Magnetic Resonance Elastography Data Analysis

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Waves and imaging in complex media,
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Colleagues of the study


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J. MacLaughlin
Outline of my talk

1. A little bit about MRE
2. MRE data and PDE model
3. Modified integral method
4. Least square method
5. Model independent method and directional filter
Q: which one has the cirrhotic?

Magnitude image  Wave image  Elastogram  Answer

A (a)  (b)  (c)  Normal

B (d)  (e)  (f)  Cirrhotic


- wavelength: short → soft
- wavelength: long → hard
Magnetic Resonance Elastography, MRE

- a newly developed **non-invasive** technique
- measure the **viscoelasticity** of human tissue
  (Muthupillai et al., *Science*, 269, 1854-1857, 1995.)

⇒ enable us to virtually realize a doctor’s palpation

- **Diagnosis:**
  - the stage of liver cirrhosis, fibrosis
  - early stage cancer: breast cancer, pancreatic cancer, prostate cancer, etc.
  - neurological diseases: Alzheimer’s disease, hydrocephalus, multiple sclerosis, etc.

- **Aid of surgery, postoperative observation, evaluation of treatment**

- **Nondestructive testing:** biological material, polymer material
Micro MRE System in Hokkaido University

- 0.3 T Micro-MRI (MR Technology)
- “MRE vibration system” (Japanese Patent: 2010-188514)
Data analysis (inversion schemes) for MRE

MRE data $\rightarrow$ recover stiffness (storage modulus)
MRE phantom: (agarose or) PAAm gel

\[ F = f(x) \cos(2\pi ft) \]

- \( F \) --- time harmonic external vibration (3D vector)
- \( f \) --- frequency of external vibration (50~500Hz)
- \( f \) --- amplitude of external vibration (~500μm)
Viscoelastic wave in soft tissues

- time harmonic external vibration \( F = f(x) \cos(2\pi ft) \)

- interior viscoelastic wave

\[
U(t, x) \approx \text{Re}(u(x)e^{i2\pi ft}) \quad \text{(after some time)}
\]

\[
= \text{Re}((\phi(x) - i\psi(x))e^{i2\pi ft})
\]

\[
= \phi(x) \cos(2\pi ft) + \psi(x) \sin(2\pi ft)
\]

- \( u \) --- amplitude of viscoelastic wave
  \( (\phi: \text{real part, } \psi: \text{imaginary part}) \)
MRE pulse sequence: SE+MSG

- motion-sensitizing gradients (MSG)

\[ G(t) = G_0 \cos(2\pi ft - \beta) \]

- \( \beta \) --- phase offset between external vibration and MSG
- \( G_0 \) --- amplitude of magnetic field
MRE measurements: phase image

MRI signal

real part: R
imaginary part: I

magnitude image

\[ M = \sqrt{|R|^2 + |I|^2} \]

phase image

\[ \theta_\beta = \tan^{-1} \left( \frac{I}{R} \right) \]
MRE measurements: phase image

MRI signal: \( M e^{i\theta_\beta} \), \( M, \theta_\beta \) : magnitude, phase image

\[
G_0 = (G_0, 0, 0) \text{ or } (0, G_0, 0) \text{ or } (0, 0, G_0)
\]

\[
\beta = 0, \pi/2, \pi, 3\pi/2
\]

\[
\theta_0^{(i)}(x) = \frac{N\gamma G_0}{2f} \phi_i(x) + \theta_{err},
\]

\[
\theta_\pi^{(i)}(x) = -\frac{N\gamma G_0}{2f} \phi_i(x) + \theta_{err},
\]

\[
\theta_{\pi/2}^{(i)}(x) = \frac{N\gamma G_0}{2f} \psi_i(x) + \theta_{err},
\]

\[
\theta_{3\pi/2}^{(i)}(x) = -\frac{N\gamma G_0}{2f} \psi_i(x) + \theta_{err},
\]
MRE measurements: phase image

Wave images:

\[ \phi_i(x) = \frac{f}{N \gamma G_0} \theta_{\text{real}}^{(i)} = \frac{f}{N \gamma G_0} (\theta_0^{(i)} - \theta_{\pi}^{(i)}), \]

\[ \psi_i(x) = \frac{f}{N \gamma G_0} \theta_{\text{imag}}^{(i)} = \frac{f}{N \gamma G_0} (\theta_{\pi/2}^{(i)} - \theta_{3\pi/2}^{(i)}), \]

\[ (i = 1, 2, 3) \]
Two types of data analysis (inversion schemes)

- **Model independent data analysis**
  i.e. Don’t need to model tissues, but just assume that the waves are superposition of sinusoidal waves with attenuation.

  1) Local wave vector estimation by the continuous wavelet transform (cf. LFE method)

  2) Local phase estimate
     (cf. phase gradient method)

- **Model dependent data analysis**
  i.e. Need to model tissues.
Model dependent data analysis for MRE

- viscoelasticity of soft tissues
- interior wave displacement

Step 1: modeling

- Step 2: numerical simulation (forward problem)
- Step 3: recovery (inverse problem)

viscoelasticity models for soft tissues
Viscoelasticity models for soft tissues

- **Time**: $t > 0$; $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $3$): bounded domain;

  $\partial \Omega$: Lipschitz continuous boundary;

- **Displacement**: $\mathbf{U}(t, x) = (U_1(t, x), \ldots, U_n(t, x))$

- **General linear viscoelasticity PDE model**:

  $$\rho(x) \partial_t^2 U_i(t, x) = \sum_{l=1}^{n} \frac{\partial}{\partial x_l} \sigma_{il}(\mathbf{U}) \quad (t > 0, x \in \Omega, 1 \leq i \leq n).$$
Viscoelasticity models for soft tissues

- **Stress tensor:** \( \sigma_{il} \ (1 \leq i, \ l \leq n) \)
- **Density:** \( 0 < \delta < \rho(x) \in L^\infty(\Omega) \)
  
  \( \text{(known as } \approx 1.0 \times 10^3 \text{ kg/m}^3 \text{)} \)

- **Small deformation (micro meter)** \( \Rightarrow \) **linear strain tensor**

\[
\varepsilon_{ij}(\mathbf{U}) = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (1 \leq i, j \leq n).
\]
Constitutive equations

- **Voigt model:**
  \[ \sigma_{il}(\mathbf{U}) = \sum_{k,m=1}^{n} \mu_{ilk m} \varepsilon_{km}(\mathbf{U}) + \sum_{k,m=1}^{n} \eta_{ilk m} \partial_t \varepsilon_{km}(\mathbf{U}). \]

- **Maxwell model:**
  \[ \partial_t \varepsilon_{il}(\mathbf{U}) = \sum_{k,m=1}^{n} \mu^{-1}_{ilk m} \partial_t \sigma_{km}(\mathbf{U}) + \sum_{k,m=1}^{n} \eta^{-1}_{ilk m} \sigma_{km}(\mathbf{U}). \]

- **Zener model:**
  \[
  \sum_{k,m=1}^{n} \mu_{ilk m}^{(2)} \sigma_{km}(\mathbf{U}) + \sum_{k,m=1}^{n} \eta_{ilk m} \partial_t \sigma_{km}(\mathbf{U}) = \sum_{k,m=1}^{n} \mu_{ilk m}^{(1)} \mu_{ilk m}^{(2)} \varepsilon_{km}(\mathbf{U}) + \sum_{k,m=1}^{n} (\mu_{ilk m}^{(1)} + \mu_{ilk m}^{(2)}) \eta_{ilk m} \partial_t \varepsilon_{km}(\mathbf{U}).
  \]
Viscoelasticity tensors

- $0 < \delta < A_{ilmn} (\mu_{ilmn}^{(1)}, \mu_{ilmn}^{(2)}, \eta_{ilmn}) \in L^\infty(\Omega)$

- full symmetries:
  \[ A_{ilmn}(x) = A_{likm}(x) = A_{ilmk}(x) = A_{kmnl}(x) \]

- strong convexity (symmetric matrix $\zeta_{il}$):
  \[
  \sum_{i,l,k,m=1}^{n} A_{ilmn}(x) \zeta_{il} \zeta_{km} \geq C \sum_{i,l=1}^{3} \zeta_{il}^2
  \]
Time harmonic wave

- **Boundary:** \( \partial \Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \quad \Gamma_D \neq \emptyset, \quad \Gamma_D \cap \Gamma_N = \emptyset \)

\( \Gamma_D, \Gamma_N \) : open subsets of \( \partial \Omega \), Lipschitz continuous

time harmonic boundary input on \( \Gamma_D \), traction free on \( \Gamma_N \) and 0 initial condition

\( U(t, x) \) conv. exp. to \( u(x)e^{i2\pi ft} \) (i.e. time harmonic wave)

Time harmonic wave

Stationary model:

\[
\begin{align*}
\sum_{l=1}^{n} \frac{\partial}{\partial x_l} \sigma_{il}(u) + \rho (2\pi f)^2 u &= 0 \quad \text{on } \Omega, \\
u = f \quad &\text{on } \Gamma_D, \\
\partial_n u &= 0 \quad \text{on } \Gamma_N
\end{align*}
\]

\[
f \in \dot{H}^{1/2}(\overline{\Gamma'_D}) \rightarrow u(x) \in H^1(\Omega)
\]

- an open subset \( \Gamma'_D \subset \Gamma_D \) with \( C^2 \) boundary away from \( \partial \Gamma_D \)
- \( \dot{H}^s(\overline{\Gamma'_D}) \) the set of distributions in the usual fractional Sobolev space \( H^s(\partial \Omega) \) compactly supported in \( \overline{\Gamma'_D} \)
Constitutive equations (stationary case)

- **Voigt model:**
  \[
  \sigma_{il}(u) = \sum_{k,m=1}^{n} \mu_{ilk,m} \varepsilon_{km}(u) + i2\pi f \sum_{k,m=1}^{n} \eta_{ilk,m} \varepsilon_{km}(u).
  \]

- **Maxwell model:**
  \[
  i2\pi f \varepsilon_{ik}(u) = i2\pi f \sum_{k,m=1}^{n} \mu^{-1}_{ilk,m} \sigma_{km}(u) + \sum_{k,m=1}^{n} \eta^{-1}_{ilk,m} \sigma_{km}(u)
  \]

- **Zener model:**
  \[
  \sum_{k,m=1}^{n} \mu^{(2)}_{ilk,m} \sigma_{km}(u) + i2\pi f \sum_{k,m=1}^{n} \eta_{ilk,m} \sigma_{km}(u)
  \]

  \[
  = \sum_{k,m=1}^{n} \mu^{(1)}_{ilk,m} \mu^{(2)}_{ilk,m} \varepsilon_{km}(u) + i2\pi f \sum_{k,m=1}^{n} (\mu^{(1)}_{ilk,m} + \mu^{(2)}_{ilk,m}) \eta_{ilk,m} \varepsilon_{km}(u).
  \]
Incompressible stationary viscoelasticity model

- **Isotropic**

\[
\mu_{\ell\ell km} := \lambda(x)\delta_{i\ell}\delta_{km} + \mu(x)(\delta_{ik}\delta_{\ell m} + \delta_{im}\delta_{\ell k}),
\]
\[
\eta_{\ell\ell km} := \zeta(x)\delta_{i\ell}\delta_{km} + \eta(x)(\delta_{ik}\delta_{\ell m} + \delta_{im}\delta_{\ell k})
\]

- **Isotropic incompressible stationary viscoelastic model:**

\[
\begin{align*}
\nabla \cdot \left[ 2(G' + iG'')\varepsilon(u) \right] + \rho(2\pi f)^2 u &= 0, \\
(\nabla \cdot u &= 0), \\
+ \text{boundary conditions}
\end{align*}
\]
Scalar model

- Isotropic incompressible stationary scalar model:

\[
\begin{aligned}
\nabla \cdot \left[ 2(G' + iG'') \nabla u \right] + \rho (2\pi f)^2 u &= 0, \\
+ \text{boundary conditions}
\end{aligned}
\]

(global uniqueness if coefficient are piecewise analytic and more than the local Holder stability if we assume more)

- Here, we assumed:

\[
\begin{aligned}
\nabla \cdot u &= 0, \\
\mu(x), \eta(x) &\in C^1(\Omega); \quad \nabla \mu \cdot \frac{\partial u}{\partial x_i} = 0, \\
\nabla \eta \cdot \frac{\partial u}{\partial x_i} &= 0 \quad (1 \leq i \leq 3)
\end{aligned}
\]
Modified Stokes model

- Isotropic+ nearly incompressible

\[ \nu = 0.49999..., \lambda \text{ (GPa)} \gg \mu \text{ (kPa)} \]

- Asymptotic analysis \( \Rightarrow \) modified Stokes model:

\[
\lambda(x) = \alpha \tilde{\lambda}(x), \quad \mu(x) = \beta \tilde{\mu}(x), \quad \kappa = \alpha / \beta, \quad |\kappa| \gg 1;
\]
\[
\tilde{\zeta} := \beta^{-1} \zeta, \quad \tilde{\eta} := \beta^{-1} \eta, \quad \tilde{\rho} := \beta^{-1} \rho,
\]
\[
p := -\lambda \nabla \cdot \mathbf{u} = -\kappa \beta \tilde{\lambda} \nabla \cdot \mathbf{u}
\]
\[
\mathbf{u} = \sum_{j=0}^{+\infty} \kappa^{-j} \mathbf{u}_{-j}, \quad p = \sum_{j=0}^{+\infty} \kappa^{-j} p_{-j},
\]
\[
\begin{cases}
\nabla \cdot [2(G' + iG''') \varepsilon(\mathbf{u})] - \nabla p + \rho (2\pi f)^2 \mathbf{u} = 0, \\
\nabla \cdot \mathbf{u} = 0,
\end{cases}
\]

+ boundary conditions

Storage modulus and loss modulus $(G', G'')$

- **Voigt model**
  \[ G' = \mu, \quad G'' = \omega \eta. \]

- **Maxwell model**
  \[ G' = \frac{\mu (\omega \eta)^2}{\mu^2 + (\omega \eta)^2}, \quad G'' = \frac{\mu^2 (\omega \eta)}{\mu^2 + (\omega \eta)^2}. \]

- **Zener model**
  \[ G' = \mu^{(1)} + \frac{(\mu^{(2)})^2 (\omega \eta)^2}{(\mu^{(2)})^2 + (\omega \eta)^2}, \quad G'' = \frac{(\mu^{(2)})^2 (\omega \eta)}{(\mu^{(2)})^2 + (\omega \eta)^2}. \]

- **Angular frequency:** \( \omega = 2\pi f \)
- **Shear modulus:** \( \mu (\mu^{(1)}, \mu^{(2)}) \)
- **Shear viscosity:** \( \eta \)
Numerical experiments

- 2D numerical simulation (Freefem++)
  - Plane strain assumption

### Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>15.8 kPa</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.4 Pa s</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.4999999</td>
</tr>
<tr>
<td>freq.</td>
<td>250 Hz</td>
</tr>
<tr>
<td>$f$</td>
<td>0.02 mm</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$1.0 \times 10^3$ kg/m$^3$</td>
</tr>
</tbody>
</table>
Modified Stokes model

wave propagation

(a) Ux_real

(b) Ux_imag

(c) Uy_real

(d) Uy_imag
Incompressible stationary viscoelasticity model

(a) Ux_real
(b) Ux_imag
(c) Uy_real
(d) Uy_imag
Modified Stokes model

- 2D numerical simulation (Freefem++)
  - Plane strain assumption
Curl operator

- **Modified Stokes model** (soft tissues: nearly incompressible, isotropic media):

  \[
  \nabla \cdot [2(G' + iG'')\varepsilon(u)] - \nabla p + \rho(2\pi f)^2 u = 0,
  \]

  \[
  \nabla \cdot u = 0,
  \]

  + boundary conditions

- **Locally homogeneous (constants)** \(G', G''\):

  Curl operator: filter of the pressure term (longitudinal wave)

  \[
  \mathbf{w} = \nabla \times \mathbf{u}
  \]

  \[
  (G' + iG'')\Delta \mathbf{w} + \rho(2\pi f)^2 \mathbf{w} = 0
  \]
Pre-treatment: denoising (for Laplacian or curl)

- **Mollifier** (Murio, D. A.: Mollification and Space Marching)

  \[
  (S_\epsilon f)(x) = \int_{\mathbb{R}^2} s_\epsilon(y - x) \hat{f}(y) dy
  \]

- Smooth function \( f \) defined in the nbd of \( \bar{D} \)
- \( D \): a bounded domain
- \( \hat{f} \): an extension of \( f \) to \( \mathbb{R}^2 \)
- \( D \subset D_\epsilon \), distance between \( D \) and the boundary of \( D_\epsilon \geq \epsilon \)
- \( \hat{f} = 0 \) outside \( D_\epsilon \)
- \( s_\epsilon(x) = \epsilon^{-2} s(\epsilon^{-1} x) \) (\( \epsilon > 0 \))
- Function \( s \): a nonnegative \( C^2 \) function over \( \mathbb{R}^2 \) such that

  \[
  \text{supp } s \subset \{ x ||x| \leq 1 \} \quad \text{and} \quad \int s(x) dx = 1.
  \]

- \( S_\epsilon f \in C^2(\bar{D}) \)
Denoising $\varepsilon = 0.01$ (relate to the SD of data)
Recovery of storage modulus

- **Constants:** $G'$, $G''$, $\rho$, $f$
- **Mollification:** $\mathcal{S}_\varepsilon u$
- **Curl operator:** $w = \nabla \times \mathcal{S}_\varepsilon u$
- $$(G' + iG'')\Delta w + \rho(2\pi f)^2 w = 0$$
- **Modified Integral Method**

$$G' - iG'' = -\rho(2\pi f)^2 \left( \frac{\int_D |w|^2 dx}{\int_D w \Delta w dx} \right)$$

$$\int_D w \Delta w dx \neq 0$$

- $D$: test region (2D or 3D)
- test region size: about one wavelength
Recovery from no noise simulated data

amplitude: 0.2 mm, freq.: 250 Hz

Inclusion: small              large          outside

- Exact value:     3.3 kPa 3.3 kPa 7.4 kPa
- Mean value:   3.787 kPa 3.768 kPa 7.436 kPa
- Stddev:           0.147             0.060          0.003
- Relative error: 0.1476           0.1418       0.00049
Recovery from noisy simulated data

10% relative error

Inclusion:       small              large          outside

Exact value:     3.3 kPa 3.3 kPa 7.4 kPa

Mean value:     4.636 kPa 3.890 kPa 7.422 kPa

StdDev:          0.328             0.129          0.322

Relative error:  0.4048          0.1788          0.00294
Recovery from experimental data
(hard 32.5, soft 9.2 kPa by a rheometer)

Layered PAAm gel:
Mean value: hard (left) 31.100 kPa, soft (right) 10.762 kPa
StdDev: hard 0.535, soft 0.201

Frequency: 250 Hz
Amplitude: 0.3 mm
Least square method

- Scalar model

\[
\begin{aligned}
\nabla \cdot \left[(G' + iG'')\nabla u\right] + \rho \omega^2 u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \Gamma_D, \\
\partial_{\nu} u := (G' + iG'')\nabla u \cdot \nu &= 0 \quad \text{on } \Gamma_N,
\end{aligned}
\]

(1)

- Least square functional

\[
J(\tilde{G}', \tilde{G}'') = \frac{1}{2} \left( \frac{\|u(\tilde{G}', \tilde{G}'') - \tilde{u}\|_{L^2(\Omega)}^2}{\|\tilde{u}\|_{L^2(\Omega)}^2} + \alpha \frac{\|\tilde{G}' - \tilde{G}'\|_{L^2(\Omega)}^2}{\|\tilde{G}'\|_{L^2(\Omega)}^2} + \beta \frac{\|\tilde{G}'' - \tilde{G}''\|_{L^2(\Omega)}^2}{\|\tilde{G}''\|_{L^2(\Omega)}^2} \right)
\]

- Constraint:

\[
\mathcal{K} := \left\{(G'(x), G''(x)) \in [L^\infty(\Omega)]^2 \mid 0 < G'_* \leq G'(x) \leq G'^*_*, \ 0 < G''_* \leq G''(x) \leq G'''^* \right\}
\]
The used notations

\( u(\tilde{G}', \tilde{G}'') \): the solution to (1) for \((G', G'') = (\tilde{G}', \tilde{G}'')\)

\( \tilde{u} \): MRE data

\( 0 \leq \alpha < 1 \) and \( 0 \leq \beta < 1 \): two regularization parameters that enforces stability,

\( \tilde{G}' \) and \( \tilde{G}'' \): \textit{a-priori} information about the unknown \( G' \) and \( G'' \), for example some information about the position of inclusion we can get from MRI data.

Try to minimize by the projected gradient method
Numerical tests (without noise, 500 Hz)

Figure 5. Non-regularized recovery of viscoelasticity from simulated data without noise ($\alpha = \beta = 0$): (a) $G'$ (kPa); (b) $G''$ ($\times \omega$ Pa).

true values: $G'$=7.4 kPa (outside), 3.3 kPa (inside), $G''=0.4$ ($\times \omega$ Pa)
Numerical tests (without noise, 500 Hz)

Figure 6. Regularized recovery of viscoelasticity from simulated data without noise ($\alpha = \beta = 0.00002$): (a) $G''$ (kPa); (b) $G'''$ ($\omega$ Pa).
Numerical tests *(20% relative random error, 500 Hz)*

**Figure 8.** Non-regularized recovery of viscoelasticity from noisy simulated data *(α = β = 0)*:

(a) $G'$ (kPa); (b) $G''$ $(\times \omega$ Pa).

*Table 1. Initial parameters*

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G'$</td>
<td>1.0 kPa</td>
<td>$G'^*$</td>
</tr>
<tr>
<td>$G''$</td>
<td>0.1×ω Pa</td>
<td>$G''^*$</td>
</tr>
<tr>
<td>$\tilde{G}'_0$</td>
<td>7.6 kPa</td>
<td>$\tilde{G}''_0$</td>
</tr>
</tbody>
</table>
Numerical tests (20% relative random error, 500 Hz)

![Image](image_url)

**Figure 9.** Regularized recovery of viscoelasticity from noisy simulated data ($\alpha = \beta = 0.00002$): (a) $G'$ (kPa); (b) $G''$ ($\times \omega$ Pa).

<table>
<thead>
<tr>
<th>viscoelasticity</th>
<th>outside inclusions</th>
<th>inside inclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\widetilde{G}'$ (kPa)</td>
<td>7.6 ($=\widetilde{G}'_0$)</td>
<td>3.5</td>
</tr>
<tr>
<td>$\widetilde{G}''$ (Pa)</td>
<td>$0.45 \times \omega$ ($=\widetilde{G}''_0$)</td>
<td>$0.45 \times \omega$ ($=\widetilde{G}''_0$)</td>
</tr>
</tbody>
</table>
Numerical tests (experimental data, 250 Hz)

Figure 14. Regularized recovery of viscoelasticity from experimental data: (a) $G'$ (kPa); (b) $G'' \times \omega$ (Pa).
3 D forward solver for finite element least square method

0.01mm, 500 Hz

By Prof. Suga (Chiba Univ.)
Computation time is 36 minutes and the number of tetrahedrons is 151,096.

3D Mesh Generator: Gmsh-2.5.0 (http://geuz.org/gmsh)
For wave image measured by micro MRE machine, we want to know the \textit{local wavelength} $\lambda$, or equivalently, the \textit{local frequency} $\omega$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{Example image of wave patterns.}
\end{figure}
Key idea of our method

- For given $f(x) \in \mathbb{C}$, $x \in \mathbb{R}^2$, we introduce the Fourier transform:

$$F[f](\xi) := \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} f(x) dx.$$

- Define the Gaussian function:

$$g(x, x_0, b) := \exp \left\{ -\frac{(x - x_0)^2}{2b^2} \right\}.$$
For any given $x_0$, we consider

$$\xi_0 := \arg\max_{\xi \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} f(x) g(x, x_0, b) \, dx \right|.$$ 

Then the local frequency at $x_0$ can be evaluated by

$$\omega(x_0) := |\xi_0|.$$
The arrow points at the propagation direction of the wave, and its length reflects the local wavelength.
Directional filtering
Summary and discussions

1. We discussed an appropriate viscoelastic model for MRE data analysis (i.e. modified Stokes model).

2. As model dependent data analysis, we considered the modified integration method and least square method.

3. The recovery of storage modulus was good for both methods, but further studies are necessary to recover loss modulus.

4. We gave a model independent data analysis and directional filtering which may be useful for pre-processing the data.
Thank you for your attention.