic?

Based on previous works (CPDE 2002, JMPA 2007) and ongoing works (Milan journal of maths, submitted).
Consider a set of points \( \{X_i\}_{i \in \mathbb{N}} \) s.t.

(H1) \( \sup_{x \in \mathbb{R}^3} \#\{ i \in \mathbb{N} \mid |x - X_i| < 1 \} < +\infty \),

(H2) \( \exists R_0 > 0, \inf_{x \in \mathbb{R}^3} \#\{ i \in \mathbb{N}, |x - X_i| < R_0 \} > 0 \),

(H3) the following limit exists in \( L^\infty(\mathbb{R}^n) \):

\[
\lim_{R \to \infty} \frac{1}{|B_R|} \# \left\{ (i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1}, \quad |X_{i_0}| \leq \delta_0 R, \quad |X_{i_0} - X_{i_1} - h_1| \leq \delta_1, \ldots, |X_{i_0} - X_{i_n} - h_n| \leq \delta_n \right\}.
\]
With \( \{X_i\} \) now defined, we introduce, for all \( \varphi \in \mathcal{D}(\mathbb{R}^{3n}) \), the functions (for \( \varphi \in \mathcal{D}(\mathbb{R}^{3n}) \))

\[
f(x) = \sum_{i_1 \in \mathbb{N}} \cdots \sum_{i_n \in \mathbb{N}} \varphi(x - X_{i_1}, \ldots, x - X_{i_n})
\]

Assumptions (Hi) allow these functions to have averages

\[
\langle f \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3(n-1)}} \varphi(x, x - h_1, \ldots, x - h_{n-1}) dl^{n-1}(h_1, \ldots h_{n-1}) dx.
\]

Definition: Set \( \mathcal{A}^{k,p}(\{X_i\}) \) the closure, for \( \| \cdot \|_{W_{\text{unif}}^{k,p}} \), of the vector space generated by

\[
f(x) = \sum_{i_1 \in \mathbb{N}} \sum_{i_2 \in \mathbb{N}} \cdots \sum_{i_n \in \mathbb{N}} \varphi(x - X_{i_1}, x - X_{i_2}, \ldots, x - X_{i_n}), \quad (1)
\]

with \( \varphi \in \mathcal{D}(\mathbb{R}^{3n}) \). It is also the closure for the same norm of the algebra generated by the

\[
f(x) = \sum_{i \in \mathbb{N}} \varphi(x - X_i), \quad \varphi \in \mathcal{D}(\mathbb{R}^3).
\]
The question is homogenization for

\[-\text{div} \left( a\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \right) = f,\]

where \(a\) is a function of the algebra, for instance

\[a(y) = 1 + \sum_{i \in \mathbb{N}} \varphi(x - X_i).\]

Homogenization does hold (because the case falls under the setting of, say, H-convergence). But the issue we examine is the existence of an explicit expression for the limit (thus the uniqueness, and the convergence without considering extractions).

A general theory by N’Guetseng covers (for (H3’)) our setting, but the formulae are not sufficiently explicit (averages, etc...):

\[\forall v \in \mathcal{A}, \quad \langle A(\nabla w_p + p)\nabla v \rangle = 0,\]

\[a^*_i = \langle a_{ij} + a_{ik}\partial_k w_{e_j} \rangle,\]
Examples of sets $\{X_i\}$

Examples of sets $\{X_i\}$:

- Compactly perturbed periodic systems: $\{X_i\}_{i \in \mathbb{N}}$ is a periodic set, except for a finite number of points. For instance, $\mathbb{Z}^3 \setminus \{0\}$.

\[ \mathcal{A}^p(\{X_i\}) = L^p_{\text{per}}(\mathbb{Z}^3) + L^p_0(\mathbb{R}^3), \quad \text{(2)} \]

where $L^0_0(\mathbb{R}^3) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^3), \lim_{|x| \to \infty} \| f \|_{L^p(B+x)} = 0 \}$. The algebra consists of periodic functions up to local perturbations. Note that $\langle | \cdot | \rangle$ is not a norm!

Two semi-crystals:

\[ \mathcal{A}^p(\{X_i\}) = \left( L^p_{\text{per},1}(\mathbb{Z}^3-), L^p_{\text{per},2}(\mathbb{Z}^3+) \right) + L^p_0(\mathbb{R}^3). \]

Deformed periodic lattices:

Set $X_i(\omega) = \Phi(i, \omega)$ where $\Phi$ is a stationary diffeomorphism, and consider the associated algebra. Then $A = B(\Phi^{-1})$ where $B$ is stationary and we may apply the results of our theory.
A motivation

\[- \frac{d}{dx} \left( (a_{per}(x/\varepsilon)) + b(x/\varepsilon) \right) \frac{d}{dx} u_\varepsilon \right) = f\]

where \(b\) is a compact perturbation of the periodic function \(a_{per}\). It homogenizes into \(-a^*(u^*)'' = f\) with \(a^* = a^*_{per} = (a^{-1}_{per})^{-1}\). Using the corrector \(w'_{per}(y) = -1 + a^*_{per} (a_{per})^{-1}(y)\) solution to

\[- \frac{d}{dx} \left( a_{per}(y) \left( 1 + \frac{d}{dy} w_{per}(y) \right) \right) = 0\]

we note that (for \(F(x) = \int_0^x f\) and appropriate integration constants)

\[
[u'_\varepsilon - (1 + w'_{per}(\cdot/\varepsilon)) (u^*)'](x) = (a_{per} + b)^{-1}(x/\varepsilon) (F(x) + c_\varepsilon) - (a_{per})^{-1}(x/\varepsilon) (F(x) + c^*)
\]

\[
= \left[ (a_{per} + b)^{-1} - (a_{per})^{-1} \right] (x/\varepsilon) (F(x) + c_\varepsilon)
\]

\[
+ (a_{per})^{-1}(x/\varepsilon) (c_\varepsilon - c^*)
\]

Consider \(\varepsilon x\) instead of \(x\) (that is, micro instead of macro scale), the r.h.s. does not vanish (because of the first term of the r.h.s.).

It however does in the perfect periodic case...
Consider now:

\[- \frac{d}{dy} \left( (a_{per} + b)(y) \left( 1 + \frac{d}{dy} w(y) \right) \right) = 0\]

that is \(w'(y) = -1 + a_{per} (a_{per} + b)^{-1}(y)\), then:

\[
\left[ u'_\varepsilon - (1 + w'(./\varepsilon)) (u^*)' \right] (x) = (a_{per} + b)^{-1}(x/\varepsilon) \left(c_\varepsilon - c^*\right)
\]

Bingo!

The "quality" of the approximation is identical to that obtained in the perfect periodic case: one can accurately approximate \(u^\varepsilon\) close to the defects.
Example: periodic lattice with a defect

<table>
<thead>
<tr>
<th>$1/\varepsilon$</th>
<th>$\delta^{\varepsilon}_{\text{per}}$</th>
<th>$\delta^{\varepsilon}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.198112</td>
<td>0.0850091</td>
</tr>
<tr>
<td>5</td>
<td>0.191952</td>
<td>0.0425306</td>
</tr>
<tr>
<td>10</td>
<td>0.183784</td>
<td>0.0266084</td>
</tr>
<tr>
<td>20</td>
<td>0.175248</td>
<td>0.0139564</td>
</tr>
</tbody>
</table>

Relative errors using the periodic corrector (left column) and the corrector adapted to the case with defect (right column).

$$
\delta^{\varepsilon}_{\text{per}} = \frac{\| \nabla u^\varepsilon (\varepsilon \cdot) - \nabla u^\varepsilon,1_{\text{per}} (\varepsilon \cdot) \|_{L^2(\Omega)}}{\| \nabla u^\varepsilon (\varepsilon \cdot) \|_{L^2(\Omega)}}, \quad \delta^{\varepsilon} = \frac{\| \nabla u^\varepsilon (\varepsilon \cdot) - \nabla u^\varepsilon,1 (\varepsilon \cdot) \|_{L^2(\Omega)}}{\| \nabla u^\varepsilon (\varepsilon \cdot) \|_{L^2(\Omega)}}.
$$

$$
-\text{div} ((a_{\text{per}}(x/\varepsilon) + b(x/\varepsilon)) \nabla u_\varepsilon) = f.
$$

$$
-\text{div} ((a_{\text{per}}(y) + b(y)) (p + \nabla w_p)) = 0
$$
Ongoing work

Blanc/LB/Lions, Milan journal of maths, submitted.
Objective: Generalize this to other specific cases, to the general case, in one dimension and in dimensions higher than one.

Current difficulty (work in progress – Blanc-LB-Lions–): show that the corrector problem is well posed in the algebra, that is, if \( a \in \mathcal{A} \) then the corrector problem

\[-\text{div} \left( a(y) \left( p + \nabla w_p \right) \right) = 0\]

is uniquely solvable for \( \nabla w_p \in \mathcal{A} \) and \( < \nabla w_p > = 0 \).
http://www-rocq.inria.fr/MICMAC/

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