Coherent Interferometric Imaging for Synthetic Aperture Radar in the Presence of Noise and Dispersion

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Outline

→ On SAR (Synthetic Aperture Radar) configuration and scales.
→ Deep probing with random index of refraction.
→ Image stabilization with CINT (Coherent-Interferometric imaging).
→ The case of dispersive media.
Acquisition Issues

**SAR:** (Single) Transducer (on satellite/plane) moves along known trajectory and emits a train of electromagnetic pulses (chirps) and records the scattered signal.

- High resolution via synthetic aperture and bandwidth.
- Resolution ‘gearing’ gives sensitivity to noise, optimal accuracy require accurate phase recordings.
- Analysis of clutter particular for SAR acquisition.
- Some noise sources:
  - Medium clutter.
  - Medium background variations.
  - Inaccurate transducer location estimates.
  - Time variations in medium.
The Chirped Source and Scaling Regime

- The chirped source pulse:

\[ s(t) = \frac{1}{2}a \left( \frac{t}{T_p} \right) e^{-i\omega_c t - i\pi\gamma t^2} \]

- ”Example scaling”:
  ERS-1: \( \omega_c = 2\pi 5.3 \times 10^9 \text{ s}^{-1}; T_p = 37.1 \times 10^{-6} \text{ s}; \gamma = 4.2 \times 10^{11} \text{ s}^{-2}; \)
  \( \omega_c \gg \pi\gamma T_p \gg T_p^{-1} \) (3.3 \( \times 10^{10} \gg 4.9 \times 10^7 \gg 2.7 \times 10^4 \)).

- This gives scaling regime:
  \( \omega_c \gg \pi\gamma T_p \gg T_p^{-1} \);

Large Carrier; “high” chirp bandwidth, and wide source envelope function.

(chirp allows for emission of high bandwidth signals a relatively small power as large \( T_p \) may give sufficient signal energy for acceptable SNR).
Aperture and Resolution

- For a source pulse shape $a$ and with a stationary phase argument for $\pi \gamma T_p \gg T_p^{-1}$:

$$\hat{s}(\omega) \simeq e^{-\frac{i\pi}{4}} \frac{a}{2\sqrt{\gamma}} \left( \frac{\omega - \omega_c}{2\pi \gamma T_p} \right) \exp \left( i \frac{(\omega - \omega_c)^2}{4\pi \gamma} \right)$$

→ ”Synthetic” bandwidth is: $2\pi \gamma T_p$.

→ ”Synthetic” $X_a$ aperture via recordings: $x_n = (x_n, 0, 0)$, $n = 1, \ldots, N$, with $x_n = (n/N - 1/2)X_a$.

MainQ: How does medium microscale fluctuations affect resolution and SNR?

First: Some more on measurements and imaging function.

Cheney and Borden SIAM 2009.
Deramping and Measurements

- Backscattered signal: $R_n(t)$.

- Single point target at $y_s$, point antenna, homogeneous medium:

$$R_n(t) \mid_{\text{single target}} = \frac{\omega^2 c}{2\pi} \int \hat{G}(\omega, x_n, y_s) v_s \hat{G}(\omega, y_s, x_n) \hat{s}(\omega) e^{-i\omega(t-nT)} d\omega$$

$$= \frac{\omega^2 v_s}{2\pi} \int \hat{G}(\omega, x_n, y_s)^2 \hat{s}(\omega) e^{-i\omega(t-nT)} d\omega$$

for $v_s$ the reflectivity (using Born approximation).
(jargon: $n \sim \text{“slow-time”}; t \sim \text{“fast-time”};$ we used “start-stop approximation”; more generally one may use array antennas that gives specific beam patterns).

- Deramping, multiplication by the opposite quadratic phase “recompresses” signal:

$$S_n(t) = \exp[i\omega_c(t - nT - 2\tau_0) + i\pi\gamma(t - nT - 2\tau_0)^2] R_n(t),$$

where $\tau_0 = |x_{N/2} - y_0|/c_0$ for $c_0$ background velocity and $y_0$ is center of search area.
(typically done in hardware to lower frequency band and allow analogue-to-digital conversion, here “strip-map mode”).

- SAR data: $S_n(t), n = 1, \ldots, N$. 
On Structure of Deramped Signal

- In the Fourier domain:

\[ \hat{S}_n(\omega) |_{\text{homo}} = \omega_c^2 \nu_s \hat{H}(\omega, x_n, y_s) \exp(i \omega (nT + 2\tau_0)) \]

\[ \hat{H}(\omega, x_n, y_s) = \frac{T_p}{32\pi^2 |y_s - x_n|^2} \hat{a} \left[ T_p \left( 4\pi \gamma \left( \frac{|x_n - y_s|}{c_0} - \tau_0 \right) + \omega \right) \right] \]

\[ \exp \left[ 2i\omega_c \left( \frac{|x_n - y_s|}{c_0} - \tau_0 \right) \right] \exp \left[ 2i\omega \left( \frac{|x_n - y_s|}{c_0} - \tau_0 \right) + 4i\pi \gamma \left( \frac{|x_n - y_s|}{c_0} - \tau_0 \right)^2 \right] \]

- the centered & deramped model response function \( \hat{H}(\omega, x_n, y_s) \) depends on \( x_n, y_s \) only through \( |x_n - y_s| \),

- Concentration at \( |x_n - y_s| \approx c_0 (\tau_0 - \frac{\omega}{4\pi \gamma}) \), with resolution \( O(c_0/4\gamma T_p) \).

\[ \iff \] gives range resolution (high chrip bandwidth).

- \( \omega_c \gg \pi \gamma T_p \iff \) slow amplitude modulation.

\[ \iff \] Phase variation with source point gives azimuthal resolution.
Classical SAR Ambiguity Function

• The point spread function:

\[ I_n(y) = \int \overline{H}(t - nT - 2\tau_0, x_n, y)S_n(t) dt \]

\[ = \frac{1}{2\pi} \int \overline{H}(\omega, x_n, y) \exp(-i\omega(nT + 2\tau_0)) \hat{S}_n(\omega) d\omega, \]

which is matched filter with time reversal, back-propagation or “adjoint” interpretations, that is, match signal with received signal in case of homogeneous medium and point scatterer at search point at \( y \). Best signal-to-noise ratio in presence of white noise.

• The SAR ambiguity function:

\[ I(y) = \left| \sum_n I_n(y) \right|^2 = \frac{1}{4\pi^2} \sum_{n,n'=1}^N \int \int \overline{H}(\omega, x_n, y) \hat{H}(\omega', x_{n'}, y) \]

\[ \times \hat{S}_n(\omega) \hat{S}_{n'}(\omega') \exp(i\omega'(n'T + 2\tau_0) - i\omega(nT + 2\tau_0)) d\omega d\omega'. \]

**Optimal Resolution**

- Assume scatterer in plane through antenna trajectory and $\mathbf{r} = (y, z)$ the range (3rd dimension via beam pattern).

- Ambiguity function

$$I(y) = I_0 F\left(\frac{4\pi\gamma T_p}{c_0} (r - r_s)\right) \text{sinc}^2\left(\frac{\omega_c X_a}{c_0 |y_0|} (x_s - x)\right),$$

$$F(r) = \left| \int \overline{\hat{a}(r+u)} \hat{a}(u) du \right|^2, \quad I_0 = \frac{|v_s|^2 \omega_c^4 T_p^2 N^2}{2^{22} \pi^{10} |y_0|^8}$$

- “Optimal” range and azimuthal resolutions from synthetic bandwidth and Rayleigh resolution associated with synthetic aperture and carrier wavelength:

$$\Delta r = \frac{c_0}{4\gamma T_p} \quad \text{and} \quad \Delta x = \frac{\pi c_0 |y_0|}{\omega_c X_a} = \frac{\lambda_c |y_0|}{2 X_a}.$$

with $y_0$ center of search area.

$\hookrightarrow$ For instance for ers-1 $X_a \approx 2km$ this gives $\Delta r \approx 5m$ and $\Delta x = 10m$. 
The Role of the Noise

- Measurement errors and clutter may lead to *"white noise measurements errors"* "additive noise" in scaling limit.
- An important noise source is *"wave front perturbations"* or "multiplicative noise".
- **Example**: Recorded travel times perturbed by an additive zero-mean random vector $\tau^{(r)}_n$ with Gaussian statistics:

$$\mathbb{E}[\tau^{(r)}_n \tau^{(r)}_{n'}] = \sigma_t^2 \exp(- (x_n - x_{n'})^2 / l_c^2).$$

$\rightarrow$ **Q**: How can we describe the statistical scaling structure of the observations given the statistics of the medium?

$\rightarrow$ **Q**: How is resolution and stability affected?
On the propagation model

We consider propagation of high-frequency ("transverse electromagnetic waves in dilute plasma (earth’s ionosphere) governed by the Klein-Gordon equation")\(^{†}\):

\[
c^{-2} \frac{\partial^2 E_\perp}{\partial t^2} - \Delta E_\perp + \left( \frac{\omega'_{pe}}{c'_0} \right)^2 E_\perp = A_n^\varepsilon.
\]

\(\rightarrow\) We consider first the non-dispersive case with Langmuir frequency \(\omega'_{pe} = 0\), but fluctuations in index of refraction.

• Forcing term \(A_n^\varepsilon\) of form:

\[
A_n^\varepsilon(t, x_\perp, z) = \varepsilon^{\frac{p(2-d)}{2}} \exp \left( -\frac{i\omega'_{c}t}{\varepsilon p} - \frac{\pi\gamma't^2}{\varepsilon^2 p} \right) a \left( \frac{t}{T'_p \varepsilon p}, \frac{x_\perp - (x_n, 0)}{\varepsilon p} \right) \delta(z - z_s) |_{z_s=0}.
\]

• Medium fluctuations of form:

\[
c^{-2}(x_\perp, z) = (c'_0)^{-2} \left( 1 + \varepsilon^{p-1} \nu \left( \frac{x_\perp}{\varepsilon^{\eta}}, \frac{z}{\varepsilon^2} \right) 1_{(0,L')} \right),
\]

• Non-dimensionalized units: \(c'_0 = O(1)\); \(L' = O(1)\).

• Assume point reflector at \(y'_s = (x'_s, y'_s, L')\).

\(\rightarrow\) Scaling regime:

\[\eta < \frac{p}{2}, \quad 0 < \eta \leq 2, \quad 1/T'_p \ll \gamma'T'_p \ll \omega'_{c} = O(1) \quad |x'_s|, |y'_s| \ll L'.\]

\(^{†}\) SIAM J. Imag. 2009, On SAR Imaging through the Earth’s Ionosphere; S.V. Tsynkov.
Specific Scaling Regimes (ers-1)

Height/distance from target: $L = 780 km$. Carrier $\omega_c = 2\pi 5.3 \times 10^9 s^{-1}$.

Parameters:

- Relative longitudinal medium correlation length: $\varepsilon^2 = l_z/L$.
- Carrier ($f_0 = c_0/L$): $(\omega_c/f_0)\varepsilon^p = \omega'_c = 2\pi$.
- Synthetic bandwidth: $(\gamma T_p/f_0)\varepsilon^p = (\gamma' T'_p) = 10^{-3}$.
- Pulse duration: $(1/T_p f_0)\varepsilon^p = (1/T'_p) = 10^{-5}$.

$\rightarrow$ We choose:

Correlation length aspect ratio $l_x/l_z = 10$. & **homogeneous** medium statistics & Relative medium fluctuations $10^{-5}$. (corresponds to $c_n^2 \propto 10^{-13} |L_0=1 km|$)

<table>
<thead>
<tr>
<th>correlation regime</th>
<th>$l_z$</th>
<th>$\varepsilon$</th>
<th>$p$</th>
<th>$\eta$</th>
<th>$C(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>short</td>
<td>20m</td>
<td>$5 \times 10^{-3}$</td>
<td>3.1</td>
<td>1.5</td>
<td>0.5</td>
</tr>
<tr>
<td>medium</td>
<td>200m</td>
<td>$10^{-2}$</td>
<td>4.0</td>
<td>1.4</td>
<td>4.9</td>
</tr>
<tr>
<td>long</td>
<td>2km</td>
<td>$5 \times 10^{-2}$</td>
<td>5.5</td>
<td>1.2</td>
<td>49</td>
</tr>
</tbody>
</table>

- Here: $C(x_\perp) = \int_{-\infty}^{\infty} \mathbb{E}[v(0,0)v(x_\perp,z)]dz < \infty$, (integrable covariance).
- For $l_z = .1 m$ we have $p = 2$. 


Statistical Structure of Transmitted and Reflected Wave

Schematic showing the transmitted wave field distortions: after a time of order one; front distortions have a lateral correlation radius of order $\varepsilon\eta$ and is of order $\varepsilon^p$, the carrier wavelength.

→ A spatial random travel time model that captures multifrequency correlations and delay spread ($p=2$).
Front Moment Description

\[
\left\{ R_{\text{back},n}^{\epsilon}(\tau_{n}^{(0)} + \epsilon p s_{n}) \right\}_{n=1}^{N} \sim \left\{ \left( A_{\text{back},n} * s \, r_{\text{back},n}^{0} \right) \left( s_{n} - \Theta_{\text{back},n} \right) \right\}_{n=1}^{N}.
\]

Here

1. The field \( r_{\text{back},n}^{0} (s) \) is the centered reflected field observed in the absence of random medium fluctuations (\( \tau_{n}^{(0)} \): two way travel time to scatterer in homogeneous medium).

2. The process \( (\Theta_{\text{back},n}) \) is a Gaussian process with mean zero. The covariance function of the process \( (\Theta_{\text{back},n}) \) is

\[
\mathbb{E} [\Theta_{\text{back},n_1} \Theta_{\text{back},n_2}] = \frac{L'}{(\frac{c'}{c_0})^2 \cos^2(\theta)} \tilde{C}(n_1 - n_2),
\]

where

\[
\cos(\theta) = \frac{L}{|y|},
\]

and we assume \( (x_1 - x_0) = O(\epsilon^n L) \) and \( \tilde{C} \) is defined in terms of medium statistics.

Garnier and S. 2012
Details of Covariance of Travel Time Shift

\[
\tilde{C}(l) = \begin{cases} 
\int_{0}^{1} \int_{-\infty}^{\infty} C(le_1(1-w),z) \, dz \, dw & \text{if } \eta < 2, \\
\int_{0}^{1} \int_{-\infty}^{\infty} C(le_1(1-w)-y_{\perp,s},z) \, dz \, dw & \text{if } \eta = 2.
\end{cases}
\]

The convolution kernel \( (\mathcal{A}_{\text{back}}(s))_{s \in \mathbb{R}} \) is deterministic and its Fourier transform is given by

\[
\hat{\mathcal{A}}_{\text{back}}(\omega) = \exp \left[ -\frac{\omega^2 \Gamma^{(p)}_{\omega \cos(\theta)/c_0} L'}{4(c'_0)^2 \cos^2(\theta)} \right],
\]

where

\[
\Gamma^{(p)}_k = \begin{cases} 
2 \int_{0}^{\infty} C(0,z) \exp(ikz) \, dz & \text{if } p = 2, \\
2 \int_{0}^{\infty} C(0,z) \, dz & \text{if } p < 2, \\
0 & \text{if } p > 2.
\end{cases}
\]

\( \rightarrow \) In above examples only travel time perturbation.
Connection to random travel time model

- Medium perturbations:

\[ \frac{1}{c^2(\mathbf{x})^2} = \frac{1}{c_1^2} \left( 1 + v^\varepsilon(\mathbf{x}) \right), \quad v^\varepsilon(\mathbf{x}) = \varepsilon^{p-1} v \left( \frac{\mathbf{x}_\perp}{\varepsilon^2}, \frac{z}{\varepsilon^2} \right). \]

Greens function with random travel time perturbation:

\[ \hat{G}^\varepsilon(\omega, \mathbf{x}, \mathbf{y}) \approx \alpha_1(\mathbf{x}, \mathbf{y}, \omega) \exp \left( i\omega \left[ \tau_1(\mathbf{x}, \mathbf{y}) + v^\varepsilon(\mathbf{x}, \mathbf{y}) \right] \right). \]

Here

\[ v^\varepsilon(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x} - \mathbf{y}|}{2c_1} \int_0^1 v^\varepsilon(\mathbf{y} + (\mathbf{x} - \mathbf{y})w) \, dw. \]

is a zero-mean random travel time perturbation acquires Gaussian statistics in the limit \( \varepsilon \to 0 \), and its covariance is described by

\[ \varepsilon^{-2p} \mathbb{E} \left[ v^\varepsilon(\mathbf{0}, 0, (\mathbf{x}_\perp, L)) v^\varepsilon(\mathbf{0}, 0, (\mathbf{x}_\perp + \varepsilon^2 \mathbf{y}_\perp, L)) \right] \]

\[ \to_{\varepsilon \to 0} \frac{L}{4c_1^2 \cos^2(\theta)} \int_0^1 \int_{-\infty}^{\infty} C(\mathbf{y}_\perp w, z) \, dz \, dw. \]

As in the case of the wide angle approximation corresponding to the high frequency regime of the travel time model \( (p > 4) \).
The antenna coherence length

• For $l_c$ and $\sigma_t$ the correlation length and magnitude of the travel-time correction along the antenna we have

$$X_c = \frac{l_c}{\omega_c \sigma_t} = \frac{l_x \cos(\theta)}{2\pi \sqrt{\tilde{C}_{back,y}(0)}} = o(\varepsilon^\eta).$$

$\xrightarrow{} X_c \sim$ the “maximum separation distance” on the antenna before observations becomes incoherent.

• For the above example this gives $(\cos(\theta) = 1)$:

<table>
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<th>$X_c$</th>
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<tr>
<td>short</td>
<td>200m</td>
<td>50m</td>
</tr>
<tr>
<td>medium</td>
<td>2km</td>
<td>150m</td>
</tr>
<tr>
<td>long</td>
<td>20km</td>
<td>500m</td>
</tr>
</tbody>
</table>
The SAR ambiguity function:

\[
I(y) = \left| \sum_n I_n(y) \right|^2 = \frac{1}{4\pi^2} \sum_{n,n'=1}^N \int \int \overline{H}(\omega, x_n, y) \hat{H}(\omega', x_{n'}, y) \\
\times \hat{S}_n(\omega; \tau_n^{(r)}) \hat{S}_n(\omega'; \tau_{n'}^{(r)}) \exp(i\omega'(n'T + 2\tau_0) - i\omega(nT + 2\tau_0)) d\omega d\omega' 
\]

\[\hookrightarrow\] Introduce \( X_d \) and \( \Omega_d \) cut-off parameters in space (along the antenna trajectory) and in frequency, then the CINT imaging functional becomes:

\[
I_{\text{CINT}}(y) = \frac{1}{4\pi^2} \sum_{|n-n'|\leq n_d/2} \int \int |\omega-\omega'|\leq \Omega_d/2 \overline{H}(\omega, x_n, y) \hat{H}(\omega', x_{n'}, y) \\
\times \hat{S}_n(\omega) \hat{S}_n'(\omega') \exp(i\omega'(n'T + 2\tau_0) - i\omega(nT + 2\tau_0)) d\omega d\omega' 
\]

where \( n_d = \lfloor NX_d/X_a \rfloor \).

- \( X_a, |y - y_0| \ll |y_0| \implies \) only frequencies \((\omega, \omega')\) with small offsets:
  \[
  |\omega - \omega'| \leq O\left(1/T_p\right) \text{ contributes to ambiguity function.}
  \]

Borcea et al Geophysics 2006, Inverse Problems 2011, ...
On CINT Parameters

• For $X_d > X_a$ and $\Omega_d > 2T_p^{-1}$, we recover SAR ambiguity function $\rightarrow$ ”optimal resolution”, but very noisy images.

• Reduce $X_d$ and $\Omega_d \Rightarrow$ resolution is reduced as well, but enhanced signal-to-noise ratio. Only coherent data are being used in backpropagation of correlations.

• Due to diagonal concentration of ambiguity kernel only space filtering is effective for CINT in SAR setting in our regime.
Coherence Recovered with CINT

- We find that

\[
\mathbb{E} \left[ I_{\text{CINT}}(y) \right] \propto \tilde{F} \left( \frac{4\pi\gamma T_p}{c_0} (r - r_s) \right) \text{sinc} \left( \frac{\omega_c \min(X_a, X_c, X_d)}{c_0 |y_0|} (x_s - x) \right).
\]

→ Optimal resolution \((X_c \ll X_a)\):

\[
\Delta r = \frac{c_0}{4\gamma T_p}, \quad \Delta (c)x = \frac{\lambda_c |y_0|}{2X_c}.
\]

→ Effective synthetic aperture \(X_c\).

↔ By choosing \(X_d\) of the order of \(X_c\), we obtain a compromise between resolution and noise reduction, and the signal to noise ratio \((X_c \ll X_a)\):

\[
\frac{\mathbb{E} \left[ I_{\text{CINT}}(y) \right]}{\text{Var} \left( I_{\text{CINT}}(y) \right)^{1/2}} \approx \frac{\sqrt{X_a}}{\min \left( \max \left( \sqrt{X_c}, \sqrt{X_d} \right), \sqrt{X_a} \right)}.
\]

↔ For SAR ambiguity function we have \(\text{SNR} = O(1)\).

↔ As \(X_d\) is reduced: \(X_a \geq X_d \geq X_c\), & “optimal resolution preserved”.
Statistical structure of image fluctuations

- Assume \( \frac{X_a}{N} \ll X_c \ll X_a \):

\[ \leftrightarrow \text{Image fluctuations localized on support of coherent image} \]

“\( \sim \max(\Delta x^{(c)}, \Delta x^{(d)}) \times \Delta r \).”

\[ \leftrightarrow \text{: Correlations structure:} \]

\textbf{SAR}: Lateral and range correlation lengths of image fluctuations:
\[ l_x^{(I)} = \Delta x^{(a)} \quad \& \quad l_r^{(I)} = \Delta r \]

\textbf{CINT}: \( X_d \ll X_a \)
\[ l_x^{(I)} = \Delta x^{(d)} \quad \& \quad l_r^{(I)} = \Delta r \]

\[ \leftrightarrow \text{Optimal filtering gives spatial noise structure on optimal resolution scale.} \]
Role of White Noise

- **Noise:** $\mathbb{E} \left[ \hat{W}_n(\omega)\hat{W}_{n'}(\omega') \right] = \sigma_w^2 \tilde{I}_\phi(T_p(\omega - \omega')) \delta_{nn'}$.

- **Resulting CINT imaging functional:**

\[
I_{\text{CINT}}(y) = I_{\text{CINT}}^{\text{targ}}(y) + I_{\text{CINT}}^{i}(y) + I_{\text{CINT}}^{ii}(y).
\]

- $I_{\text{CINT}}^{i}$: zero mean and:
  $\rightarrow$ **Support on coherent image** and correlation lengths on resolution scales.

- $I_{\text{CINT}}^{ii}$: with non-zero mean $\propto \sigma_w / \sqrt{N}$ and:
  $\rightarrow$ **Extended support** and correlation lengths on resolution scales.

- **SNR**$^{(i)} = \sqrt{N} / \sigma_w$ & **SNR**$^{(ii)} = (N / \sigma_w^2) \sqrt{X_d / X_a}$.
Remark on the "Generation" of White Noise

• Medium: collection of small scatterers, then after deramping we get contribution:

\[ \hat{W}_n(\omega) = -\omega^2 c \sum_j v_j \hat{H}(\omega, x_n, y_j) \exp(i\omega(nT + \tau_0)) \]

with \( v_j \) reflectivity of the \( j \)-th scatterer and \( y_j \) its position.

• If \( x_2 - x_1 > \lambda_c \) we get a white-noise model in \( n \):

\[ \mathbb{E} \left[ \hat{W}_n(\omega) \bar{\hat{W}}_{n'}(\omega') \right] = \sigma^2 \mathcal{IF}^{\frac{1}{2}} \left( T_p(\omega - \omega') \right) \delta_{nn'} . \]
Simulation Setup

Swath center: \( y_0 = (0, y_0, 0) \) with \( y_0 = 440 \text{ m} \).
Antenna is 11 m.
Pulse width \( T_p = 2 \times 10^{-6} \text{ s} \).
Carrier frequency is \( \omega_c = 2\pi \times 35.3 \times 10^9 \text{ s}^{-1} \).
Chirp parameter is \( \gamma = 5 \times 10^{14} \text{ s}^{-2} \).
Background velocity is \( c_0 = 3 \times 10^8 \text{ ms}^{-1} \).

\[
\omega_c \gg \pi \gamma T_p \gg T_p^{-1}: 2.2 \times 10^{11} \gg 3.1 \times 10^9 \gg 5 \times 10^5.
\]

Optimal range resolution: \( c_0/(4\gamma T_p) \sim 0.1 \text{ m} \).
Optimal azimuthal resolution: \( \lambda_c |y_0|/(2X_a) \sim 0.2 \text{ m} \).

Targets in plane \( z = 0 \) (the imaging plane).

\[\leftrightarrow \text{Here, } X_c = 0.4m; \Delta X^{(c)} = 5.5m; \tau^{(0)} \sim \text{travel distance of 3cm}.\]
Imaging function in dB, travel time perturbation only.
$X_d = 0.33$
On the Dispersive Case

• Propagation via Klein-Gordon:
\[ c^{-2} \frac{\partial^2 E_\perp}{\partial t^2} - \Delta E_\perp + \left( \frac{\omega_{pe}}{c_0} \right)^2 E_\perp = A_n. \]

• Medium fluctuations of form \((\nu, \mu, \text{are independent, } (p + 1)/2 < q < 3p/4)\):
\[
\left( \frac{\omega_{pe}}{\omega_c} \right)^2 (x_\perp, z) = \left( \frac{k'_{pe}}{2\pi} \right)^2 \varepsilon^{2(p-q)} \left( 1 + \varepsilon^{2q-p-1} \mu \left( \frac{x_\perp}{\varepsilon \eta}, \frac{z}{\varepsilon^2} \right) 1_{(0,L')}(z) \right),
\]

\[
c^{-2} (x_\perp, z) = (c'_0)^{-2} \left( 1 + \varepsilon^{p-1} \nu \left( \frac{x_\perp}{\varepsilon \eta}, \frac{z}{\varepsilon^2} \right) 1_{(0,L')} (z) \right).
\]

\(\leftrightarrow\) Example relative magnitude of dispersion:
\[
\left( \frac{\omega_{pe}}{\omega_c} \right)^2 \approx \left( \frac{10^7}{3 \times 10^{10}} \right)^2 \approx 3 \times 10^{-7}.
\]
Relative Wave Front Perturbations

- Asymptotics of backscattered field (filtered w.r.t. coherent frequency dispersion):

\[
\left\{ \tilde{R}^{\varepsilon}_{\text{back}, n}(\varepsilon^p s_n(L/c_0)) \right\}_{n=1}^N \sim \left\{ \left( A_{\text{back}, n} \ast s_n^0 \right) \left( s_n - \Theta^V_{\text{back}, n} \right) \right\}_{n=1}^N,
\]

for

\[
\tilde{A}_{\text{back}, n}(\omega) = \exp \left[ -\frac{\omega^2 \Gamma^{(p, v)}_{2\omega \cos(\theta)/c'_0} L'}{4(c'_0)^2 \cos^2(\theta)} - \frac{\Gamma^{(p, \mu)}_{2\omega \cos(\theta)/c'_0} L'(k'_{pe})^4 (c'_0)^2}{4\omega^2 \cos^2(\theta)} + \frac{i(k'_{pe})^2 (c'_0)^2 \Theta^\mu_{\text{back}, n}}{2\omega \cos(\theta)} \right].
\]

→ In principle one could approximate filter with respect to the coherent dispersive effect. However, now the choice \( \Omega_d \propto \frac{1}{T_p} \varepsilon^{2q - p} \ll \frac{1}{T_p} \), serves to filter this part, giving an interferometric imaging function in the limit (frequency gating efficient!):

\[
I_{\text{INT}}(y) = \frac{1}{4\pi^2} \sum_{n,n'=1}^N \int \tilde{H}(\omega, x_n, y) \tilde{H}(\omega, x_{n'}, y) \hat{S}_n(\omega) \hat{S}'_{n'}(\omega) \exp(i\omega(n' T - n T)) d\omega
\]

\[
= \frac{1}{2\pi} \sum_{n,n'=1}^N \int S_{n, \text{back}}(t - n T) \overline{S_{n', \text{back}}(t - n' T)} dt.
\]

→ Choosing also \( X_d = X_c^d \) gives optimal resolution and \( \text{SNR} = \sqrt{X_a/X_d^c} \) as before.
Array decorrelation from dispersion

- For the previous example this then gives (no fluctuations in refractive index, \( \cos(\theta) = 1 \)) an antenna (de)coherence length due to random dispersion:

<table>
<thead>
<tr>
<th>correlation regime</th>
<th>( l_x )</th>
<th>( q )</th>
<th>( k'_{pe} )</th>
<th>( X^d_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>short</td>
<td>200m</td>
<td>2.3</td>
<td>0.1</td>
<td>110m</td>
</tr>
<tr>
<td>medium</td>
<td>2km</td>
<td>2.8</td>
<td>0.3</td>
<td>630m</td>
</tr>
<tr>
<td>long</td>
<td>20km</td>
<td>3.6</td>
<td>0.4</td>
<td>3.5km</td>
</tr>
</tbody>
</table>

Here, we used \( \omega_{pe} = 9 \times 10^{-6} \text{ s}^{-1} \) and relative fluctuations in \( \omega_{pe} = 0.1 \).
Remark on turbulent medium fluctuations

• The case of Kolmogorov turbulence, corresponding to a “Hurst exponent” $H = 1/3$ and rapid (super-linear) decay for the spectrum of medium fluctuations at origin.

• Medium covariance still integrable and travel-time correction Gaussian in scaling limit.

• The coherence of antenna measurements now has a sharp decay at the origin (medium spectrum in tensor product form):

$$
\mathbb{E} \left[ e^{i\omega c (\tau^{(r)}_n - \tau^{(r)}_{n'})} \right] = e^{-\left(\omega_c \sigma_t\right)^2 \frac{|l_n - l_{n'}|}{l_c}^{2H}}
$$

• Qualitatively similar behavior, but roughness of medium fluctuations affects performance.
Implementational Issues

- Averaging radius may be chosen adaptively using feature preserving norms.
- CINT is not effective for SAR in the context of a pure white noise.
- “Redundant phase information” via several carriers or via several scatterers and Phase Gradient Autofocus may be important.
Concluding Remarks

• Have analyzed scale contents in SAR acquisition.

• Correlation scales and coherent averaging dramatically changes SNR and detectability.

• Resolution and variability sensitive to spatial weighting.

• Non-stationary medium fluctuations, like in turbulent atmosphere, should be associated with adaptive data segmentation.

• Dispersion constrains optimal frequency weighting.