Euclidean random matrices for waves in random media

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The problem: Waves in 3D random media

- Resonant scatterers
- Scattering cross-section \( \sigma \sim \lambda^2 \)

![Diagram showing wave propagation in 3D random media with different regimes: Single scattering, Multiple scattering, Diffusion, and Anderson localization.](image)
Random matrices in quantum (wave) mechanics

**Conjecture:** Certain properties of complex systems are determined by their fundamental symmetries (e.g., time-reversal symmetry) and not by precise details of their structure.

**Idea:** Replace the complicated (and often unknown) Hamiltonian of the system by a random matrix that has the same symmetry.

RMT prediction (Wigner surmise)

Energy dependence of the total cross-section of neutron scattering on $^{238}\text{U}$ (fragment) 

Garg et al. (1964)

Probability density of normalized spacings between nearest levels

Bohigas et al. (1983)

Niels Bohr's toy model for compound-nucleus scattering

*Nature* 137, 351 (1936)
Random matrices for open quantum (wave) systems

“Standard model”:

\[ \hat{H}_{\text{eff}} = \hat{H}_0 - \frac{i}{2} \hat{V} \hat{V}^\dagger \]

- \( N \times N \) random matrix with i.i.d. elements (Hamiltonian of the closed system)
- \( N \times M \) random matrix with i.i.d. elements (describes \( M \) scattering channels)

For disordered systems, this model has a number of drawbacks:

- Scattering channels are not always easy to identify
- Valid under quite restrictive conditions
- Cannot be always justified microscopically

Scattering by $N$ point-like resonant scatterers

\[
\hat{H}_{\text{eff}} = (\omega - i \frac{\Gamma}{2}) \hat{I}_N - \frac{\Gamma}{2} \hat{G}
\]

Random Green matrix: \( G_{ij} = G(r_i, r_j) = \frac{e^{ik_0|r_i-r_j|}}{k_0|r_i-r_j|} \)

\[
\nabla^2 G(r, r') + k_0^2 G(r, r') = -\frac{4\pi}{k_0} \delta(r - r')
\]

\( \omega_0 \) — resonance frequency
\( k_0 = \omega_0/c = 2\pi/\lambda_0 \)
\( \Gamma_0 \ll \omega_0 \) — resonance width
\( \Gamma_0(2R/c) \ll 1 \)

Eigenvalue problem: \( \hat{H}_{\text{eff}} \tilde{\psi}_n = \Lambda_n^H \tilde{\psi}_n \)
\( \hat{G} \tilde{\psi}_n = \Lambda_n^G \tilde{\psi}_n \)

\[
\Lambda_n^H = \omega_0 - i \frac{\Gamma}{2} - \frac{\Gamma}{2} \Lambda_n^G
\]
Chaos versus disorder

\[ \hat{H}_{\text{eff}} = \hat{H}_0 - \frac{i}{2} \hat{V} \hat{V}^\dagger \]

Hermitian (\(\hat{H}_0\)) and anti-Hermitian (\(-\frac{i}{2} \hat{V} \hat{V}^\dagger\)) parts of \(\hat{H}_{\text{eff}}\) are independent

\(\hat{H}_0\) and \(\hat{V}\) have i.i.d. elements → simple theory

\[ \hat{H}_{\text{eff}} = (\omega_0 - \frac{i}{2} \frac{\Gamma_0}{2}) \hat{I}_N - \frac{\Gamma_0}{2} \hat{G} \]

Hermitian (\(\omega_0 \hat{I}_N - \frac{\Gamma_0}{2} \text{Re} \hat{G}\)) and anti-Hermitian (\(-\frac{i}{2} \frac{\Gamma_0}{2} [\hat{I}_N + \text{Im} \hat{G}]\)) parts of \(\hat{H}_{\text{eff}}\) are correlated

\(\hat{G}\) has correlated elements

We will see that the “disordered” case reduces to the “chaotic” one for weak scattering
Green matrix for $N = 2$

$$G_{ij} = (1 - \delta_{ij})G(r_i, r_j) = (1 - \delta_{ij})\frac{e^{ik_0|r_i - r_j|}}{k_0|r_i - r_j|}$$

$$\tilde{G}\tilde{\psi}_n = \Lambda_n \tilde{\psi}_n$$

$$\Lambda_{1,2} = \pm G_{12}$$
$$\psi_{1,2} = (\pm 1, 1)$$

... but the main interest is in $N \gg 1$ case
Random Euclidean matrices

$N \times N$ matrix $F_{ij} = f(r_i, r_j)$

Mézard, Parisi and Zee
Random Euclidean matrices

$N \times N$ matrix $F_{ij} = f(r_i, r_j)$ ← Arbitrary function

$= f(r_i - r_j)$

$= f(|r_i - r_j|)$

Eigenvalue problem:

$\hat{F} \tilde{\Psi}_n = \Lambda_n \tilde{\Psi}_n$

$\Lambda_n$ ← eigenvalues

$\tilde{\Psi}_n = \{\psi_n^i\}$ ← eigenvectors

$n = 1, \ldots, N$

We will be interested in the random Green matrix:

$f(r_i, r_j) = G(r_i, r_j), \quad G_{ij} = G(r_i, r_j) = \frac{e^{ik_0|r_i-r_j|}}{k_0|r_i-r_j|}$

Mézard, Parisi and Zee
Our central result

$$\hat{G} \vec{\phi}_n = \Lambda_n \vec{\phi}_n$$

Analytic theory for the eigenvalue distribution of an arbitrary non-Hermitian Euclidean matrix in the limit of large size $N \to \infty$

EPL 96, 34005 (2011)

Main advances of our approach with respect to previous theories of random Euclidean matrices:

- **Non-perturbative** in $\rho$ or $1/\rho$
- **Full account** for the finite sample size and the breakdown of translational invariance
- **General equations** derived for non-Hermitian matrices
Reminder: resolvent for Hermitian matrices

\[ \hat{F} \hat{\Psi}_n = \Lambda_n \hat{\Psi}_n, \quad \hat{F}^+ = \hat{F}, \quad \Lambda_n - \text{real eigenvalues} \]

Probability density of eigenvalues \( \Lambda \):

\[
p(\Lambda) = \left\langle \frac{1}{N} \sum_{n=1}^{N} \delta(\Lambda - \Lambda_n) \right\rangle = -\left\langle \frac{1}{N} \text{Im} \sum_{n=1}^{N} \frac{1}{\pi} \frac{1}{\Lambda + i\epsilon - \Lambda_n} \right\rangle
\]

\[
\delta(x) = -\frac{1}{\pi} \text{Im} \frac{1}{x + i\epsilon}, \quad \epsilon \to 0^+ 
\]

\[
p(\Lambda) = -\frac{1}{\pi} \text{Im} G(z = \Lambda + i\epsilon) 
\]

Green’s function (resolvent):

\[
G(z) = \frac{1}{N} \left\langle \text{Tr} \frac{1}{z - \hat{F}} \right\rangle 
\]
How to deal with non-Hermitian matrices?

\[ \hat{F} \tilde{\Psi}_n = \Lambda_n \tilde{\Psi}_n, \quad \hat{F}^+ \neq \hat{F}, \quad \Lambda_n - \text{complex eigenvalues} \]

Probability density of eigenvalues \( \Lambda \):

\[
p(\Lambda) = \left\langle \frac{1}{N} \sum_{n=1}^{N} \delta^2(\Lambda - \Lambda_n) \right\rangle = \frac{1}{\pi} \left. \frac{\partial}{\partial \bar{z}} g(z, \bar{z}) \right|_{z = \Lambda} = \frac{1}{\pi} \frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2}, \quad \epsilon \to 0^+ \]

Green’s function (resolvent):

\[
g(z, \bar{z}) = \frac{1}{N} \left\langle \text{Tr} \frac{\bar{z} - \hat{F}^+}{(z - \hat{F}^+)(\bar{z} - \hat{F}^+) + \epsilon^2} \right\rangle \]

Non-Hermitian matrices: matrix Green’s function

\[ \hat{F}_{N \times N} \rightarrow \hat{F}_2 = \begin{pmatrix} \hat{F} & 0 \\ 0 & \hat{F}^+ \end{pmatrix}_{2N \times 2N} \]

\[ \hat{G} = \left< \frac{1}{\hat{z}_\epsilon - \hat{F}_2} \right> = \begin{pmatrix} \hat{G}_{qq} & \hat{G}_{q\bar{q}} \\ \hat{G}_{\bar{q}q} & \hat{G}_{\bar{q}\bar{q}} \end{pmatrix} \quad \text{where} \quad \hat{z}_\epsilon = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \]

Green’s function (resolvent):

\[ g(z, \bar{z}) = \frac{1}{N} \left< \text{Tr} \frac{\bar{z} - \hat{F}^+}{(z - \hat{F})(\bar{z} - \hat{F}^+) + \epsilon^2} \right> = \frac{1}{N} \text{Tr}\hat{G}_{qq} \]

Correlator of right and left eigenvectors:

\[ O(z) = \frac{1}{N} \left< \sum_{n=1}^{N} \langle L_n | L_n \rangle \langle R_n | R_n \rangle \delta^2(z - \lambda_n) \right> = -\frac{N}{\pi} (\text{Tr}\hat{G}_{q\bar{q}})(\text{Tr}\hat{G}_{\bar{q}q}) \]

Euclidean matrices: $HTH^\dagger$ representation

$N \times N$ matrix $F_{ij} = f(r_i, r_j)$

$$\hat{F} = \hat{H} \hat{T} \hat{H}^\dagger$$

Eigenmodes of the box:

$$\psi_m(r) = \frac{1}{\sqrt{V}} e^{i q_m r}$$

$$q_m = \left\{ \frac{2\pi}{L} m_x, \frac{2\pi}{L} m_y, \frac{2\pi}{L} m_z \right\}$$

$$T_{mn} = \frac{\rho}{V} \int_V \int_V d^3r d^3r' e^{-i q_m r} f(r, r') e^{i q_n r'}$$

$$H_{im} = \sqrt{\frac{V}{N}} \psi_m(r_i) = \frac{1}{\sqrt{N}} e^{i q_m \cdot r_i}$$

$$\langle H_{im} \rangle = 0 \quad \langle H_{im} H_{jn}^* \rangle = \frac{1}{N} \delta_{ij} \delta_{mn}$$

$i, j = 1, \ldots, N$, $m, n = 1, \ldots, M$

Two sources of complexity are now separated:

- $\hat{T}$ depends on $f$ but is not random
- $\hat{H}$ is random but does not depend on $f$
Non-Hermitian matrices: diagrammatic expansion

\[ \tilde{G} = \left\langle \frac{1}{\hat{z}_\epsilon - \hat{F}_2} \right\rangle = \left\langle \frac{1}{\hat{z}_\epsilon} + \frac{1}{\hat{z}_\epsilon} \hat{F}_2 \frac{1}{\hat{z}_\epsilon} + \frac{1}{\hat{z}_\epsilon} \hat{F}_2 \frac{1}{\hat{z}_\epsilon} \hat{F}_2 \frac{1}{\hat{z}_\epsilon} + \ldots \right\rangle \]

\[ \hat{z}_\epsilon = \left( \begin{array}{cc} z & i\epsilon \\ i\epsilon & \bar{z} \end{array} \right) \quad \hat{F}_2 = \left( \begin{array}{cc} \hat{F} & 0 \\ 0 & \hat{F}^+ \end{array} \right)_{2N \times 2N} \]

\[ \hat{F} = \hat{H} \hat{T} \hat{H}^+ \]

The diagrams can be calculated using the properties of the matrix \( \hat{H} \):

\[ \left\langle H_{im} \right\rangle = 0 \]

\[ \left\langle H_{im} H_{jn}^* \right\rangle = \frac{1}{N} \delta_{ij} \delta_{mn} \]

assuming that:
- \( H_{im} \) are Gaussian random variables (not essential)
- \( H_{im} \) are independent (breaks down when \( |\mathbf{r}_i - \mathbf{r}_j| < \lambda_0 \))

and keeping only the leading (planar) diagrams in the limit of \( N \to \infty \)
Non-Hermitian matrices: results

\[ \hat{\mathcal{G}} = \left\langle \frac{1}{\tilde{z}_e - \tilde{F}_2} \right\rangle = \begin{pmatrix} \hat{g}_{qq} & \hat{g}_{q\bar{q}} \\ \hat{g}_{q\bar{q}} & \hat{g}_{\bar{q}\bar{q}} \end{pmatrix}_{2N \times 2N} \]

\[ \tilde{g} = \frac{1}{N} \text{Tr}_{\text{block}} \hat{\mathcal{G}} = \begin{pmatrix} \frac{1}{N} \text{Tr} \hat{g}_{qq} & \frac{1}{N} \text{Tr} \hat{g}_{q\bar{q}} \\ \frac{1}{N} \text{Tr} \hat{g}_{q\bar{q}} & \frac{1}{N} \text{Tr} \hat{g}_{\bar{q}\bar{q}} \end{pmatrix}_{2 \times 2} = \begin{pmatrix} g & g_2 \\ g_2 & g \end{pmatrix} = \frac{1}{z - \hat{\Sigma}} \]

\[ \hat{\Sigma} = \begin{pmatrix} \Sigma & \Sigma_2 \\ \Sigma_2 & \Sigma \end{pmatrix} \]

\[ \begin{align*}
\Sigma &= \frac{1}{N} \text{Tr} \left( \frac{(1 - \tilde{g} \hat{T}^+)(\hat{T})}{|1 - g \hat{T}|^2 - |g_2 \hat{T}|^2} \right) \\
\Sigma_2 &= \frac{1}{N} \text{Tr} \left( \frac{g_2 \hat{T} \hat{T}^+}{|1 - g \hat{T}|^2 - |g_2 \hat{T}|^2} \right)
\end{align*} \]

Reminder:
\[ p(\Lambda) = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} g(z, \bar{z}) \bigg|_{z = \Lambda} \]
Borderline of the eigenvalue domain

Correlator of right and left eigenvectors:

\[
O(z) = \frac{1}{N} \left\langle \sum_{n=1}^{N} \langle L_n | L_n \rangle \langle R_n | R_n \rangle \delta^2(z - \lambda_n) \right\rangle = -\frac{N}{\pi} \left( \text{Tr} \hat{G}_{qq} \right) \left( \text{Tr} \hat{G}_{qq}^{-1} \right)
\]

\[= 0\]

\[g_2 = 0\]

\[
\frac{1}{|g|^2} = \frac{1}{N} \text{Tr} \left( \frac{\hat{T} \hat{T}^+}{|1 - g \hat{T}|^2} \right)
\]

\[z = \frac{1}{g} + \frac{1}{N} \text{Tr} \left( \frac{\hat{T}}{1 - g \hat{T}} \right)\]

In the \(r\)-representation:

\[
\frac{1}{|g|^2} = \frac{\rho}{V} \int_V d^3r \int_V d^3r' \left| \frac{f(r, r')}{1 - \rho g f(r, r')} \right|^2
\]

\[z = \frac{1}{g} + \frac{\rho g}{V} \int_V d^3r \int_V d^3r' \frac{f(r, r')^2}{1 - \rho g f(r, r')}
\]

Support of the distribution on the complex plane

$N = 10^4$
$\rho \chi_0^3 = 0.01$

Density of eigenvalues of $\hat{G}$
(numerical)

Borderline of the domain of existence of eigenvalues of $\hat{G}$ (analytical)

$N \gg 1$ points in a sphere:
Support of the distribution on the complex plane

\[ N = 10^4 \]
\[ \rho \lambda_0^3 = 0.01 \]

\[ N = 10^4 \]
\[ \rho \lambda_0^3 = 10 \]

\[ N = 10^4 \]
\[ \rho \lambda_0^3 = 50 \]

\[ N = 10^4 \]
\[ \rho \lambda_0^3 = 500 \]
Full distribution function

\[
N = 10^4 \\
\rho \lambda^3 = 1.0 \\
\gamma = 1.6
\]

Numerical simulation

Analytical prediction
Typical eigenvectors

\[ \tilde{G} \tilde{\psi}_n = \Lambda_n \tilde{\psi}_n \]

\[ N = 1000 \]
\[ \rho \lambda^3_0 = 30 \]

Extended

Localized

Localized on 2 points
Application 1: Collective spontaneous emission

High-density limit: $\rho \lambda_0^3 \to \infty$

Decay rate of the symmetric 1-photon state:

$$\Gamma \sim \Gamma_0 \cdot \frac{N \lambda_0^2}{R^2}$$

Collective Lamb shift:

$$\Delta \omega \sim \Gamma_0 \cdot \rho \lambda_0^3$$

Scully, Phys. Rev. Lett. 102, 143601 (2009)
Application 1: Collective spontaneous emission

\[
p(\text{Re } \Lambda) = \frac{\rho \lambda^3}{\gamma} = 0.01, \quad \gamma = 0.07
\]

\[
p(\text{Im } \Lambda) = \frac{\rho \lambda^3}{\gamma} = 0.1, \quad \gamma = 0.33
\]

\[
p(\text{Im } \Lambda) = \frac{\rho \lambda^3}{\gamma} = 1.0, \quad \gamma = 1.53
\]
Application 2: Anderson localization

\[ \rho \lambda_0^3 = 10 \]

\[ \rho \lambda_0^3 = 30 \]

\[ \rho \lambda_0^3 = 50 \]

States localized due to collective effects

\[
\text{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(r_i)|^4}{\left[ \sum_{i=1}^{N} |\psi_n(r_i)|^2 \right]^2}
\]
Application 2: Anderson localization

\[ \hat{G} \tilde{\Psi}_n = \Lambda_n \tilde{\Psi}_n \]

Thouless number \( g = \frac{\delta \omega}{\Delta \omega} \)

Scaling function \( \beta(g) = \frac{\partial \ln g}{\partial \ln k_0 R} \)

\( \delta \omega = 1 + \text{Im} \Lambda \)

\( \Delta \omega \)

\( \nu \sim 1.5 \)

Numerical data
Application 3: Random lasing in a cloud of cold atoms

The same atoms scatter and amplify light!
Application 3: Random lasing in a cloud of cold atoms

Laser threshold:

\[ \Lambda_n = \frac{1}{t} \]

The dimensionless scattering matrix of a single atom:

\[ t(\omega) = \frac{1}{2} \times \frac{W - 1}{W + 1} \times \frac{1}{\omega - \omega_0 + \frac{i}{2}(W + 1)} \]

An eigenvalue of the matrix \( \tilde{G} \):

\[ \tilde{G} \tilde{\Psi}_n = \Lambda_n \tilde{\Psi}_n \]

The true scattering matrix is \( \tilde{\tau} = (4\pi/k_0)t \)
[see Savels, Mosk and Lagendijk (2005)]

Rate equations beyond threshold:

\[
\frac{dI_n}{dt} = -2\kappa_n I_n + \sum_m \gamma_{nm} I_n I_m
\]

\[
\kappa_n = \frac{\Gamma}{2} \times \frac{W - 1}{W + 1} \left[ \frac{(W + 1)^2}{W - 1} - \text{Im}\Lambda_n \right]
\]

\[
\gamma_{nm} = 4\Gamma \frac{W - 1}{(W + 1)^3} \times \text{Im} \left[ \Lambda_n \sum_{i=1}^{N} |\psi_n^i|^2 |\psi_m^i|^2 \sum_{i=1}^{N} |\psi_n^i|^2 \right]
\]
Application 3: Random lasing in a cloud of cold atoms

Domain spanned by $1/t(\omega)$ when $\omega$ and the pump $W$ are varied

Laser threshold:
$$\Lambda = \frac{1}{t}$$
Application 3: Random lasing in a cloud of cold atoms

\[ \rho \lambda_0^3 = 1 \]
\[ N = 10^4 \]

Domain spanned by \( 1/t(\omega) \) when \( \omega \) and the pump \( W \) are varied

Borderline of the domain of existence of eigenvalues of \( \tilde{G} \) (analytical)

Eigenvalues of \( \tilde{G} \) (numerical)

Laser threshold:
\[ \Lambda = \frac{1}{t} \]

Well below threshold...
Application 3: Random lasing in a cloud of cold atoms

Approaching threshold...

\[ \rho \lambda_0^3 = 5 \]
\[ N = 10^4 \]

Domain spanned by \( 1/t(\omega) \) when \( \omega \) and the pump \( W \) are varied.

Borderline of the domain of existence of eigenvalues of \( \hat{G} \) (analytical).

Eigenvalues of \( \hat{G} \) (numerical).

Laser threshold:
\[ \Lambda = \frac{1}{t} \]

Approaching threshold...
Application 3: Random lasing in a cloud of cold atoms

\[ \rho \lambda_0^3 = 10 \]
\[ N = 10^4 \]

Domain spanned by \( 1/t(\omega) \) when \( \omega \) and the pump \( W \) are varied.

Borderline of the domain of existence of eigenvalues of \( \hat{G} \) (analytical).

Eigenvalues of \( \hat{G} \) (numerical).

Laser threshold:
\[ \Lambda = \frac{1}{t} \]

Threshold reached!
Application 3: Random lasing in a cloud of cold atoms

Laser threshold

Intensity beyond threshold

\[ b_{0\text{cr}} \text{ (minimal } b_0 \text{ for lasing)} \]

\[ \text{Pump } W \]

\[ \text{Intensity } (I/N) \]

(a) diffusion approximation

Green matrix theory

(b) \[ b_0 = 40 \]
Application 4: Instability in a nonlinear medium

\[ G_{ij} = G(r_i, r_j) = \frac{e^{ik_0|r_i-r_j|}}{k_0|r_i - r_j|} \]

\[ \psi(r_i) = \psi_0(r_i) - \frac{k_0}{4\pi} \sum_{j=1}^{N} G_{ij} t_j \psi(r_j) \]

Intensity-dependent scattering matrix: \[ t_j = -\frac{2\pi i}{k_0} \left( e^{i\alpha|\psi(r_j)|^2} + 1 \right) \]

\[ \vec{\psi} = \vec{\psi}_0 + t\vec{G}\vec{\psi} \rightarrow \text{stability analysis} \rightarrow \text{Im}\Lambda \simeq -1 + \frac{1}{2} \alpha \ |I\ | \text{Re}\Lambda \]
Application 4: Instability in a nonlinear medium

\[ \tilde{\psi} = \tilde{\psi}_0 + t \tilde{G} \tilde{\psi} \]  

stability analysis  

\[ \text{Im}\Lambda \simeq -1 + \frac{1}{2} \alpha I \text{Re}\Lambda \]

Linear medium:  
\[ \alpha = 0 \]
Application 4: Instability in a nonlinear medium

\[ \tilde{\psi} = \tilde{\psi}_0 + t \tilde{G} \tilde{\psi} \]  

\[ \text{stability analysis} \rightarrow \text{Im} \Lambda \simeq -1 + \frac{1}{2} \alpha \text{ I Re} \Lambda \]

Nonlinear medium:
\[ \alpha \neq 0 \]

Instability due to pairs of closely located atoms
Application 4: Instability in a nonlinear medium

\[ \tilde{\psi} = \tilde{\psi}_0 + t \tilde{G} \tilde{\psi} \quad \text{stability analysis} \quad \text{Im}\Lambda \simeq -1 + \frac{1}{2} \alpha I \text{ Re}\Lambda \]

Nonlinear medium: \( \alpha \neq 0 \)

Instability due to collective effects
Conclusions

Collective spontaneous emission
2-level atom

Random Green matrix

Random lasing
Pump

Laser emission

Anderson localization

Nonlinear media
\[ t_j = -\frac{2\pi i}{\kappa_0} \left( e^{i\alpha\psi(r_j)} |\psi(r_j)|^2 + 1 \right) \]
From waves to imaging...

Understand propagation of waves in a complex environment

Formulate a mathematical model

Subject of this talk

Test that the model yields reasonable results

Focus of this workshop

Solve the inverse problem

Formulate the inverse problem

Subject of this talk

Focus of this workshop