Sensor Array Imaging in a Noisy Environment

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• Principle of imaging with waves: waves are used to probe an unknown medium.
  1) record the waves generated by sources on a sensor array.
  2) process the data (backpropagation) in order to estimate relevant features of the medium (source or reflector locations, ...).

• Applications: medical imaging, geophysical exploration, non-destructive testing.

• In the presence of clutter noise (i.e. due to the scattering medium):
  - one usually observes a loss of resolution and signal-to-noise ratio in imaging.
  - one also observes “super-resolution” phenomena and “enhanced refocusing” in time-reversal experiments.
Time reversal experiment (1/3)


Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.
Time reversal experiment (2/3)


Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.
One can observe a spatial refocusing at the original source location.
Sensor Array Imaging in a Noisy Environment

- Motivation (time-reversal experiment).
- Some useful tools: Fourier identities, Green’s functions, Helmholtz-Kirchhoff theorem, geometric optics.
- Migration and least squares imaging.
- Resolution and stability analysis of Reverse-Time migration and time reversal:
  - homogeneous medium,
  - random medium.
- Coherent interferometric imaging.
Some useful tools
Fourier identities

Let $f(t)$ be a “nice” real-valued function. Its Fourier transform is

$$\hat{f}(\omega) = \int f(t) e^{i\omega t} dt$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int \hat{f}(\omega)e^{-i\omega t} d\omega$$

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$\hat{f}(\omega)$</th>
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<tbody>
<tr>
<td>$\frac{d^n f}{dt^n}$</td>
<td>$(-i\omega)^n \hat{f}(\omega)$</td>
</tr>
<tr>
<td>$f \ast g(t) = \int f(s)g(t - s)ds$</td>
<td>$2\pi \hat{f}(\omega)\hat{g}(\omega)$</td>
</tr>
<tr>
<td>$f(-t)$</td>
<td>$\hat{f}(\omega)$</td>
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<tr>
<td>$\int f(s)g(t + s)ds$</td>
<td>$2\pi \frac{\hat{f}(\omega)\hat{g}(\omega)}{\hat{f}(\omega)\hat{g}(\omega)}$</td>
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time reversal

cross correlation
Wave equation and Green’s function (1/3)

• Scalar wave model:

\[
\frac{1}{c^2(x)} \frac{\partial^2 u}{\partial t^2}(t, x) - \Delta_x u(t, x) = n(t, x)
\]

\[n(t, x): \text{source.}\]
\[c(x): \text{propagation speed (parameter of the medium), assumed to be constant outside a domain with compact support.}\]

• The time-dependent Green’s function \(G(t, x, y)\) is the solution of

\[
\frac{1}{c^2(x)} \frac{\partial^2 G}{\partial t^2}(t, x, y) - \Delta_x G(t, x, y) = \delta(t)\delta(x - y)
\]

Assume that \(G(t, x, y) = 0 \quad \forall t < 0 \implies\) unique solution (causal Green’s function).
Interpretation: Emission from a point source at \(y\) emitting a Dirac pulse at time 0.

If the medium is homogeneous \(c(x) \equiv c_0\), then

\[
G(t, x, y) = \frac{1}{4\pi|x - y|} \delta\left(t - \frac{|x - y|}{c_0}\right), \quad t > 0
\]

\(\implies\) spherical wave propagating at speed \(c_0\).
Wave equation and Green’s function (2/3)

• The time-harmonic Green’s function

\[ \hat{G}(\omega, x, y) = \int G(t, x, y) e^{i\omega t} dt \]

is the solution of the Helmholtz equation

\[ \Delta_x \hat{G}(\omega, x, y) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y) = -\delta(x - y), \]

with the Sommerfeld radiation condition (\(c(x) = c_0\) at infinity):

\[ \lim_{|x| \to \infty} |x| \left( \frac{x}{|x|} \cdot \nabla_x - i \frac{\omega}{c_0} \right) \hat{G}(\omega, x, y) = 0 \]

If the medium is homogeneous \(c(x) \equiv c_0\), then

\[ \hat{G}(\omega, x, y) = \frac{1}{4\pi|x - y|} e^{i\frac{\omega}{c_0}|x - y|} \]
Wave equation and Green’s function (3/3)

- The solution of the wave equation with source $n(t, x)$ has the form:

\[
    u(t, x) = \iint G(t - s, x, y)n(s, y)dyds
\]

In the Fourier domain:

\[
    \hat{u}(\omega, x) = \int u(t, x)e^{i\omega t}dt
\]

we have

\[
    \hat{u}(\omega, x) = \int \hat{G}(\omega, x, y)\hat{n}(\omega, y)dy
\]
Reciprocity

\[ \hat{G}(\omega, x, y) = \hat{G}(\omega, y, x) \]
Proof of reciprocity (1/2)

We consider the equations satisfied by the Green’s function with the source at \( y_2 \) and with the source at \( y_1 \) \((y_2 \neq y_1)\):

\[
\Delta_x \hat{G}(\omega, x, y_2) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y_2) = -\delta(x - y_2)
\]

\[
\Delta_x \hat{G}(\omega, x, y_1) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y_1) = -\delta(x - y_1)
\]

We multiply the first equation by \( \hat{G}(\omega, x, y_1) \) and subtract the second equation multiplied by \( \hat{G}(\omega, x, y_2) \):

\[
\nabla_x \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right]
\]

\[
= \hat{G}(\omega, x, y_2)\delta(x - y_1) - \hat{G}(\omega, x, y_1)\delta(x - y_2)
\]

\[
= \hat{G}(\omega, y_1, y_2)\delta(x - y_1) - \hat{G}(\omega, y_2, y_1)\delta(x - y_2)
\]

We next integrate over the ball \( B(0, L) \) which contains both \( y_1 \) and \( y_2 \) and use the divergence theorem:

\[
\int_{\partial B(0,L)} n \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right] dS(x)
\]

\[
= \hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
\]

where \( n \) is the unit outward normal to the ball \( B(0, L) \), which is \( n = x/|x| \).
Proof of reciprocity (2/2)

\[
\int_{\partial B(0,L)} n \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right] dS(x)
= \hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
\]

where \( n = x/|x| \).

If \( x \in \partial B(0, L) \) and \( L \to \infty \), then by the Sommerfeld radiation condition:

\[
n \cdot \nabla_x \hat{G}(\omega, x, y) = i \frac{\omega}{c_0} \hat{G}(\omega, x, y) + o\left( \frac{1}{L} \right)
\]

Therefore, for \( L \to \infty \),

\[
\hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
= i \frac{\omega}{c_0} \int_{\partial B(0,L)} \hat{G}(\omega, x, y_1) \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \hat{G}(\omega, x, y_1) dS(x) + o(1)
= 0
\]
If the medium is homogeneous (velocity $c_0$) outside $B(0, D)$, then $\forall x_1, x_2 \in B(0, D)$ we have for $L \gg D$:

$$\hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(y) \hat{G}(\omega, x_1, y) \hat{G}(\omega, x_2, y)$$

Proof: Green’s identity and Sommerfeld radiation condition.

Useful for: scattering theory, time reversal experiment, and cross correlation.
Proof of Helmholtz-Kirchhoff theorem (1/2)

Consider

\[ \Delta_y \hat{G}(\omega, y, x_2) + \frac{\omega^2}{c^2(y)} \hat{G}(\omega, y, x_2) = -\delta(y - x_2) \]

\[ \Delta_y \hat{G}(\omega, y, x_1) + \frac{\omega^2}{c^2(y)} \hat{G}(\omega, y, x_1) = -\delta(y - x_1) \]

Multiply the first equation by \( \hat{G}(\omega, y, x_1) \) and subtract the second equation multiplied by \( \hat{G}(\omega, y, x_2) \):

\[ \nabla_y \cdot \left[ \hat{G}(\omega, y, x_1) \nabla_y \hat{G}(\omega, y, x_2) - \hat{G}(\omega, y, x_2) \nabla_y \hat{G}(\omega, y, x_1) \right] = \hat{G}(\omega, y, x_2) \delta(y - x_1) - \hat{G}(\omega, y, x_1) \delta(y - x_2) \]

using the reciprocity property \( \hat{G}(\omega, x_2, x_1) = \hat{G}(\omega, x_1, x_2) \).

Integrate over the ball \( B(0, L) \) and use the divergence theorem:

\[ \int_{\partial B(0, L)} n \cdot \left[ \hat{G}(\omega, y, x_1) \nabla_y \hat{G}(\omega, y, x_2) - \hat{G}(\omega, y, x_2) \nabla_y \hat{G}(\omega, y, x_1) \right] dS(y) = \hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2) \]

where \( n \) is the unit outward normal to the ball \( B(0, L) \), which is \( n = y/|y| \).
Proof of Helmholtz-Kirchhoff theorem (2/2)

\[
\int_{\partial B(0, L)} n \cdot \left[ \hat{G}(\omega, y, x_1) \nabla_y \hat{G}(\omega, y, x_2) - \hat{G}(\omega, y, x_2) \nabla_y \hat{G}(\omega, y, x_1) \right] dS(y)
= \hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2)
\]

where \( n = y / |y| \).

This equality can be viewed as an application of the second Green’s identity.

The Green’s function also satisfies the Sommerfeld radiation condition

\[
\lim_{|y| \to \infty} |y| \left( \frac{y}{|y|} \cdot \nabla_y - \frac{i\omega}{c_0} \right) \hat{G}(\omega, y, x_1) = 0
\]

uniformly in all directions \( y / |y| \). Substitute \( i(\omega/c_0) \hat{G}(\omega, y, x_2) \) for \( n \cdot \nabla_y \hat{G}(\omega, y, x_2) \) in the surface integral over \( \partial B(0, L) \), and \( -i(\omega/c_0) \hat{G}(\omega, y, x_1) \) for \( n \cdot \nabla_y \hat{G}(\omega, y, x_1) \), which gives the desired result:

\[
\hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(y) \hat{G}(\omega, x_1, y) \hat{G}(\omega, x_2, y)
\]
A quick introduction to geometric optics (1/2)

We look for an approximate expression as $\varepsilon \to 0$ for $\hat{G}(\frac{\omega}{\varepsilon}, x, y)$ solution of

$$\Delta_x \hat{G}(\frac{\omega}{\varepsilon}, x, y) + \frac{\omega^2}{c^2(x)\varepsilon^2} \hat{G}(\frac{\omega}{\varepsilon}, x, y) = -\delta(x - y)$$

Note that, if $c(x) = c_0$, then

$$\hat{G}(\frac{\omega}{\varepsilon}, x, y) = \frac{1}{4\pi|x - y|} e^{i\frac{\omega |x - y|}{c_0}}$$

Consider a smoothly varying $c(x)$ and look for an expansion of the form:

$$\hat{G}(\frac{\omega}{\varepsilon}, x, y) = e^{i\frac{\omega \mathcal{T}(x, y)}{\varepsilon}} \sum_{j=0}^{\infty} \frac{\varepsilon^j A_j(x, y)}{\omega^j}$$

Substitute the ansatz into Helmholtz equation and collect the terms with the same powers in $\varepsilon$:

$$O\left(\frac{1}{\varepsilon^2}\right) : \quad |\nabla_x \mathcal{T}|^2 - \frac{1}{c^2(x)} = 0$$

$$O\left(\frac{1}{\varepsilon}\right) : \quad 2\nabla_x \mathcal{T} \cdot \nabla_x A_0 + A_0 \Delta_x \mathcal{T} = 0$$

$\leftrightarrow$ Eikonal equation for the quantity $\mathcal{T}$ (that turns out to be the travel time) + transport equation for the amplitude $A_0$.

Solve by method of characteristics (ray equations).
A quick introduction to geometric optics (2/2)

Geometric optics approximation of the Green’s function:

\[ \hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) \sim A(x, y)e^{i\frac{\omega}{\varepsilon}T(x, y)} \]

valid when \( \varepsilon \ll 1 \), where the travel time is

\[ T(x, y) = \inf \left\{ T \text{ s.t. } \exists (X_t)_{t \in [0, T]} \in C^1, \ X_0 = x, \ X_T = y, \ |\frac{dX_t}{dt}| = c(X_t) \right\} \]

The curve(s) that minimizes this functional are called ray(s).

Simple geometry hypothesis: \( c(x) \) is smooth and there is a unique ray between any pair of points (in the region of interest).

In the homogeneous case \( c(x) \equiv c_0 \):

\[ \hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) = A(x, y)e^{i\frac{\omega}{\varepsilon}T(x, y)}, \text{ with } A(x, y) = \frac{1}{4\pi|x - y|}, \ T(x, y) = \frac{|x - y|}{c_0} \]
Passive array imaging
Passive array imaging: data acquisition

Passive array $\iff$ The array sensors are used only as receivers.

The source $y$ emits a pulse.

The sensors $(x_r)_{r=1,\ldots,N}$ record the waves.

The data set is $(P(t, x_r))_{t \in \mathbb{R}, r=1,\ldots,N}$.

Goal: find the source position $y$ (more generally, find the source region).
Passive array imaging: simulation

Configuration

Data set \((P(t, x_r))_{80 \leq t \leq 200, r=1, \ldots, 270}\)

Simulations carried out by C. Tsogka (University of Crete).
Passive array imaging: imaging functional

Data acquisition  Search region

Goal: find the source point $y$ (more generally, find the source region).

The data set is $(P(t, x_r))_{t \in \mathbb{R}, r = 1, \ldots, N}$.

$\rightarrow$ Given the data set, build an imaging functional in the search region $\Omega \subset \mathbb{R}^3$:

$I : \Omega \rightarrow \mathbb{R}^+$  which plots an image of the search region.

$y^S \mapsto I(y^S)$
Passive array imaging - the linear forward operator

The source term is of the form $n(t, y) = \rho_{\text{real}}(y)\delta(t)$. Goal: find the source function $\rho_{\text{real}}$.
Here: the Green’s function is known.

The data set is $\hat{P}(\omega) = (\hat{P}(\omega, x_r))_{r=1,\ldots,N}$ with $\hat{P}(\omega, x_r) = \int \hat{G}(\omega, x_r, y)\rho_{\text{real}}(y)dy$.

We define

$[\hat{A}(\omega)\rho](x_r) = \int \hat{G}(\omega, x_r, y)\rho(y)dy$.

$\hat{A}(\omega)$ is the frequency-dependent, linear operator that maps the source function to the array data $\hat{P}(\omega)$:

$\hat{P}(\omega) = \hat{A}(\omega)\rho_{\text{real}}$.
Passive array imaging - the adjoint operator

The least squares inverse problem is to minimize $J[\rho]$ where

$$J[\rho] = \int d\omega \sum_r |\hat{P}(\omega, x_r) - [\hat{A}(\omega)\rho](x_r)|^2$$

Solution:

$$\rho_{LS} = \left( \int \hat{A}^H(\omega)\hat{A}(\omega)d\omega \right)^{-1} \left( \int \hat{A}^H(\omega)\hat{P}(\omega)d\omega \right)$$

The adjoint operator is (using reciprocity)

$$[\hat{A}^H(\omega)\hat{P}(\omega)](y) = \sum_r \overline{\hat{G}(\omega, y, x_r)}\hat{P}(\omega, x_r)$$

Remember: complex conjugation = time reversal.
Adjoint operator = Backpropagation to the test point $y$. 
Passive array imaging - the reverse-time imaging functional

• Least-squares imaging functional:

\[
\mathcal{I}_{\text{LS}}(y^S) = \left[ \left( \int \hat{A}^H(\omega) \hat{A}(\omega) d\omega \right)^{-1} \left( \int \hat{A}^H(\omega) \hat{P}(\omega) d\omega \right) \right] (y^S)
\]

with

\[
\left[ \hat{A}^H(\omega) \hat{P}(\omega) \right](y) = \sum_r \hat{G}(\omega, y, x_r) \hat{P}(\omega, x_r)
\]

• The operator \( \int \hat{A}^H(\omega) \hat{A}(\omega) d\omega \) is proportional to the identity operator \( \rightarrow \) drop the normalizing factor in the LS functional.

\( \leftrightarrow \) Reverse Time imaging functional for the search point \( y^S \):

\[
\mathcal{I}_{\text{RT}}(y^S) = \int d\omega \left[ \hat{A}^H(\omega) \hat{P}(\omega) \right](y^S) = \int d\omega \sum_r \hat{G}(\omega, y^S, x_r) \hat{P}(\omega, x_r)
\]
Kirchhoff Migration (or travel-time migration) for passive array imaging

- Reverse Time imaging functional for the search point $y^S$:

$$\mathcal{I}_{RT}(y^S) = \int d\omega \sum_r \hat{G}(\omega, y^S, x_r) \hat{P}(\omega, x_r)$$

- If we take $\hat{G}(\omega, x, y) \simeq \exp[i\omega T(x, y)]$, where $T(x, y)$ is the travel time from $x$ to $y$, then we get the Kirchhoff Migration imaging functional:

$$\mathcal{I}_{KM}(y^S) = \int d\omega \sum_r \exp[-i\omega T(x_r, y^S)] \hat{P}(\omega, x_r)$$

$$= 2\pi \sum_r P(T(x_r, y^S), x_r)$$

Kirchhoff Migration (or travel-time migration) has been analyzed in detail (Beylkin, Burridge, Symes, Bleistein, ...) and is used extensively in practice.
Kirchhoff migration for passive array imaging: simulation

Data \((P(t, x_r))_{t\in[40,150], r=1,...,264}\)

KM imaging functional \(I_{KM}(y^S)\)
Active array imaging
Active array imaging: data acquisition

Active array $\leftrightarrow$ The array sensors can be used as sources and/or as receivers.

For each $s = 1, \ldots, N$:
- The source $x_s$ emits a pulse.
- The sensors $(x_r)_{r=1,\ldots,N}$ record the waves.

The data set is $(P(t, x_r, x_s))_{t \in \mathbb{R}, r, s=1,\ldots,N}$ (the time-dependent response matrix).

Goal: find the reflector position $y$ (more generally, find the reflectivity function of the medium).
Active array imaging: simulation

Configuration

Data set \( (P(t, x_{r}, x_{135}))_{80 \leq t \leq 200, r=1,\ldots,270} \) (traces recorded for a central illumination)
Active array imaging: imaging functional

Data acquisition

Search region

Goal: find the reflector position $y$ (more generally, find the reflectivity function of the medium).

The data set is $(P(t, x_r, x_s))_{t \in \mathbb{R}, r, s=1,\ldots,N}$.

$\leftarrow$ Given the data set, build an imaging functional in the search region $\Omega \subset \mathbb{R}^3$:

$\mathcal{I} : \Omega \rightarrow \mathbb{R}^+ \\
\boldsymbol{y}^S \mapsto \mathcal{I}(\boldsymbol{y}^S)$ which plots an image of the search region.
Passive and active array imaging: comparison

Passive array of sensors
The sensors \((x_r)_{r=1,...,N}\) record
\(y\) is a source
Data: \((P(t, x_r))_{t \in \mathbb{R}, r=1,...,N}\)

Active array of sensors/sources
The sensors \((x_r)_{r=1,...,N}\) record
\(y\) is a reflector, \(x_s\) is a source
Data: \((P(t, x_r, x_s))_{t \in \mathbb{R}, r,s=1,...,N}\)

Goal: process the data to find \(y\)
Goal: find the propagation speed $c_{\text{real}} = (c_{\text{real}}(y))_{y \in \mathbb{R}^d}$.

The time Fourier transform of the data $\hat{P}(\omega, x_r, x_s)$ is

$$\hat{P}(\omega, x_r, x_s) = \hat{G}(\omega, x_r, x_s; c_{\text{real}})$$

$\hat{G}(\omega, x, y; c)$ is the Green’s function that solves the Helmholtz equation

$$\Delta \hat{G}(\omega, x, y; c) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y; c) = -\delta(x - y),$$

with the Sommerfeld radiation condition. It depends on the velocity $c = (c(x))_{x \in \mathbb{R}^d}$. 

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Active array imaging - nonlinear inversion

Data:
\[ \hat{P}(\omega) = (\hat{P}(\omega, x_r, x_s))_{r,s=1,...,N} = (\hat{G}(\omega, x_r, x_s, c_{\text{real}}))_{r,s=1,...,N} \]

Goal: find the propagation speed \( c_{\text{real}} \).

The inverse problem, the array least squares problem, is:

Minimize
\[ J[c] + \alpha \|c\|_{\text{REG}}, \]

where
\[ J[c] = \int d\omega \sum_{r,s} |\hat{P}(\omega, x_r, x_s) - \hat{G}(\omega, x_r, x_s; c)|^2 \]

and \( \alpha \) is a strength of regularization parameter.

→ this is a nonlinear problem for the unknown propagation speed \( c = (c(x))_{x \in \mathbb{R}^d} \).
Active array imaging - linearization

\[
\frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( n_0^2(x) + \rho(x) \right)
\]

where
- \( c_0 \) is a reference speed,
- \( n_0(x) \) is a smooth background index of refraction (known, typically constant),
- \( \rho(x) \) is the target reflectivity (unknown but small).

The Green’s function satisfies:

\[
\Delta \hat{G}(\omega, x, y; \rho) + \frac{\omega^2}{c_0^2} \left( n_0^2(x) + \rho(x) \right) \hat{G}(\omega, x, y; \rho) = -\delta(x - y)
\]

The background Green’s function is defined by:

\[
\Delta \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} n_0^2(x) \hat{G}_0(\omega, x, y) = -\delta(x - y)
\]

Born approximation:

\[
\hat{G}(\omega, x, y; \rho) = \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz
\]

First term: direct waves.
Second term: single-scattered waves \( y \rightarrow z \rightarrow x \).
Proof of the Born approximation (1/2)

Consider

\[\Delta_z \hat{G}(\omega, z, x; \rho) + \frac{\omega^2}{c_0^2} n_0^2(z) \hat{G}(\omega, z, x; \rho) = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x; \rho) - \delta(z - x)\]

\[\Delta_z \hat{G}_0(\omega, z, y) + \frac{\omega^2}{c_0^2} n_0^2(z) \hat{G}_0(\omega, z, y) = -\delta(z - y)\]

We multiply the first equation by \(\hat{G}_0(\omega, x, y)\) and subtract the second equation multiplied by \(\hat{G}(\omega, x, z; \rho)\):

\[\nabla_z \cdot \left[ \hat{G}_0(\omega, z, y) \nabla_z \hat{G}(\omega, z, x; \rho) - \hat{G}(\omega, z, y) \nabla_z \hat{G}_0(\omega, z, x) \right] = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x; \rho) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, x, y) \delta(z - x) + \hat{G}(\omega, z, x; \rho) \delta(z - y)\]

\[= -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x; \rho) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, x, y) \delta(z - x) + \hat{G}(\omega, y, x; \rho) \delta(z - y)\]

\[\text{reciprocity} \quad = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, x, z; \rho) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, x, y) \delta(z - x) + \hat{G}(\omega, x, y; \rho) \delta(z - y)\]

We integrate over \(B(0, L)\) (with \(L\) large enough so that it encloses the support of \(\rho\)):

\[0 = -\frac{\omega^2}{c_0^2} \int \hat{G}(\omega, x, z; \rho) \rho(z) \hat{G}_0(\omega, z, y) dz - \hat{G}_0(\omega, x, y) + \hat{G}(\omega, x, y; \rho)\]
Proof of the Born approximation (2/2)

• Lippmann-Schwinger equation (exact):

\[ \hat{G}(\omega, x, y; \rho) = \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}(\omega, x, z; \rho) \rho(z) \hat{G}_0(\omega, z, y) dz \]

• Born approximation (approximate):

\[ \hat{G}(\omega, x, y; \rho) \simeq \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz \]
Active array imaging - the linearized forward operator

The data set is modeled by:

\[ \hat{P}(\omega, x_r, x_s) = \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x_r, z) \rho_{\text{real}}(z) \hat{G}_0(\omega, z, x_s) dz \]

(remove \( \hat{G}_0(\omega, x_r, x_s) \)).

We define

\[ [\hat{A}(\omega)\rho](x_r, x_s) = \int \hat{G}_0(\omega, x_r, z) \rho(z) \hat{G}_0(\omega, z, x_s) dz \]

It is the frequency dependent, linear operator that maps the reflectivity function to the array data.
Active array imaging - inversion

The least squares linearized inverse problem is to minimize $J_{LS}[\rho]$ where

$$J_{LS}[\rho] = \int d\omega \sum_{r,s} \left| \hat{P}(\omega, x_r, x_s) - [\hat{A}(\omega)\rho](x_r, x_s) \right|^2$$

Solution:

$$\rho_{LS} = \left( \int \hat{A}^H(\omega)\hat{A}(\omega)d\omega \right)^{-1} \left( \int \hat{A}^H(\omega)\hat{P}(\omega)d\omega \right)$$

The adjoint operator is

$$[\hat{A}^H(\omega)\hat{P}(\omega)](y) = \sum_{r,s} \hat{G}_0(\omega, y, x_r)\hat{G}_0(\omega, x_s, y)\hat{P}(\omega, x_r, x_s)$$

Remember: complex conjugation = time reversal.
Adjoint operator = Backpropagation both from $x_r$ and from $x_s$ to the test point $y$. 

Heraklion June, 2012
Kirchhoff migration (or travel-time migration) for active array imaging

- Reverse-time migration (using the full Green’s function for migration).

Imaging functional for the search point $y^S$:

$$I_{RT}(y^S) = \int d\omega \sum_{r,s} \hat{G}_0(\omega, y^S, x_r)\hat{G}_0(\omega, x_s, y^S)\hat{P}(\omega, x_r, x_s)$$

- Kirchhoff migration (or travel time migration).

If we take $\hat{G}_0(\omega, x, y) \simeq \exp[i\omega T(x, y)]$, where $T(x, y)$ is the travel time from $x$ to $y$, then we get the KM imaging functional:

$$I_{KM}(y^S) = \int d\omega \sum_{r,s} \exp[-i\omega(T(x_r, y^S) + T(x_s, y^S))]\hat{P}(\omega, x_r, x_s) = 2\pi \sum_{r,s} P(T(x_r, y^S) + T(x_s, y^S), x_r, x_s)$$
Kirchhoff migration for active array imaging: simulation

Data

$P(t, x_r, x_{128})$ $t \in [80, 200], r = 1, \ldots, 264$

KM imaging functional

$\mathcal{I}_{KM}(y^S)$
Time reversal
Time reversal experiment

Originally: Not for imaging, but for energy focusing (kidney stone destruction).

A Time-Reversal Mirror (TRM) is used first as a sensor array, then as a source array.

The source $y$ emits a pulse
The TRM records the signals
The TRM emits the time-reversed signals
A sensor array probes the region around the original source location

→ looks like reverse-time migration for passive array imaging! The difference: backpropagation is performed physically in a TR experiment and numerically in RT migration.
Comparison reverse-time migration vs time reversal

In both cases the recorded data are back-propagated from the array.

In RT migration, backpropagation is carried out numerically, in a fictitious medium.

In TR, backpropagation is carried out physically, in the real medium.

$\rightarrow$ No difference when the medium is perfectly known.
Random medium
Random medium

- We need a model for the medium that incorporates
  (a) a background velocity that is known,
  (b) the clutter that we do not know and can only estimate statistically.

- This motivates the model:

\[
\frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( n_0^2(x) + \mu(x) \right)
\]

\(c_0\) is a reference speed,
\(n_0(x)\) is a smooth background index of refraction (known),
\(\mu(x)\) is a zero-mean, stationary random process that represents the clutter.
Passive array imaging: simulation with clutter

- Array size: $a$ units
- Distance between array elements: $d=6\lambda_0$
- Array length: $L=90\lambda_0$
- Absorbing medium

![Image of simulation results with color scale and time in msec]
Random medium

- We need a model for the medium that incorporates
  (a) a background velocity that is known,
  (b) the clutter that we do not know and can only estimate statistically.

- This motivates the model:

\[
\frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( n_0^2(x) + \mu(x) \right)
\]

- \( c_0 \) is a reference speed,
- \( n_0(x) \) is a smooth background index of refraction (known),
- \( \mu(x) \) is a zero-mean, stationary random process that represents the clutter.

- The background Green’s function (known):

\[
\Delta \hat{G}_b + \frac{\omega^2}{c_0^2} n_0^2(x) \hat{G}_b = -\delta(x - y)
\]

The physical Green’s function (random and unknown):

\[
\Delta \hat{G} + \frac{\omega^2}{c_0^2} \left( n_0^2(x) + \mu(x) \right) \hat{G} = -\delta(x - y)
\]
A remark: Cluttered noise is not an additive white noise

The structure of $\hat{G} - \hat{G}_b$ depends on the propagation regime (single scattering, multiple scattering, diffusive, randomly layered media, paraxial, ...)

Example: In the single-scattering regime, the response matrix of the sensor array $(x_j)_{j=1,...,n}$ is $(\hat{G}(\omega, x_i, x_j) - \hat{G}_b(\omega, x_i, x_j))_{i,j=1,...,n}$. It has a Hankel structure (coherence along the antidiagonals) if the array is linear.

Experimental response matrix $(\text{Re} (\hat{G}(\omega, x_i, x_j) - \hat{G}_b(\omega, x_i, x_j)))_{i,j=1,...,64}$ in a scattering medium (in the absence of strong reflector) [Aubry and Derode, PRL 102, 084301 (2009)].
Comparison reverse-time migration vs time reversal

- In RT migration, backpropagation is carried out numerically, in a fictitious medium.
- In TR, backpropagation is carried out physically, in the real medium.

No difference when the medium is perfectly known.

But: dramatic difference when the medium is not known or known partially.

\[
\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \sum_r \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{P}(\omega, \mathbf{x}_r) = \sum_r \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \hat{f}(\omega)
\]

\[
\hat{I}_{\text{RT}}(\omega, \mathbf{y}^S) = \sum_r \hat{G}_b(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{P}(\omega, \mathbf{x}_r) \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \hat{f}(\omega)
\]

Heraklion

June, 2012
Time reversal experiment with a full-aperture mirror (1/3)

First part:
A point source at \( y \) emits a pulse \( f(t) \).
The waves are recorded at the surface \( \partial B(0, L) \):
\[
\hat{u}(\omega, x) = \hat{G}(\omega, x, y) \hat{f}(\omega), \quad x \in \partial B(0, L)
\]

Second part:
The recorded signals are time-reversed
and sent back into the medium.
The signal received at \( y^S \) is
\[
\hat{u}_{\text{TR}}(\omega, y^S) = \int_{\partial B(0, L)} dS(x) \hat{G}(\omega, y^S, x) \hat{G}(\omega, x, y) \hat{f}(\omega)
\]
Time reversal experiment with a full-aperture mirror (2/3)

The signal received at $\mathbf{y}^S$ is (using reciprocity: $\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \hat{G}(\omega, \mathbf{y}, \mathbf{x})$):

$$\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \int_{\partial B(0, L)} dS(\mathbf{x}) \hat{G}(\omega, \mathbf{y}, \mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}) \hat{f}(\omega)$$

By Helmholtz-Kirchhoff theorem:

$$\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \hat{G}(\omega, \mathbf{y}, \mathbf{y}^S) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(\mathbf{x}) \hat{G}(\omega, \mathbf{y}, \mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x})$$

we get

$$\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \frac{\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \hat{G}(\omega, \mathbf{y}, \mathbf{y}^S)}{2i\omega/c_0} \hat{f}(\omega) = \frac{c_0}{\omega} \text{Im}(\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S)) \hat{f}(\omega)$$

[Remember: $\mathbf{y}$ is the original source location].
Time reversal experiment with a full-aperture mirror (3/3)

The focal spot is determined by the imaginary part of the Green’s function:

$$\hat{u}_{TR}(\omega, y^S) = \frac{c_0}{\omega} \text{Im}(\hat{G}(\omega, y, y^S)) \hat{f}(\omega)$$

- In a homogeneous medium:

$$\hat{G}(\omega, y, y^S) = \frac{1}{4\pi|y - y^S|} e^{i \frac{\omega|y - y^S|}{c_0}} \implies \hat{u}_{TR}(\omega, y^S) = \frac{1}{4\pi} \text{sinc}\left(\frac{\omega|y - y^S|}{c_0}\right) \hat{f}(\omega)$$

$\rightarrow$ refocusing with a focal spot of diameter $\lambda/2$ (diffraction limit).

- In a complex medium: $\text{Im}(\hat{G}(\omega, y, y^S))$ can be sharper than in a homogeneous medium

“Super-resolution effect” [Lerosey et al, Science 315 (2007), 1120]: if a micro-structured medium surrounds the original source $y$, then the focal spot can be smaller than the diffraction limit $\lambda/2$!

Main effect of the micro-structured medium: modify the effective wavelength (homogeneization result) [Ammari et al, SIAP 70 (2009) 1428], [Gomez, SIAM MMS 7 (2009) 1348]).
Time reversal experiment with a finite-aperture mirror (1/3)

Square array \([−D/2, D/2] \times [−D/2, D/2] \times \{0\}\). Point source \(\mathbf{y} = (0, 0, L)\).

Refocused wave: 
\[
\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \sum_{r=1}^{N} \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r)\hat{G}(\omega, \mathbf{x}_r, \mathbf{y})\hat{f}(\omega)
\]

- In the homogeneous paraxial regime \((\lambda \ll D \ll L)\), for a dense square array:

\[
\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S = (x_1, x_2, L)) = \text{sinc} \left( \frac{\pi x_1}{r_c} \right) \text{sinc} \left( \frac{\pi x_2}{r_c} \right) \hat{f}(\omega)
\]

\(r_c = \frac{\lambda L}{D}\) (Rayleigh resolution formula).

\(\rightarrow\) The diameter of the focal spot is determined by the aperture \(D/L\) of the focusing cone.
Time reversal experiment with a finite-aperture mirror (2/3)

- In the random paraxial regime, for a square TR array with diameter $D$:

$$
\mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, y^S = (x_1, x_2, L)) \right] = \text{sinc} \left( \pi \frac{x_1}{r_c} \right) \text{sinc} \left( \pi \frac{x_2}{r_c} \right) \exp \left( - \frac{x_1^2 + x_2^2}{2r_a^2} \right) \hat{f}(\omega)
$$

$$
\quad r_a = \frac{\lambda \sqrt{6}}{\pi \sqrt{\gamma_2 L}} \quad \text{and} \quad \gamma_2 = \int_{-\infty}^{\infty} \mathbb{E} [\nabla_{\perp} \mu(0, 0) \cdot \nabla_{\perp} \mu(0, z)] dz
$$

Rayleigh resolution formula with an effective array diameter $D_{\text{eff}} \sim \sqrt{D^2 + \gamma_2 L^3}$. The diameter of the focal spot is smaller because the effective focusing cone is larger.

- Statistical stability of the refocused focal spot ensured by frequency decorrelation:

$$
\mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, y^S) \hat{u}_{\text{TR}}(\omega', y^S) \right] \simeq \mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, y^S) \right] \mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega', y^S) \right] \quad \text{for} \quad |\omega - \omega'| > \Omega_c
$$

where $\Omega_c$ is the decoherence frequency (=frequency gap beyond which the signals are not correlated).

$\leftrightarrow u_{\text{TR}}(t, y^S)$ is statistically stable provided its bandwidth is larger than $\Omega_c$:

$$
\frac{\text{Var}(u_{\text{TR}}(t, y^S))}{\mathbb{E}[u_{\text{TR}}(t, y^S)]^2} \ll 1
$$

Note: other mechanisms (lateral diversity) can ensure statistical stability.

Heraklion June, 2012
Time reversal experiment with a finite-aperture mirror (3/3)
One can observe a spatial refocusing at the original source location.

Passive array imaging in a cluttered medium

Square array $[-D/2, D/2] \times [-D/2, D/2] \times \{0\}$. Point source $\mathbf{y} = (0, 0, L)$.

Reverse-time migration: $\hat{u}_{RT}(\omega, \mathbf{y}^S) = \sum_{r=1}^{N} \hat{G}_b(\omega, \mathbf{y}^S, \mathbf{x}_r)^* \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \hat{f}(\omega)$

Mean “focal spot” (in the random paraxial regime):

$$
\mathbb{E} \left[ \hat{u}_{RT}(\omega, \mathbf{y}^S = (x_1, x_2, L)) \right] = \text{sinc} \left( \pi \frac{x_1}{r_c} \right) \text{sinc} \left( \pi \frac{x_2}{r_c} \right) \exp(-d(\omega)) \hat{f}(\omega)
$$

where $d(\omega) = \frac{\gamma_0 L}{8c_0^2} \omega^2$ and $\gamma_0 = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(0, z)]dz$

No statistical stability (standard deviation larger than the mean):

$$
\frac{\text{Var}(I_{RT}(t, \mathbf{y}^S))}{\mathbb{E}[I_{RT}(t, \mathbf{y}^S)]^2} \gg 1
$$
Passive array imaging in a cluttered medium: simulation
Kirchhoff migration: simulation without and with clutter

How can we improve imaging in a cluttered medium?

Heraklion June, 2012
Comparison reverse-time migration vs time reversal

- The true Green’s function for the random medium is not known and so cannot be used for imaging, but is used in physical TR.

- Reverse-Time migration (and Kirchhoff migration) does not work well in clutter and it is statistically unstable. The reason is that migration tries to cancel the random phases of the signals arriving at the array with a deterministic phase using travel times.

- Time Reversal works all the better (resolution enhancement) as the clutter is important!

- Precise results in the random medium case are based on the statistical analysis of $\hat{G}\hat{G}$ (useful for imaging, time reversal, and cross correlation imaging).

- Idea for imaging in random media: use and backpropagate cross correlations (Coherent Interferometric Imaging).
Coherent Interferometric Imaging (CINT)
Coherent Interferometric Imaging (CINT): principle

- RT (or KM) imaging in scattering media (passive array): backpropagate the data

\[ \mathcal{I}_{RT}(y^S) = \sum_r \hat{G}_b(\omega, y^S, x_r) \hat{P}(\omega, x_r) \]

(remember \( \hat{P}(\omega, x_r) \sim \hat{G}(\omega, x_r, y) \)). The problem is that \( \hat{G}_b \hat{G} \) is not stable.

- CINT imaging: - cross correlate the data (\( \rightarrow \hat{G} \hat{G} \)),
- backpropagate (with \( \hat{G}_b \hat{G}_b \)) the cross correlations of the data.

First guess:

\[ \mathcal{I}(y^S) = \sum_{r,r'=1}^{N} \int \hat{P}(\omega, x_r) \hat{P}(\omega, x_{r'}) e^{-i\omega \tau(x_r, y^S) + i\omega \tau(x_{r'}, y^S)} d\omega \]

but not so good (pairing of \( \hat{G} \) and \( \hat{G}_b \) is not correct).

- The key idea is to migrate the cross correlations of the data rather than the data.
- It is important to compute the cross correlations \textit{locally} in time and space, and not over the whole time interval and the whole set of pairs of sensors, in order to pair \( \hat{G} \hat{G} \) and \( \hat{G}_b \hat{G}_b \) correctly.
Coherent Interferometric Imaging: implementation (1/2)

• Consider the square of the KM functional:

\[ |I_{\text{KM}}(y^S)|^2 = \sum_{r,r'=1}^N \int \int \hat{P}(\omega, x_r)\overline{\hat{P}(\omega', x_{r'})}e^{-i\omega T(x_r, y^S) + i\omega' T(x_{r'}, y^S)}d\omega d\omega' \]

→ pairing (of \( \hat{G} \) and \( \hat{G}_b \)) is wrong.

• Decoherence frequency \( \Omega_c \): frequency gap beyond which the frequency components of the recorded signals are not correlated. Remark: The reciprocal of the decoherence frequency is the delay spread (duration of the coda).

\[ I_{\text{CINT}}(y^S, \Omega_d) = \sum_{r,r'=1}^N \int \int \hat{P}(\omega, x_r)\overline{\hat{P}(\omega', x_{r'})}e^{-i\omega T(x_r, y^S) + i\omega' T(x_{r'}, y^S)}d\omega d\omega' \]

Compare with the square of the KM functional: The CINT functional and the square of the KM functional differ only in that the frequencies \( |\omega - \omega'| > \Omega_d \) are eliminated in CINT.

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Coherent Interferometric Imaging: implementation (2/2)

- Decoherence length $X_c$: distance between sensors beyond which the signals recorded at them are not correlated.

$$\mathcal{I}_{\text{CINT}}(y^S, \Omega_d, X_d) = \sum_{r,r'=1}^{N} \int \int \hat{P}(\omega, x_r)\hat{P}(\omega', x_{r'}) e^{-i\omega T(x_r, y^S) + i\omega' T(x_{r'}, y^S)} d\omega d\omega'$$

- The cross range resolution of CINT is of the order of $\lambda_0 L/X_d$ (Rayleigh resolution formula with the effective array diameter $D_{\text{eff}} = X_d$).

- The range resolution of CINT is of the order of $c_0/\Omega_d$.

- Statistical stability ensured provided bandwidth and/or diameter array are large enough.

- The optimal values for the parameters $\Omega_d$ and $X_d$ can be found by a statistical analysis (depends on the propagation regime).

- An adaptive procedure for estimating optimally the parameters $\Omega_d$ and $X_d$ is based on minimizing a suitable norm of the image to improve its quality, both in terms of resolution and signal-to-noise ratio.
Coherent Interferometric Imaging: simulation

Optimal choice of $X_d, \Omega_d$ based on the image (obtained by minimizing an appropriate objective functional, a suitable norm of the image).

Trade-off between resolution and signal-to-noise ratio.

Simulations carried out by C. Tsogka (University of Crete).