A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation

The current research has been co-financed by the European Union (European Social Fund- ESD) and by national resources through the operational programme "Education and Lifelong Learning" of the National Strategic Research Frame (NSRF) - Financed Research Project: Herakleitos II. Investment in the society of knowledge through the European Social Fund.

Heraklion, June 21, 2013
Introduction

- Hydrodynamic Limits in the Presence of Condensation
Introduction

Hydrodynamic Limits in the Presence of Condensation

The object of study of the theory of hydrodynamic limits is the macroscopic description of microscopic systems consisting of a large number of particles, through a system of partial differential equations that describe the time evolution of the macroscopic densities of the conserved quantities, such as mass, energy and momentum.
Introduction

**Hydrodynamic Limits in the Presence of Condensation**

The object of study of the theory of hydrodynamic limits is the macroscopic description of microscopic systems consisting of a large number of particles, through a system of partial differential equations that describe the time evolution of the macroscopic densities of the conserved quantities, such as mass, energy and momentum.

In this work we are mainly interested in the case that the mass is allowed to exhibit phase transition via the emergence of a condensate.
Introduction

- **Hydrodynamic Limits in the Presence of Condensation**

  The object of study of the theory of hydrodynamic limits is the macroscopic description of microscopic systems consisting of a large number of particles, through a system of partial differential equations that describe the time evolution of the macroscopic densities of the conserved quantities, such as mass, energy and momentum.

  In this work we are mainly interested in the case that the mass is allowed to exhibit phase transition via the emergence of a condensate.

  Since we are mainly interested in the phenomenon of mass condensation we work in systems with one conserved quantity, the mass.
Introduction

- *Hydrodynamic Limits in the Presence of Condensation*

The object of study of the theory of hydrodynamic limits is the macroscopic description of microscopic systems consisting of a large number of particles, through a system of partial differential equations that describe the time evolution of the macroscopic densities of the conserved quantities, such as mass, energy and momentum.

In this work we are mainly interested in the case that the mass is allowed to exhibit phase transition via the emergence of a condensate.

Since we are mainly interested in the phenomenon of mass condensation we work in systems with one conserved quantity, the mass. In particular we work with in the context of the n.n. symmetric Zero Range Process, the simplest interacting particle system exhibiting phase separation.
The Zero Range Process on the discrete torus $\mathbb{T}^d_N$

Let $\mathbb{T}^d_N := \{0, 1, \ldots, N - 1\}^d$, $N \in \mathbb{N}$, denote the discrete torus.
Let $\mathbb{T}_N^d := \{0, 1, \ldots, N-1\}^d$, $N \in \mathbb{N}$, denote the discrete torus. A Zero Range Process (ZRP) on $\mathbb{T}_N^d$ is a Markov jump process on the state space

$$\mathcal{M}_N^d := \mathbb{Z}_+^{\mathbb{T}_N^d}$$

of all configurations of particles on $\mathbb{T}_N^d$, defined through a local jump rate function $g$ and an elementary step distribution $p$. 
The Zero Range Process on the discrete torus $\mathbb{T}^d_N$

Let $\mathbb{T}^d_N := \{0, 1, \ldots, N - 1\}^d$, $N \in \mathbb{N}$, denote the discrete torus. A Zero Range Process (ZRP) on $\mathbb{T}^d_N$ is Markov jump process on the state space

$$\mathcal{M}^d_N := \mathbb{Z}^{\mathbb{T}^d_N}$$

of all configurations of particles on $\mathbb{T}^d_N$, defined through a local jump rate function $g$ and an elementary step distribution $p$.

For simplicity we consider the symmetric nearest neighbor (n.n.) elementary step distribution $p \in \mathcal{M}_+(\mathbb{T}^d_N) \subseteq \mathcal{M}_+(\mathbb{Z}^d)$, given by

$$p = \sum_{j=1}^d (1_{\{e_j\}} + 1_{\{-e_j\}})$$
Let $\mathbb{T}_N^d := \{0, 1, \ldots, N - 1\}^d$, $N \in \mathbb{N}$, denote the discrete torus.

A Zero Range Process (ZRP) on $\mathbb{T}_N^d$ is a Markov jump process on the state space

$$\mathcal{M}_N^d := \mathbb{Z}_+^\mathbb{T}_N^d$$

of all configurations of particles on $\mathbb{T}_N^d$, defined through a local jump rate function $g$ and an elementary step distribution $p$.

For simplicity we consider the symmetric nearest neighbor (n.n.) elementary step distribution $p \in \mathcal{M}_+(\mathbb{T}_N^d) \subseteq \mathcal{M}_+(\mathbb{Z}^d)$, given by

$$p = \sum_{j=1}^{d} (\mathbb{1}_{\{e_j\}} + \mathbb{1}_{\{-e_j\}}),$$

where by convention we consider $p$ to be of total probability $p(\mathbb{Z}^d) = 2d$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
A function $g : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ is called a local jump rate function if

$$g(k) = 0 \iff k = 0, \quad (1)$$

$$\|g'\|_u := \sup_{k \in \mathbb{Z}_+} |g(k+1) - g(k)| < +\infty, \quad (2)$$

and

$$\varphi_c := \liminf_{k \to \infty} \sqrt[k]{g!(k)} > 0, \quad (3)$$

where $g!(0) := 1$ and $g!(k) := g(1) \cdot g(2) \cdots \cdot g(k)$. 

For simplicity we restrict our attention to bounded local jump rate functions.
The Zero Range Process on the discrete torus $\mathbb{T}^d_N$

**Definition**

A function $g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is called a local jump rate function if

$$g(k) = 0 \iff k = 0,$$

$$\|g'\|_u := \sup_{k \in \mathbb{Z}_+} |g(k + 1) - g(k)| < +\infty,$$

and

$$\varphi_c := \liminf_{k \to \infty} \sqrt[k]{g!(k)} > 0,$$

where $g!(0) := 1$ and $g!(k) := g(1) \cdot g(2) \cdots \cdot g(k)$.

For simplicity we restrict our attention to bounded local jump rate functions.
The Zero Range Process on the discrete torus $\mathbb{T}^d_N$

**Definition**

Let $g$ be a bounded local jump rate function. The symmetric n.n. ZRP of local jump rate function $g$ on the discrete torus $\mathbb{T}^d_N$ is the unique Markov jump process on the Skorohod space $D(\mathbb{R}_+; \mathcal{M}_N^d)$ with bounded generator $L_N : B(\mathcal{M}_N^d) \to B(\mathcal{M}_N^d)$ given by the formula

$$L_N f(\eta) = \sum_{x,y \in \mathbb{T}^d_N} \{ f(\eta^{x,y}) - f(\eta) \} g(\eta(x)) p(y - x),$$

where

$$\eta^{x,y}(z) = \begin{cases} 
\eta(z), & \text{if } z \notin \{x, y\} \\
\eta(x) - 1, & \text{if } z = x \\
\eta(y) + 1, & \text{if } z = y
\end{cases}.$$
Suppose we start from an initial configuration $\eta_0 \in \mathbb{M}_N^d$. 

• Remarks
  1. The total number of particles is conserved by the dynamics since for all configurations $\eta$ we obviously have that $|\eta| := \sum_{z \in T^d} \eta(z) = |\eta_{x,y}|$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
Suppose we start from an initial configuration \( \eta_0 \in \mathbb{M}_N^d \). The system waits an exponential time \( \tau_1 \) of parameter

\[
\lambda_N(\eta_0) := \sum_{x \in \mathbb{T}_N^d} g(\eta_0(x)),
\]

then the process starts afresh with \( \eta_{\tau_1} \) in place of \( \eta_0 \).

Remarks
1. The total number of particles is conserved by the dynamics since for all configurations \( \eta \) we obviously have that

\[
|\eta| := \sum_{z \in \mathbb{T}_N^d} \eta(z) = |\eta_{x,y}|.
\]
Suppose we start from an initial configuration $\eta_0 \in \mathbb{M}_N^d$. The system waits an exponential time $\tau_1$ of parameter

$$\lambda_N(\eta_0) := \sum_{x \in \mathbb{T}_N^d} g(\eta_0(x)),$$

at which time the system shifts to the new state

$$\eta_{\tau_1} = \eta_0^{x,y}, \quad \text{with probability} \quad \frac{g(\eta_0(x))}{\lambda_N(\eta_0)} p(y - x).$$
The Stochastic Microscopic Dynamics of the ZRP

Suppose we start from an initial configuration \( \eta_0 \in \mathbb{M}_N^d \).
The system waits an exponential time \( \tau_1 \) of parameter

\[
\lambda_N(\eta_0) := \sum_{x \in \mathbb{T}_N^d} g(\eta_0(x)),
\]

at which time the system shifts to the new state

\[
\eta_{\tau_1} = \eta_0^{x,y}, \quad \text{with probability } \frac{g(\eta_0(x))}{\lambda_N(\eta_0)} p(y - x).
\]

Then the process starts afresh with \( \eta_{\tau_1} \) in place of \( \eta_0 \).
The Stochastic Microscopic Dynamics of the ZRP

Suppose we start from an initial configuration $\eta_0 \in \mathbb{M}^d_N$.

The system waits an exponential time $\tau_1$ of parameter

$$\lambda_N(\eta_0) := \sum_{x \in \mathbb{T}_N^d} g(\eta_0(x)),$$

at which time the system shifts to the new state

$$\eta_{\tau_1} = \eta_0^{x,y}, \quad \text{with probability } \frac{g(\eta_0(x))}{\lambda_N(\eta_0)} p(y - x).$$

Then the process starts afresh with $\eta_{\tau_1}$ in place of $\eta_0$.

- **Remarks**
The Stochastic Microscopic Dynamics of the ZRP

Suppose we start from an initial configuration $\eta_0 \in \mathbb{M}_N^d$.

The system waits an exponential time $\tau_1$ of parameter

$$\lambda_N(\eta_0) := \sum_{x \in \mathbb{T}_N^d} g(\eta_0(x)),$$

at which time the system shifts to the new state

$$\eta_{\tau_1} = \eta_0^{x,y}, \quad \text{with probability } \frac{g(\eta_0(x))}{\lambda_N(\eta_0)} p(y - x).$$

Then the process starts afresh with $\eta_{\tau_1}$ in place of $\eta_0$.

**Remarks**

1. The total number of particles is conserved by the dynamics since for all configurations $\eta$ we obviously have that

$$|\eta| := \sum_{z \in \mathbb{T}_N^d} \eta(z) = |\eta^{x,y}|.$$
2. If the local jump rate function $g$ is increasing, then particles jump with higher rate from sites with a large number of particles than from sites with a small number of particles.
2. If the local jump rate function \( g \) is increasing, then particles jump with higher rate from sites with a large number of particles than from sites with a small number of particles.

On the other hand if the local jump rate is decreasing, then particles jump with lower rate from sites with a large number of particles than from sites with a small number of particles.
2. If the local jump rate function $g$ is increasing, then particles jump with higher rate from sites with a large number of particles than from sites with a small number of particles. On the other hand if the local jump rate is decreasing, then particles jump with lower rate from sites with a large number of particles than from sites with a small number of particles. This behaviour of ZR process with decreasing local jump rate functions may lead to the emergence of a condensate.
The Grand Canonical Ensemble

The grand canonical ensemble of the ZRP is the monoparametric family of its extremal translation invariant equilibrium distributions.
The Grand Canonical Ensemble

The grand canonical ensemble of the ZRP is the monoparametric family of its extremal translation invariant equilibrium distributions. Suppose that $\nu \in \mathbb{P} M^d_N$ a translation invariant product equilibrium distribution for the ZRP with local rate function $g$ of the form

$$
\nu = \nu_1 \otimes T^d_N
$$

for some $\nu_1 \in \mathbb{P} \mathbb{Z}_+$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
The Grand Canonical Ensemble

The grand canonical ensemble of the ZRP is the monoparametric family of its extremal translation invariant equilibrium distributions. Suppose that $\nu \in \mathbb{P} M_N^d$ a translation invariant product equilibrium distribution for the ZRP with local rate function $g$ of the form

$$\nu = \nu_1 \otimes T_N^d$$

for some $\nu_1 \in \mathbb{P} \mathbb{Z}_+$. Then $\nu L^N = 0$ and so for any $k \in \mathbb{Z}_+$,

$$\int L^N \left[ \mathbb{1}_{\{k\}} (\eta(x)) \right] d\nu = 0. \quad (4)$$
The Grand Canonical Ensemble

The grand canonical ensemble of the ZRP is the monoparametric family of its extremal translation invariant equilibrium distributions.

Suppose that $\nu \in \mathbb{P}M_{dN}^{d}$ a translation invariant product equilibrium distribution for the ZRP with local rate function $g$ of the form

$$\nu = \nu_1^{\otimes T_{dN}}$$

for some $\nu_1 \in \mathbb{P}Z_+$. Then $\nu L^N = 0$ and so for any $k \in Z_+$,

$$\int L^N \left[ \mathbb{1}_{\{k\}}(\eta(x)) \right] d\nu = 0. \tag{4}$$

By a standard calculation,

$$L^N \left[ \mathbb{1}_{\{k\}}(\eta(x)) \right] = \left( \mathbb{1}_{\{\eta(x)=k+1\}} - \mathbb{1}_{\{\eta(x)=k\}} \right) \cdot g(\eta(x))$$

$$+ \left( \mathbb{1}_{\{\eta(x)=k-1\}} - \mathbb{1}_{\{\eta(x)=k\}} \right) \sum_{y \neq x} g(\eta(y))p_N(x - y).$$
Since $\nu$ is a translation invariant product measure, the r.v. $\eta(x)$, $x \in \mathbb{T}_N^d$, are i.i.d. with common marginal $\nu_1$. 
Since \( \nu \) is a translation invariant product measure, the r.v. \( \eta(x) \), \( x \in \mathbb{T}_N^d \), are i.i.d. with common marginal \( \nu_1 \).

Consequently the quantities \( \int g(\eta(x))d\nu \) are non-negative real numbers independent of \( x \in \mathbb{T}_N^d \), say

\[
\int g(\eta(x))d\nu = \int_{\mathbb{Z}^+} gd\nu_1 =: \varphi \in \mathbb{R}_+, \quad \forall \ x \in \mathbb{T}_N^d.
\]
The Grand Canonical Ensemble

Since $\nu$ is a translation invariant product measure, the r.v. $\eta(x)$, $x \in \mathbb{T}^d_N$, are i.i.d. with common marginal $\nu_1$.

Consequently the quantities $\int g(\eta(x))d\nu$ are non-negative real numbers independent of $x \in \mathbb{T}^d_N$, say

$$\int g(\eta(x))d\nu = \int_{\mathbb{Z}^+} gd\nu_1 =: \varphi \in \mathbb{R}_+, \quad \forall \, x \in \mathbb{T}^d_N.$$ 

It follows by (4) and the formula of $L^N[\mathbb{1}_{\{k\}}(\eta(x))]$ that

$$g(k + 1)\nu_1(k + 1) - g(k)\nu_1(k) = \varphi \nu_1(k) - \varphi \nu_1(k - 1) \quad (5)$$

for all $k \in \mathbb{Z}_+$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
The Grand Canonical Ensemble

It follows that

\[ \nu_1(k) = \nu_1(0) \frac{\phi^k}{g!(k)}, \quad \forall k \in \mathbb{N}. \]  

(6)
The Grand Canonical Ensemble

It follows that

\[ \nu_1(k) = \nu_1(0) \frac{\varphi^k}{g!(k)}, \quad \forall \ k \in \mathbb{N}. \]  

(6)

Since \( \nu_1 \) is a probability measure we must have that \( \nu_1(0) > 0 \) and

\[ 1 = \nu_1(0) + \sum_{k=1}^{\infty} \nu_1(k) = \nu_1(0) \left( 1 + \sum_{k=1}^{\infty} \frac{\varphi^k}{g!(k)} \right) = \nu_1(0) \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)}. \]
The Grand Canonical Ensemble

It follows that

\[ \nu_1(k) = \nu_1(0) \frac{\varphi^k}{g!(k)}, \quad \forall \ k \in \mathbb{N}. \quad (6) \]

Since \( \nu_1 \) is a probability measure we must have that \( \nu_1(0) > 0 \) and

\[ 1 = \nu_1(0) + \sum_{k=1}^{\infty} \nu_1(k) = \nu_1(0) \left( 1 + \sum_{k=1}^{\infty} \frac{\varphi^k}{g!(k)} \right) = \nu_1(0) \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)}. \]

In particular we have that the series \( \sum_{k=0}^{\infty} \varphi^k / g!(k) \) must be convergent and that if an equilibrium distribution \( \nu \) is to be of the form \( \nu = \nu_1 \otimes \mathbb{T}^d_N \) for some \( \nu_1 \in \mathbb{P}\mathbb{Z}_+ \), then the power series

\[ Z(\varphi) \equiv Z_g(\varphi) := \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)} \quad (7) \]

must be finite at \( \varphi := \int gd\nu_1. \)
The Grand Canonical Ensemble

The one site marginal $\nu_1$ must be given by the formula

$$\nu_1(k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+. \quad (8)$$
The Grand Canonical Ensemble

The one site marginal $\nu_1$ must be given by the formula

$$\nu_1(k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+. \quad (8)$$

**Definition**

The function $Z \equiv Z_g : \mathbb{R}_+ \rightarrow [1, \infty]$ defined by the power series in (7) is called the normalizing partition function associated to $g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$. We will denote by $\mathcal{D}_Z = \{ \varphi \in \mathbb{R}_+ | Z(\varphi) < +\infty \}$ the proper domain of the partition function $Z$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
The Grand Canonical Ensemble

The one site marginal $\nu_1$ must be given by the formula

$$
\nu_1(k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+.
$$

(8)

Definition

The function $Z \equiv Z_g : \mathbb{R}_+ \rightarrow [1, \infty]$ defined by the power series in (7) is called the normalizing partition function associated to $g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$. We will denote by $\mathcal{D}_Z = \{ \varphi \in \mathbb{R}_+ | Z(\varphi) < +\infty \}$ the proper domain of the partition function $Z$.

The radius of convergence of $Z$ is

$$
\varphi_c = \liminf \sqrt[k]{g!(k)}
$$

and so assumption (3) of local jump rate functions guaranties that $Z$ has non-trivial domain of convergence $[0, \phi_c)$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
The Grand Canonical Ensemble

Definition

Let $g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ be a local rate function and let $Z$ be the partition function associated to $g$. For any $\varphi \in \mathcal{D}_Z$, the distribution $\bar{\nu}_1^\varphi \equiv \bar{\nu}_1^\varphi, g \in \mathbb{P}\mathbb{Z}_+$ defined by

$$
\bar{\nu}_1^\varphi\{k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+
$$

will be called the one-site zero range distribution of local rate $g$ and fugacity $\varphi$.

The product distribution $\bar{\nu}_1^N \varphi, g \in \mathbb{P}\mathbb{M}_N^d$ with common marginal $\bar{\nu}_1^\varphi, g \in \mathbb{P}\mathbb{Z}_+, \varphi \in \mathcal{D}_Z$, will be called the zero range equilibrium distribution on the discrete torus $\mathbb{T}_N^d$ of rate $g$ and fugacity $\varphi$. 

The Grand Canonical Ensemble

Definition

Let \( g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+ \) be a local rate function and let \( Z \) be the partition function associated to \( g \). For any \( \varphi \in \mathcal{D}_Z \), the distribution \( \bar{\nu}^1_{\varphi} \equiv \bar{\nu}^1_{\varphi,g} \in \mathbb{P}\mathbb{Z}_+ \) defined by

\[
\bar{\nu}^1_{\varphi}\{k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g!(k)}, \quad k \in \mathbb{Z}_+ \quad (9)
\]

will be called the one-site zero range distribution of local rate \( g \) and fugacity \( \varphi \).

The product distribution \( \bar{\nu}^N_{\varphi,g} \in \mathbb{P}\mathbb{M}_N \) with common marginal \( \bar{\nu}^1_{\varphi,g} \in \mathbb{P}\mathbb{Z}_+, \varphi \in \mathcal{D}_Z, g \), will be called the zero range equilibrium distribution on the discrete torus \( \mathbb{T}_N^d \) of rate \( g \) and fugacity \( \varphi \).

Obviously the family \( (\bar{\nu}^N_{\varphi})_{\varphi \in \mathcal{D}_Z} \subseteq \mathbb{P}\mathbb{M}_N \) is a family of equilibrium distributions for the ZRP.
For all $\varphi \in [0, \varphi_c)$ we have that

$$R(\varphi) := \int \eta(0) d\bar{\nu}^N_{\varphi} = \frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} k \frac{\varphi^k}{g!(k)} = \frac{\varphi Z'(\varphi)}{Z(\varphi)}. \quad (10)$$

Consequently $R(\varphi) = \varphi$ iff $Z(\varphi) = Z'(\varphi)$,
For all \( \varphi \in [0, \varphi_c) \) we have that

\[
R(\varphi) := \int \eta(0) d\tilde{\nu}_\varphi^N = \frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} \frac{\varphi^k}{g!(k)} = \frac{\varphi Z'(\varphi)}{Z(\varphi)}.
\]

Consequently \( R(\varphi) = \varphi \) iff \( Z(\varphi) = Z'(\varphi) \), or in other words the family \((\tilde{\nu}_\varphi^N)\) is parametrized by the density iff \( g = id_{\mathbb{Z}_+} \).
The Grand Canonical Ensemble

The following results allow us to reparametrize the family \((\tilde{\nu}^N_\varphi)_{\varphi \in \mathcal{D}_Z}\) by the density.
The Grand Canonical Ensemble

The following results allow us to reparametrize the family 
\((\bar{\nu}^N_\varphi)_{\varphi \in D_Z}\) by the density.

Proposition

Let \(g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+\) be a local rate function and let \(\varphi_c > 0\) be the radius of convergence of the partition function \(Z : \mathbb{R}_+ \rightarrow [1, \infty]\) defined by \(g\). Then

\[ [0, \varphi_c) \subseteq D_Z \subseteq [0, \varphi_c], \]

(11)

the partition function \(Z : \mathbb{R}_+ \rightarrow [1, +\infty]\) is a lower semicontinuous, strictly increasing and strictly convex function, \(C^\infty\) on \([0, \varphi_c)\) with all of it’s derivatives strictly positive.
The Grand Canonical Ensemble

Proposition

The mean density function \( R : [0, \varphi_c) \rightarrow \mathbb{R}_+ \) defined in (10) is a smooth and strictly increasing function of the fugacity \( \varphi \).
The mean density function \( R : [0, \varphi_c) \rightarrow \mathbb{R}_+ \) defined in (10) is a smooth and strictly increasing function of the fugacity \( \varphi \).

Since the density of particles \( R : [0, \varphi_c) \rightarrow [0, \infty) \) is strictly increasing, it is injective with image the subinterval \([0, \rho_c)\) of \( \mathbb{R}_+ \), where

\[
\rho_c := \lim_{\varphi \uparrow \varphi_c} R(\varphi) = \sup_{0 \leq \varphi < \varphi_c} R(\varphi)
\]

and it’s inverse \( \Phi := R^{-1} : [0, \rho_c) \rightarrow [0, \phi_c) \) is well defined.
The Grand Canonical Ensemble

Proposition

The mean density function $R : [0, \varphi_c) \rightarrow \mathbb{R}_+$ defined in (10) is a smooth and strictly increasing function of the fugacity $\varphi$.

Since the density of particles $R : [0, \varphi_c) \rightarrow [0, \infty)$ is strictly increasing, it is injective with image the subinterval $[0, \rho_c)$ of $\mathbb{R}_+$, where

$$\rho_c := \lim_{\varphi \uparrow \varphi_c} R(\varphi) = \sup_{0 \leq \varphi < \varphi_c} R(\varphi)$$

and it’s inverse $\Phi := R^{-1} : [0, \rho_c) \rightarrow [0, \phi_c)$ is well defined. Consequently for any $0 \leq \rho < \rho_c$ we can define a a product translation invariant equilibrium distribution $\nu^N_\rho$ of the ZRP with density $\rho$, by setting

$$\nu^N_\rho := \bar{\nu}^N_{\Phi(\rho)}.$$
The following proposition deals with the case of the critical density $\rho_c$ when it is finite.
The following proposition deals with the case of the critical density \( \rho_c \) when it is finite.

**Proposition**

Let \( \{\nu_{\varphi}^{1}, g\}_{\varphi \in D_Z} \subseteq \mathbb{P}\mathbb{Z}_+ \) be the family of one-site ZR distributions associated to the local rate function \( g : \mathbb{Z}_+ \rightarrow \mathbb{R}_+ \) and let \( R : [0, \varphi_c) \rightarrow \mathbb{R}_+ \) be the mean density function. If

\[
\rho_c := \lim_{\varphi \uparrow \varphi_c} R(\varphi) < +\infty, \quad (12)
\]

then \( \varphi_c \in D_Z \cap D_{Z'} \) and

\[
\rho_c = R(\varphi_c) = \frac{\varphi_c Z'(\varphi_c)}{Z(\varphi_c)} = \int k d\tilde{\alpha}_{\varphi_c,g}(k). \quad (13)
\]
The Grand Canonical Ensemble

So when $\rho_c$ is finite we have that $\varphi_c \in \mathcal{D}_Z$ and so we can also define a translation invariant product equilibrium distribution of the ZRP with finite density $\rho_c$, by $\nu^N_{\rho_c} := \bar{\nu}^N_{\varphi_c}$.
The Grand Canonical Ensemble

So when $\rho_c$ is finite we have that $\varphi_c \in \mathcal{D}_Z$ and so we can also define a translation invariant product equilibrium distribution of the ZRP with finite density $\rho_c$, by $\nu^N_{\rho} := \tilde{\nu}^N_{\varphi_c}$. Consequently, in any case, the Grand Canonical Ensemble of the ZRP is the family of distributions defined by

$$\nu^N_{\rho} := \tilde{\nu}^N_{\Phi(\rho)}, \quad \rho \in [0, \rho_c] \cap \mathbb{R}^+.$$
The Grand Canonical Ensemble

So when $\rho_c$ is finite we have that $\varphi_c \in D_Z$ and so we can also define a translation invariant product equilibrium distribution of the ZRP with finite density $\rho_c$, by $\nu^N_{\rho_c} := \tilde{\nu}^N_{\varphi_c}$. Consequently, in any case, the Grand Canonical Ensemble of the ZRP is the family of distributions defined by

$$\nu^N_{\rho} := \tilde{\nu}^N_{\Phi(\rho)}, \quad \rho \in [0, \rho_c] \cap \mathbb{R}^+.$$ 

It is worth noting that the fugacity function $\Phi = R^{-1}$ is the one-site mean jump rate under the grand canonical ensemble:
So when $\rho_c$ is finite we have that $\varphi_c \in \mathcal{D}_Z$ and so we can also define a translation invariant product equilibrium distribution of the ZRP with finite density $\rho_c$, by $\nu^N_{\rho_c} := \bar{\nu}^N_{\varphi_c}$. Consequently, in any case, the Grand Canonical Ensemble of the ZRP is the family of distributions defined by

$$\nu^N_{\rho} := \bar{\nu}^N_{\Phi(\rho)}, \quad \rho \in [0, \rho_c] \cap \mathbb{R}_+.$$ 

It is worth noting that the fugacity function $\Phi = R^{-1}$ is the one-site mean jump rate under the grand canonical ensemble:

$$\int g(k) d\nu^1_{\rho}(k) = \frac{1}{Z(\Phi(\rho))} \sum_{k=1}^{\infty} \frac{\Phi(\rho)^k}{g!(k)} = \Phi(\rho).$$
Large Deviations of the Occupation Variables with respect to the Grand Canonical Ensemble

The moment generating function of $\nu^1_\rho$ is given by

$$M_{\nu^1_\rho}(\theta) := \int e^{\theta k} d\nu^1_\rho(k) = \frac{1}{Z(\Phi(\rho))} \sum_{k=0}^{\infty} \frac{(\Phi(\rho)e^\theta)^k}{g!(k)} = \frac{Z(\Phi(\rho)e^\theta)}{Z(\Phi(\rho))},$$

and therefore $\nu^1_\rho$ has finite exponential moments iff $\rho < \rho_c$. 
Large Deviations of the Occupation Variables with respect to the Grand Canonical Ensemble

The moment generating function of $\nu^1_\rho$ is given by

$$M_{\nu^1_\rho}(\theta) := \int e^{\theta k} d\nu^1_\rho(k) = \frac{1}{Z(\Phi(\rho))} \sum_{k=0}^{\infty} \frac{(\Phi(\rho)e^{\theta})^k}{g!(k)} = \frac{Z(\Phi(\rho)e^{\theta})}{Z(\Phi(\rho))},$$

and therefore $\nu^1_\rho$ has finite exponential moments iff $\rho < \rho_c$. In this case the radius of the exponential moments is $\log \frac{\varphi_c}{\Phi(\rho)} > 0$, that is

$$[0, \log \frac{\varphi_c}{\Phi(\rho)}] \subseteq D_{M_{\nu^1_\rho}} \subseteq [0, \log \frac{\varphi_c}{\Phi(\rho)}],$$

and therefore $\nu^1_\rho$ has full exponential moments iff $\varphi_c = +\infty$. 
Large Deviations of the Occupation Variables with respect to the Grand Canonical Ensemble

Since for each subcritical density $\rho < \rho_c$ the distribution $\nu_\rho^1$ has finite exponential moments, it follows by Cramer's theorem that with respect to $\nu_\rho^\infty \in \mathbb{PZ}_+^{\mathbb{Z}^d}$ i.i.d. sequence $\{\eta(x)\}_{x \in \mathbb{Z}^d}$ of the occupation variables satisfies the large deviations principle with rate function the Legendre transform $\Lambda^*_{\nu_\rho^1}$ of the logarithmic moment generating function $\Lambda_{\nu_\rho^1} := \log M_{\nu_\rho^1}$. 
Large Deviations of the Occupation Variables with respect to the Grand Canonical Ensemble

Since for each subcritical density $\rho < \rho_c$ the distribution $\nu^1_\rho$ has finite exponential moments, it follows by Cramer’s theorem that with respect to $\nu^\infty_\rho \in \mathbb{P} \mathbb{Z}^d_+ \text{i.i.d. sequence } \{\eta(x)\}_{x \in \mathbb{Z}^d} \text{ of the occupation variables satisfies the large deviations principle with rate function the Legendre transform } \Lambda^*_\nu^1_\rho \text{ of the logarithmic moment generating function } \Lambda_{\nu^1_\rho} := \log M_{\nu^1_\rho}. \text{ The Legendre transform } \Lambda^*_\nu^1_\rho \text{ can be computed:}

$$
\Lambda^*_\nu^1_\rho(a) = \begin{cases} 
\rho \log \frac{\Phi(a \wedge \rho_c)}{\Phi(\rho)} - \log \frac{Z(\Phi(a \wedge \rho_c))}{Z(\Phi(\rho))}, & a \geq 0 \\
+\infty, & a < 0
\end{cases}
$$
The Canonical Ensemble

It is obvious by the stochastic dynamics of the ZRP that its communication classes are the "hyperplanes" with a constant number of particles:

\[ \mathbb{M}^d_{N,K} := \{ \eta \in \mathbb{M}^d_N : |\eta| = K \}, \quad K \in \mathbb{Z}_+. \]
The Canonical Ensemble

It is obvious by the stochastic dynamics of the ZRP that its communication classes are the "hyperplanes" with a constant number of particles:

\[ \mathbb{M}_{d,N,K}^{d} := \{ \eta \in \mathbb{M}_{N}^{d} : |\eta| = K \}, \quad K \in \mathbb{Z}_{+}. \]

The sets \( \mathbb{M}_{d,N,K}^{d} \) are finite and therefore there exists a unique equilibrium distribution \( \nu_{N,K} \) concentrated on \( \mathbb{M}_{d,N,K}^{d} \), given by

\[ \nu_{N,K} := \nu_{\rho}^{N}_{\mid \mathbb{M}_{d,N,K}^{d}} := \nu_{\rho}^{N} \left( \cdot \mid \{ | \cdot |_{1} = K \} \right), \quad \rho \in (0, \rho_{c}). \]
The Canonical Ensemble

It is obvious by the stochastic dynamics of the ZRP that its communication classes are the "hyperplanes" with a constant number of particles:

\[ \mathbb{M}^d_{N,K} := \{ \eta \in \mathbb{M}^d_N : |\eta| = K \}, \quad K \in \mathbb{Z}_+. \]

The sets \( \mathbb{M}^d_{N,K} \) are finite and therefore there exists a unique equilibrium distribution \( \nu_{N,K} \) concentrated on \( \mathbb{M}^d_{N,K} \), given by

\[ \nu_{N,K} := \nu^N_\rho \big| \mathbb{M}^d_{N,K} := \nu^N_\rho \left( \cdot \middle| \{ |\cdot|_1 = K \} \right), \quad \rho \in (0, \rho_c). \]

The measure \( \nu_{N,K} \) is given by

\[ \nu_{N,K}(\eta) = \frac{1}{Z(N,K)} \prod_{x \in \mathbb{T}^d_N} \frac{1}{g!(\eta_x)}, \quad \eta \in \mathbb{M}^d_{N,K}, \]

where the so-called canonical partition function is given by

\[ Z(N,K) := \sum_{\eta \in \mathbb{M}^d_{N,K}} \prod_{x \in \mathbb{T}^d_N} \frac{1}{g!(\eta_x)}. \]
Equivalence of Ensembles

Proposition

Let \( \{ \nu_{N,K} \in \mathbb{P}M_N^d \}_{(N,K) \in \mathbb{N} \times \mathbb{Z}_+} \) and \( \{ \nu_{\rho}^N \}_{\rho \in I_c}, I_c := [0, \rho_c] \cap \mathbb{R}_+ \), be the canonical and grand canonical ensemble of the ZRP associated local rate function \( g \), let \( \pi^L : M_N^d \rightarrow M_L^d, N \geq L \), be the natural projections and set \( \nu_{N,K}^L := \pi^L_N \nu_{N,K} \). Then for fixed \( L \in \mathbb{N} \)

\[
\lim_{N \rightarrow +\infty} \mathcal{H}(\nu_{N,[\rho N^d]}^L | \nu_{\rho \wedge \rho_c}^L) = 0
\]

for all \( \rho \geq 0 \). 

Grosskinsky, Schütz, Spohn 2006.
Equivalence of Ensembles

Proposition

Let \( \{\nu_{N,K} \in \mathbb{PM}^d_{N}\}_{(N,K)\in \mathbb{N}\times \mathbb{Z}_+} \) and \( \{\nu^N_{\rho}\}_{\rho \in I_c}, \ I_c := [0, \rho_c] \cap \mathbb{R}_+ \), be the canonical and grand canonical ensemble of the ZRP associated local rate function \( g \), let \( \pi^L : \mathbb{M}^d_N \to \mathbb{M}^d_L, N \geq L \), be the natural projections and set \( \nu^L_{N,K} := \pi^L_\ast \nu_{N,K} \). Then for fixed \( L \in \mathbb{N} \)

\[
\lim_{N \to +\infty} \mathcal{H}(\nu^L_{N,[\rho N^d]}|\nu^L_{\rho \wedge \rho_c}) = 0
\]

for all \( \rho \geq 0 \).  

Recall Pinsker’s inequality: For any measurable space \( M \), the total variation norm on \( \mathbb{PM} \) is bounded by twice the relative entropy,

\[
\|\nu - \mu\|_{TV} \leq 2\mathcal{H}(\nu|\mu), \quad \forall \mu, \nu \in \mathbb{PM}.
\]
Equivalence of Ensembles

It follows that

$$\lim_{N \to +\infty} \nu_{N,[\rho N^d]} = \nu_{\rho \wedge \rho_c}$$

in the weak topology of $\mathbb{P}M^d_\infty$. 
Equivalence of Ensembles

It follows that

\[ \lim_{N \to +\infty} \nu_{N,[\rho N^d]} = \nu_{\rho \wedge \rho_c}^\infty \]

in the weak topology of \( \mathbb{P} \mathbb{M}_\infty^d \).

In the case of finite critical density \( \rho_c < \infty \), this form of the equivalence of ensembles (as well as the fact that there are no grand canonical measures of densities \( \rho > \rho_c \)) shows that the Hydrodynamic Limit via the Principle of Local Equilibrium faces difficulty in observing density values above the critical density \( \rho_c \).
It follows that
\[
\lim_{N \to +\infty} \nu_{N, [\rho N^d]} = \nu_{\rho \wedge \rho_c}^\infty
\]
in the weak topology of \( \mathbb{P} M^d_\infty \).

In the case of finite critical density \( \rho_c < \infty \), this form of the equivalence of ensembles (as well as the fact that there are no grand canonical measures of densities \( \rho > \rho_c \)) shows that the Hydrodynamic Limit via the Principle of Local Equilibrium faces difficulty in observing density values above the critical density \( \rho_c \).

This suggests that at the macroscopic level we should consider solutions \( \mu \) whose absolutely continuous part \( \mu^{ac} \) (with respect to Lebesque measure on the torus \( \mathbb{T}^d \)) satisfies \( \mu^{ac} \leq \rho_c \) in some appropriate sense.
Evans studies the ZRP with local jump rate function

$$g_b(k) = \mathbb{1}_{k \geq 1} \left( 1 + \frac{b}{k} \right), \quad b \geq 0.$$
An Example Exhibiting Condensation: The Evans Model

Evans studies the ZRP with local jump rate function

$$g_b(k) = \mathbb{1}_{\{k \geq 1\}} \left(1 + \frac{b}{k}\right), \quad b \geq 0.$$  

Then

$$g_b!(k) = \prod_{i=1}^{k} \left(1 + \frac{b}{i}\right) =: \frac{(1 + b)_k}{k!},$$
Evans studies the ZRP with local jump rate function

\[ g_b(k) = \mathbb{1}_{\{k \geq 1\}} \left( 1 + \frac{b}{k} \right), \quad b \geq 0. \]

Then

\[ g_b!(k) = \prod_{i=1}^{k} \left( 1 + \frac{b}{i} \right) =: \frac{(1 + b)^k}{k!}, \]

where

\[ (a)_k := \prod_{i=0}^{k-1} (a + i) = \frac{\Gamma(a + k)}{\Gamma(a)}, \quad a \in \mathbb{R}, \; k \in \mathbb{N}, \]

denotes the Pochhammer symbol.
An Example Exhibiting Condensation: The Evans Model

The grand canonical partition function $Z_b$ associated to $g_b$ is given by the hypergeometric function

$$Z_b(\varphi) = \sum_{k=0}^{\infty} \frac{k!\varphi^k}{(1 + b)_k} = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k}{(1 + b)_k} \frac{\varphi^k}{k!} =: \, _2F_1(1, 1; 1 + b; \varphi).$$
The grand canonical partition function $Z_b$ associated to $g_b$ is given by the hypergeometric function

$$Z_b(\varphi) = \sum_{k=0}^{\infty} \frac{k! \varphi^k}{(1 + b)_k} = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k \varphi^k}{(1 + b)_k k!} =: \, _2F_1(1, 1; 1 + b; \varphi).$$

Its radius of convergence is $\varphi_c = 1$. 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
The grand canonical partition function $Z_b$ associated to $g_b$ is given by the hypergeometric function

$$Z_b(\varphi) = \sum_{k=0}^{\infty} \frac{k! \varphi^k}{(1 + b)_k} = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k \varphi^k}{(1 + b)_k} =: \,_2\!F_1(1, 1; 1 + b; \varphi).$$

Its radius of convergence is $\varphi_c = 1$, and it has the asymptotic expansion

$$\,_2\!F_1(k, k; k + b; \varphi) = \frac{\Gamma(k + b)\Gamma(k - b)}{\Gamma(k)^2} (1 - \varphi)^{b-k}[1 + O(\varphi)]$$

$$+ \frac{\Gamma(k + b)\Gamma(b - k)}{\Gamma(b)^2} \left[1 + \frac{k^2(1 - \varphi)}{1 + k - b} + O(1 - \varphi)^2 \right]$$

as $\varphi \uparrow 1$. 
The density function $R_b$ associated to $g_b$ is given by

$$R_b(\varphi) = \frac{\varphi Z'_b(\varphi)}{Z_b(\varphi)} = \frac{\varphi \cdot 2F_1(2, 2; 2 + b; \varphi)}{(1 + b) \cdot 2F_1(1, 1; 1 + b; \varphi)}.$$
The density function $R_b$ associated to $g_b$ is given by

$$R_b(\varphi) = \frac{\varphi Z'_b(\varphi)}{Z_b(\varphi)} = \frac{\varphi \cdot \left(\begin{array}{c}2F_1(2,2; 2+b; \varphi) \end{array}\right)}{(1+b) \cdot \left(\begin{array}{c}2F_1(1,1; 1+b; \varphi) \end{array}\right)}.$$ 

For $b = 0$ the partition and density function can be computed explicitly:

$$Z_0(\varphi) = \frac{1}{1 - \varphi} \implies Z(\varphi_c) = \infty$$

$$R_0(\varphi) = \frac{\varphi}{1 - \varphi} \implies \rho_c = \infty.$$
The density function $R_b$ associated to $g_b$ is given by

$$R_b(\varphi) = \frac{\varphi Z'_b(\varphi)}{Z_b(\varphi)} = \frac{\varphi \cdot {}_2F_1(2, 2; 2 + b; \varphi)}{(1 + b) \cdot {}_2F_1(1, 1; 1 + b; \varphi)}.$$ 

For $b = 0$ the partition and density function can be computed explicitly:

$$Z_0(\varphi) = \frac{1}{1 - \varphi} \quad \Rightarrow \quad Z(\varphi_c) = \infty$$

$$R_0(\varphi) = \frac{\varphi}{1 - \varphi} \quad \Rightarrow \quad \rho_c = \infty.$$ 

For $0 < b < 1$ the asymptotic expansion of the hypergeometric funcion near $\varphi_c = 1$ gives,

$$Z_b(\varphi) \approx \Gamma(1 + b)\Gamma(1 - b)(1 - \varphi)^{b-1}, \quad \Rightarrow \quad Z(\varphi_c) = \infty,$$

$$R_b(\varphi) \approx \frac{1}{(1 + b)^2(1 - b)} \frac{\varphi}{1 - \varphi}, \quad \Rightarrow \quad \rho_c = \infty.$$
An Example Exhibiting Condensation: The Evans Model

For $b = 1$ the partition and density function can be computed explicitly:

$$Z_1(\varphi) = -\frac{\log(1 - \varphi)}{\varphi} \implies Z(\varphi_c) = \infty$$

$$R_1(\varphi) = \frac{\varphi}{(\varphi - 1) \log(1 - \varphi)} - 1 \implies \rho_c = \infty.$$
An Example Exhibiting Condensation: The Evans Model

For $b = 1$ the partition and density function can be computed explicitly:

$$Z_1(\varphi) = -\frac{\log(1 - \varphi)}{\varphi} \implies Z(\varphi_c) = \infty$$

$$R_1(\varphi) = \frac{\varphi}{(\varphi - 1) \log(1 - \varphi)} - 1 \implies \rho_c = \infty.$$ 

For $1 < b < 2$ we have

$$Z_b(\varphi) \approx \frac{b}{b - 1} + \Gamma(1+b)\Gamma(1-b)(1-\varphi)^{b-1} \implies Z(\varphi_c) = \frac{b}{b - 1} < \infty,$$

$$R_b(\varphi) = (b - 1)\Gamma(b)\Gamma(2-b)\varphi(1-\varphi)^{b-2} \implies \rho_c = \infty,$$

and for $b = 2$ the qualitative picture is the same, $Z_2(\varphi_c) = \frac{b}{b - 1}$, $R_2(\varphi) = \infty.$
An Example Exhibiting Condensation: The Evans Model

For $b > 2$, also the first moment of the grand canonical distribution $\nu_{\varphi_c}^1$ is finite as $\varphi \to 0$, thus leading to a finite critical density $\rho_c < \infty$:

\[
Z_b(\varphi) \approx \frac{b}{b-1} - \frac{b}{(b-1)(b-2)}(1 - \varphi) \quad \Rightarrow \quad Z(\varphi_c) = \frac{b}{b-1},
\]

\[
R(\varphi) \approx \frac{1}{b-2} + (b-1)\Gamma(b)\Gamma(2-b)\varphi(1-\varphi)^{b-2} \quad \Rightarrow \quad \rho_c = \frac{1}{b-2}.
\]
An Example Exhibiting Condensation: The Evans Model

For $b > 2$, also the first moment of the grand canonical distribution $\nu_{\phi_c}^1$ is finite as $\phi \rightarrow 0$, thus leading to a finite critical density $\rho_c < \infty$:

$$Z_b(\phi) \approx \frac{b}{b-1} - \frac{b}{(b-1)(b-2)} (1 - \phi) \quad \Rightarrow \quad Z(\phi_c) = \frac{b}{b-1},$$

$$R(\phi) \approx \frac{1}{b-2} + (b-1)\Gamma(b)\Gamma(2-b)\phi(1-\phi)^{b-2} \quad \Rightarrow \quad \rho_c = \frac{1}{b-2}.$$

For $b > 1$ the distribution $\bar{\nu}_{\phi_c}^1 = \nu_{\rho_c}^1 \in \mathbb{P}\mathbb{Z}_+$ is defined. By Stirling’s formula, its tails behave for large $k$ as

$$\nu_{\rho_c}^1(k) = \frac{1}{Z_b(\phi_c)g_b!(k)} \approx \Gamma(b)(b-1)k^{-b}.$$
An Example Exhibiting Condensation: The Evans Model

For $b > 2$, also the first moment of the grand canonical distribution $\nu_{\varphi_c}^1$ is finite as $\varphi \to 0$, thus leading to a finite critical density $\rho_c < \infty$:

$$Z_b(\varphi) \equiv \frac{b}{b-1} - \frac{b}{(b-1)(b-2)}(1 - \varphi) \quad \Rightarrow \quad Z(\varphi_c) = \frac{b}{b-1},$$

$$R(\varphi) \equiv \frac{1}{b-2} + (b-1)\Gamma(b)\Gamma(2-b)\varphi(1-\varphi)^{b-2} \quad \Rightarrow \quad \rho_c = \frac{1}{b-2}.$$

For $b > 1$ the distribution $\tilde{\nu}_{\varphi_c}^1 = \nu_{\rho_c}^1 \in \mathbb{P}\mathbb{Z}_+$ is defined. By Stirling’s formula, its tails behave for large $k$ as

$$\nu_{\rho_c}^1(k) = \frac{1}{Z_b(\varphi_c)g_b!(k)} \approx \Gamma(b)(b-1)k^{-b}.$$

Such a distribution has moments of order up to $b - 1$, but not of order $b - 1$. 
An Example Exhibiting Condensation: The Evans Model

Proposition

Let $b > 2$ and let $\rho > \rho_c$. Then with respect to the canonical ensemble the normalized maximum $\frac{1}{N^d} \max_{x \in \mathbb{T}_N^d} \eta(x)$ converges in probability to $\rho - \rho_c$ as $N \to \infty$, that is

$$\lim_{N \to \infty} \nu_{N,[\rho N^d]} \left\{ \left| \frac{1}{N^d} \max_{x \in \mathbb{T}_N^d} \eta(x) - (\rho - \rho_c) \right| > \epsilon \right\} = 0, \quad \forall \epsilon > 0.$$
Hydrodynamic Limit of the ZRP

In the Hydrodynamic description of the symmetric ZRP we are interested in the description of the time evolution of the empirical density of particles of the diffusively time rescaled ZRP starting from some sequence of initial distributions $\mu^N_0$ on the configuration spaces $\mathcal{M}^d_N$ associated to some macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$.
Hydrodynamic Limit of the ZRP

In the Hydrodynamic description of the symmetric ZRP we are interested in the description of the time evolution of the empirical density of particles of the diffusively time rescaled ZRP starting from some sequence of initial distributions $\mu_0^N$ on the configuration spaces $\mathbb{M}_N^d$ associated to some macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$.

**Definition**

Let $\{\mu_0^N \in \mathbb{PM}_N^d\}_{N \in \mathbb{N}}$ be a sequence of distributions. We say that the sequence $\{\mu_0^N\}_{N \in \mathbb{N}}$ is associated to the macroscopic profile $\mu \in \mathcal{M}_+(\mathbb{T}^d)$ if

$$\lim_{N \to \infty} \mu_0^N \left\{ \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} G\left( \frac{x}{N} \right) \eta(x) - \int_{\mathbb{T}^d} G(u) d\mu(u) \right| > \epsilon \right\} = 0$$

for all $G \in C(\mathbb{T}^d)$ and all $\epsilon > 0$. 

A Hydrodynamic Limit for the Zero-Range Process in the present...
Hydrodynamic Limit of the ZRP

To describe the evolution of the density of particles we define the empirical mass distribution $\pi^N(\eta)$ of a configuration $\eta \in \mathcal{M}_N^d$ on the macroscopic torus $\mathbb{T}^d$ by

$$\pi^N(\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x \delta_{x/N} \in \mathcal{M}_+(\mathbb{T}^d) =: \mathcal{M}_+,$$

where $\delta_{x/N}$ is the Dirac delta function centered at $x/N$. This empirical distribution is used to construct the empirical process $\pi^N_t(\eta)$, which describes the evolution of the system.

Let $P_{\mu^N_0} \in \mathcal{P}(\mathcal{M}^d)$ denote the diffusively rescaled law of the empirical process $(\pi^N_t(\eta))$ starting from $\mu^N_0 \in \mathcal{P}(\mathcal{M}_N^d)$.

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
Hydrodynamic Limit of the ZRP

To describe the evolution of the density of particles we define the empirical mass distribution \( \pi^N(\eta) \) of a configuration \( \eta \in \mathcal{M}_d^N \) on the macroscopic torus \( \mathbb{T}^d \) by

\[
\pi^N(\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta_x \delta_{\frac{x}{N}} \in \mathcal{M}_+(\mathbb{T}^d) =: \mathcal{M}_+,
\]

and consider the image of the ZRP under the empirical mass distribution, i.e. the process

\[
\pi^N : \mathbb{R}_+ \times D(\mathbb{R}_+; \mathcal{M}_d^N) \to D(\mathbb{R}_+; \mathcal{M}_+)
\]

given by

\[
\pi^N_t(\eta) = \pi^N_{\eta_t}.
\]
Hydrodynamic Limit of the ZRP

To describe the evolution of the density of particles we define the empirical mass distribution $\pi^N(\eta)$ of a configuration $\eta \in \mathbb{M}^d_N$ on the macroscopic torus $\mathbb{T}^d$ by

$$\pi^N(\eta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta_x \delta_{\frac{x}{N}} \in \mathcal{M}_+(\mathbb{T}^d) =: \mathcal{M}_+,$$

and consider the image of the ZRP under the empirical mass distribution, i.e. the process

$$\pi^N : \mathbb{R}_+ \times D(\mathbb{R}_+; \mathbb{M}^d_N) \rightarrow D(\mathbb{R}_+; \mathcal{M}_+)$$

given by

$$\pi^N_t(\eta) = \pi^N_{\eta_t}.$$

Let $P^{\mu_0^N} \in \mathbb{PD}(\mathbb{R}_+, \mathcal{M}_+)$ denote the diffusively rescaled law of the empirical process $(\pi^N_t)$ starting from $\mu_0^N \in \mathbb{PM}^d_N$:

$$P^{\mu_0^N} = \left((\pi^N_{tN^2})_{t \geq 0}\right)^* \text{ [Law of ZRP starting from } \mu_0^N \text{].}$$
Definition

We say that a sequence $\{\mu_0^N \in \mathbb{P} M^d_N\}$ of initial distributions satisfies the $O(N^d)$-entropy assumption if there exists $a_* \in (0, \rho_c)$ such that

$$C_{a_*} := \limsup_{N \to \infty} \frac{1}{N^d} H(\mu_0^N | \nu_{a_*}^N) < \infty.$$
Definition

We say that a sequence \( \{\mu_0^N \in \mathcal{PM}_N^d\} \) of initial distributions satisfies the \( O(N^d) \)-entropy assumption if there exists \( a_* \in (0, \rho_c) \) such that

\[
C_{a_*} := \limsup_{N \to \infty} \frac{1}{Nd} H(\mu_0^N | \nu_{a_*}^N) < \infty.
\]

The hydrodynamic limit of the ZRP is known in various cases:
Introduction - Hydrodynamic Limits

The Zero Range Process

Hydrodynamic Limit of the ZRP

Definition

We say that a sequence \( \{ \mu_0^N \in \mathcal{P}_M \} \) of initial distributions satisfies the \( O(N^d) \)-entropy assumption if there exists \( a_* \in (0, \rho_c) \) such that

\[
C_{a_*} := \limsup_{N \to \infty} \frac{1}{N^d} H(\mu_0^N | \nu_{a_*}^N) < \infty.
\]

The hydrodynamic limit of the ZRP is known in various cases: For Lipschitz local jump rate functions with super-linear growth the Papanikolaou-Guo-Varadhan aproach proves
Proposition

Let \( \{\mu_0^N \in \mathbb{P}M_N^d\} \) be a sequence of initial distributions associated to a bounded profile \( \rho_0 \in B(\mathbb{T}^d) \) satisfying the \( O(N^d) \)-entropy assumption, such that

\[
\limsup_{N \to \infty} \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \int \eta(x)^2 \, d\mu_0^N < +\infty.
\]

Then the sequence of distributions \( P^{\mu_0^N} \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+) \) converges weakly as \( N \to \infty \) to the distribution

\[
\delta_{(\rho_t \, dm_{\mathbb{T}^d})_{t \geq 0}} \in \mathbb{P}D(\mathbb{R}_+, \mathcal{M}_+),
\]

where \( \rho \) is the unique weak solution in \( L^2 \) of the non-linear heat equation

\[
\partial_t \rho = \Delta_x \Phi(\rho).
\]
One deduces then that the initial association of $\mu_0^N$ to the profile $\rho_0$ is conserved, i.e. that

$$\lim_{N \to \infty} P^{\mu_0^N} \left\{ \pi \in D(\mathbb{R}_+; \mathcal{M}_+) : \left| \int G d\pi_t - \int_{\mathbb{T}^d} G \rho_t \right| > \epsilon \right\} = 0$$

for all $t \geq 0$, $G \in C(\mathbb{T}^d)$ and $\delta > 0$. 
One deduces then that the initial association of $\mu_0^N$ to the profile $\rho_0$ is conserved, i.e. that

$$\lim_{N \to \infty} P^{\mu_0^N} \left\{ \pi \in D(\mathbb{R}_+; \mathcal{M}_+) : \left| \int G d\pi_t - \int_{\mathbb{T}^d} G \rho_t \right| > \epsilon \right\} = 0$$

for all $t \geq 0$, $G \in C(\mathbb{T}^d)$ and $\delta > 0$.

A similar result is proved for the cases $0 \leq b \leq 2$ of Evans’ model for bounded initial profiles $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ by Yau’s Relative Entropy method.
The Case of Finite Critical Density

The hydrodynamic description of the ZRP via a closed PDE involving the density without assumptions that prohibit the emergence of a condensate is not known.
The Case of Finite Critical Density

The hydrodynamic description of the ZRP via a closed PDE involving the density without assumptions that prohibit the emergence of a condensate is not known. However, without any such assumptions we can give a non-closed hyrdodynamic equation via the macroscopic time evolution of two non-conserved quantities, the empirical diffusion rate

\[ \sigma^N : M^d_N \rightarrow \mathcal{M}_+ \]

given by

\[ \sigma^N := \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} g(\eta(x)) \delta_{\frac{x}{N}} \]

and the empirical current

\[ W^N : M^d_N \rightarrow \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d) \]

with \( j \)-th coordinate

\[ \langle e_j, W^N \rangle = \frac{1}{N^{d-1}} \sum_{x \in \mathbb{T}^d_N} W^N_{x,x+e_j} \delta_{\frac{x}{N}}, \quad W^N_{x,x+e_j} := g(\eta x) - g(\eta x + e_j). \]
We consider then the induced processes: The empirical diffusion rate process

$$\sigma^N(t, \eta) = \sigma^N_{\eta t}, \quad t \geq 0, \ \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N)$$

and the empirical current process

$$W^N(t, \eta) = W^N_{\eta t}, \quad t \geq 0, \ \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N).$$
The Case of Finite Critical Density

We consider then the induced processes: The empirical diffusion rate process

$$\sigma^N(t, \eta) = \sigma^N_{\eta_t}, \quad t \geq 0, \quad \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N)$$

and the empirical current process

$$W^N(t, \eta) = W^N_{\eta_t}, \quad t \geq 0, \quad \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N).$$

Since these processes describe non-conserved quantities the Skorohod topology on their path spaces is too strong to obtain the relative compactness of their laws.
The Case of Finite Critical Density

We consider then the induced processes: The empirical diffusion rate process

$$\sigma^N(t, \eta) = \sigma_{\eta t}^N, \quad t \geq 0, \quad \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N)$$

and the empirical current process

$$W^N(t, \eta) = W_{\eta t}^N, \quad t \geq 0, \quad \eta \in D(\mathbb{R}_+, \mathbb{M}^d_N).$$

Since these processes describe non-conserved quantities the Skorohod topology on their path spaces is too strong to obtain the relative compactness of their laws.

For this reason we consider these processes as random variables taking values in appropriate $L^\infty$ spaces of Banach-valued functions equipped with their $w^*$-topology.
The Case of Finite Critical Density

For the empirical diffusion rate process $\sigma^N$ we choose the state space

$$L^\infty_{w^*}([0, T]; M(\mathbb{T}^d)) \cong L^1([0, T]; C(\mathbb{T}^d))^*$$

and for the empirical current process $W^N$ we choose the space

$$L^\infty_{w^*}([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d)^*) \cong L^1([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d))^*,$$

both equipped with their $w^*$-topologies.
The Case of Finite Critical Density

For the empirical diffusion rate process $\sigma^N$ we choose the state space

$$L^\infty_{w^*}([0, T]; \mathcal{M}(\mathbb{T}^d)) \cong L^1([0, T]; C(\mathbb{T}^d))^*$$

and for the empirical current process $W^N$ we choose the space

$$L^\infty_{w^*}([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d)^*) \cong L^1([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d))^*,$$

both equipped with their $w^*$-topologies.

We then set

$$\Omega := D([0, T]; \mathcal{M}_N^d) \times L^\infty_{w^*}([0, T]; \mathcal{M}(\mathbb{T}^d)) \times L^\infty_{w^*}([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d)^*)$$

and consider the image $R^{\mu_0^N} \in \mathbb{P}\Omega$ of the law of the diffusively rescaled ZRP starting from $\mu_0^N$ via the triple $(\pi^N, \sigma^N, W^N)$. 
The Case of Finite Critical Density

More precisely if $\mathbb{P}^{\mu_0^N}$ denotes the distribution of the diffusively rescaled ZRP starting from $\mu_0^N \in \mathbb{P}M_N^d$, then

$$R^{\mu_0^N} := (\pi^N, \sigma^N, W^N) \ast \mathbb{P}^{\mu_0^N}.$$
More precisely if $\mathbb{P}^{\mu_0^N}$ denotes the distribution of the diffusively rescaled ZRP starting from $\mu_0^N \in \mathbb{P}M^d_N$, then

$$R^{\mu_0^N} := (\pi^N, \sigma^N, W^N)_* \mathbb{P}^{\mu_0^N}.$$ 

Our main result can then be stated as follows:
The Case of Finite Critical Density

Proposition

Suppose the local jump rate $g$ is bounded and let $\mu_0^N \mathbb{P} \mathcal{M}_N^d$ be a sequence of initial distributions associated to the macroscopic profile $\mu_0 \in \mathcal{M}_+$. The sequence $\{R^{\mu_0^N}\}_{N \in \mathbb{N}} \subseteq \mathbb{P} \Omega$ is sequentially compact in the weak topology of $\mathbb{P} \Omega$. Furthermore, any limit point $R^\infty$ of the sequence $\{R^{\mu_0^N}\}$ is concentrated on trajectories $(\pi, \sigma, W)$ such that:

(a) $\pi \in C(\mathbb{R}_+; \mathcal{M}_+)$ and $\pi_0 = \mu_0$.
(b) $\sigma_t \ll m_{\mathbb{T}^d}$, $\|\sigma_t\|_{L^\infty(\mathbb{T}^d)} \leq \|g\|_u$ a.s. for all $0 \leq t \leq T$.
(c) $W_t \in \mathcal{M}(\mathbb{T}^d; \mathbb{R}^d)$ and $W_t \ll m_{\mathbb{T}^d}$ for a.s. all $t \geq 0$, and
(d) The continuity equation

$$\partial_t \pi = \Delta_x \sigma = -\text{div}W_t$$

holds in the sense of distributions.
The Case of Finite Critical Density

**Sketch of the Proof:** The relative compactness of $R^{\mu_0^N}$ is the easiest part and amounts to the relative compactness of its three marginals.
The Case of Finite Critical Density

**Sketch of the Proof:** The relative compactness of $R_{\mu_0}^N$ is the easiest part and amounts to the relative compactness of its three marginals. Following the Guo-Papanikolaou-Varadhan approach we note that the real process

$$A_{t}^{N,G} := \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \left( \partial_s + N^2 L^N \right) \langle G_s, \pi^N \rangle (\eta_s^N) \, ds$$

defined on the filtered space $(D(\mathbb{R}_+, \mathcal{M}_N^d), (\mathcal{F}_t^N)_{t \geq 0}, \mathbb{P}^{\mu_0^N})$ is a martingale,
The Case of Finite Critical Density

**Sketch of the Proof:** The relative compactness of $R^\mu_0$ is the easiest part and amounts to the relative compactness of its three marginals. Following the Guo-Papanikolaou-Varadhan approach we note that the real process

$$A_t^{N,G} := \langle G_t, \pi^N_t \rangle - \langle G_0, \pi^N_0 \rangle - \int_0^t (\partial_s + N^2 L^N) \langle G_s, \pi^N \rangle (\eta^N_s) ds$$

defined on the filtered space $(D(\mathbb{R}_+, \mathbb{M}^d_N), (\mathcal{F}_t^N)_{t \geq 0}, \mathbb{P}^\mu_0)$ is a martingale, where $(\mathcal{F}_t^N)$ is the minimal right continuous filtration to which the ZRP is adapted.

Doobs inequality shows that the martingale $A_t^{N,G}$ is asymptotically negligible:

$$\lim_{N \to \infty} \left\{ \sup_{0 \leq t \leq T} |A_t^{N,G}| \geq \delta \right\} = 0, \quad \forall \delta > 0.$$
The Case of Finite Critical Density

Computing $L^N \langle G, \pi^N \rangle$ we get

$$L^N \langle G, \pi^N \rangle = \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}^d_N} \left[ G\left( \frac{x + e_j}{N} \right) - G\left( \frac{x}{N} \right) \right] W_{x,x+e_j}.$$
The Case of Finite Critical Density

Computing $L^N \langle G, \pi^N \rangle$ we get

$$L^N \langle G, \pi^N \rangle = \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \left[ G\left( \frac{x + e_j}{N} \right) - G\left( \frac{x}{N} \right) \right] W_{x,x+e_j}^N.$$ 

By using the Taylor expansion of $G$ around $\frac{x}{N}$ one gets for $G \in C^3(\mathbb{R}_+ \times \mathbb{T}^d)$ that there exists a constant $C = C(G, g) \geq 0$ such that

$$\left| \langle G_t, \pi^N_t \rangle - \langle G_0, \pi^N_0 \rangle - \int_0^t \left[ \langle \partial_s G_s, \pi^N_s \rangle + \langle \nabla G_s, W^N_s \rangle \right] ds - A^{N,G}_t \right| \leq \frac{C}{N} \int_0^t \langle \pi^N_s, 1 \rangle ds \quad (15)$$
The Case of Finite Critical Density

One more integration by parts allows us to write \( L^N \langle G_s, \pi^N \rangle \) as

\[
L^N \langle G, \pi^N \rangle = \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \left[ G\left( \frac{x + e_j}{N} \right) + G\left( \frac{x - e_j}{N} \right) - 2G\left( \frac{x}{N} \right) \right] g(\eta_x),
\]
One more integration by parts allows us to write $L^N \langle G_s, \pi^N \rangle$ as

$$L^N \langle G, \pi^N \rangle = \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}^d_N} \left[ \frac{G(x + e_j)}{N} + \frac{G(x - e_j)}{N} - 2G\left(\frac{x}{N}\right) \right] g(\eta_x),$$

and Taylor expansion again shows again that for $G \in C^3(\mathbb{R}_+ \times \mathbb{T}^d)$ there exists a constant $C = C(G, g) \geq 0$ such that

$$\left| \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \left[ \langle \partial_s G_s, \pi_s^N \rangle + \langle \Delta G_s, \sigma_s^N \rangle \right] ds - A_{t,G}^N \right| \leq \frac{C}{N} \int_0^t \langle \pi_s^N, 1 \rangle ds. \quad (16)$$
The Case of Finite Critical Density

Inequalities (15) and (16) together with the asymptotic negligibility of the martingale $A_{t}^{N,G}$ imply that

$$\lim_{N \to \infty} \mathbb{P}^{\mu^{N}_{0}} \left\{ \sup_{0 \leq t \leq T} \left| \langle G_{t}, \pi_{t}^{N} \rangle - \langle G_{0}, \pi_{0}^{N} \rangle \right| - \int_{0}^{t} \left[ \langle \partial_{s} G_{s}, \pi_{s}^{N} \rangle + \langle \nabla G_{s}, W_{s}^{N} \rangle \right] ds \geq \delta \right\} = 0$$
The Case of Finite Critical Density

Inequalities (15) and (16) together with the asymptotic negligibility of the martingale $A_{t}^{N,G}$ imply that

$$
\lim_{N \to \infty} \mathbb{P}^{\mu_{0}^{N}} \left\{ \begin{array}{l}
\sup_{0 \leq t \leq T} \left| \langle G_{t}, \pi_{t}^{N} \rangle - \langle G_{0}, \pi_{0}^{N} \rangle \right. \\
- \int_{0}^{t} \left[ \langle \partial_{s} G_{s}, \pi_{s}^{N} \rangle + \langle \nabla G_{s}, W_{s}^{N} \rangle \right] ds \left| \geq \delta \right. \end{array} \right\} = 0
$$

and

$$
\lim_{N \to \infty} \mathbb{P}^{\mu_{0}^{N}} \left\{ \begin{array}{l}
\sup_{0 \leq t \leq T} \left| \langle G_{t}, \pi_{t}^{N} \rangle - \langle G_{0}, \pi_{0}^{N} \rangle \right. \\
- \int_{0}^{t} \left[ \langle \partial_{s} G_{s}, \pi_{s}^{N} \rangle + \langle \Delta G_{s}, \sigma_{s}^{N} \rangle \right] ds \left| \geq \delta \right. \end{array} \right\} = 0.
$$

for all $G \in C_{c}^{3}(\mathbb{R}_{+} \times \mathbb{T}^{d})$. 
The Case of Finite Critical Density

After the required relative compactness for $R^{\mu_0 N}$ has been proven, these last two inequalities will show via the portmanteau theorem that any limit point $R^\infty$ of the sequence $R^{\mu_0 N}$ is concentrated on trajectories satisfying the continuity equation (14).
The Case of Finite Critical Density

After the required relative compactness for $R^{\mu_0^N}$ has been proven, these last two inequalities will show via the portmanteau theorem that any limit point $R^\infty$ of the sequence $R^{\mu_0^N}$ is concentrated on trajectories satisfying the continuity equation (14). The compactness of the first marginal of $R^{\mu_0^N}$ by the usual criterions for the Skorohod topology.
The Case of Finite Critical Density

After the required relative compactness for $R_{\mu_0}^N$ has been proven, these last two inequalities will show via the portmanteau theorem that any limit point $R^\infty$ of the sequence $R_{\mu_0}^N$ is concentrated on trajectories satisfying the continuity equation (14).

The compactness of the first marginal of $R_{\mu_0}^N$ by the usual criterions for the Skorohod topology. By the choice of the topology on the spaces that the second and third marginals ”live”, in order to prove their relative compactness, it suffices to prove that the random variables $\sigma^N$ and $W^N$ take values in some bounded subsets of the spaces

$$L^\infty_{w^*}([0, T]; \mathcal{M}(\mathbb{T}^d)) \quad \text{and} \quad L^\infty_{w^*}([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d)^*),$$

uniformly over $N \in \mathbb{N}$. 
The Case of Finite Critical Density

After the required relative compactness for $R^{\mu_0^N}$ has been proven, these last two inequalities will show via the portmanteau theorem that any limit point $R^\infty$ of the sequence $R^{\mu_0^N}$ is concentrated on trajectories satisfying the continuity equation (14).

The compactness of the first marginal of $R^{\mu_0^N}$ by the usual criterions for the Skorohod topology. By the choice of the topology on the spaces that the second and third marginals ”live”, in order to prove their relative compactness, it suffices to prove that the random variables $\sigma^N$ and $W^N$ take values in some bounded subsets of the spaces

$$L^\infty_w([0, T]; \mathcal{M}(\mathbb{T}^d)) \quad \text{and} \quad L^\infty_w([0, T]; C^1(\mathbb{T}^d; \mathbb{R}^d))^*,$$

uniformly over $N \in \mathbb{N}$.

But these estimates are trivial since the jump rate $g$ of the ZRP is assumed to be bounded.
It then remains to prove the regularity properties (a), (b) and (c) of the trajectories \((\pi, \sigma, W)\) supporting the limiting distributions of the sequence \(\{R_{\mu_0}^N\}\).
The Case of Finite Critical Density

It then remains to prove the regularity properties (a), (b) and (c) of the trajectories \((\pi, \sigma, W)\) supporting the limiting distributions of the sequence \(\{R_{\mu_0}^N\}\).

(a) The weak continuity of \(\pi\) is a consequence of the validity of the continuity equation, and \(\pi_0 = \mu_0\) by the portmanteau theorem.
The Case of Finite Critical Density

It then remains to prove the regularity properties (a), (b) and (c) of the trajectories \((\pi, \sigma, W)\) supporting the limiting distributions of the sequence \(\{R^\mu_0^N\}\).

(a) The weak continuity of \(\pi\) is a consequence of the validity of the continuity equation, and \(\pi_0 = \mu_0\) by the portmanteau theorem.
(b) The regularity properties of \(\sigma\) are also easy to derive.
The Case of Finite Critical Density

It then remains to prove the regularity properties (a), (b) and (c) of the trajectories \((\pi, \sigma, W)\) supporting the limiting distributions of the sequence \(\{R^\mu_0N\}\).

(a) The weak continuity of \(\pi\) is a consequence of the validity of the continuity equation, and \(\pi_0 = \mu_0\) by the portmanteu theorem.

(b) The regularity properties of \(\sigma\) are also easy to derive.

(c) For the regularity properties of \(W\), one proves first that for almost all trajectories \((\pi, \sigma, W)\) we have

\[
W_t = -\nabla \sigma_t, \quad \text{a.s. for all } 0 \leq t \leq T,
\]

and then uses the following regularity estimate for \(\sigma\):
The Case of Finite Critical Density

Proposition

Let \( \{\mu_0^N \in \mathbb{P}_1 \mathbb{M}_N^d\} \) be a sequence of initial distributions satisfying
the \( O(N^d) \)-entropy assumption and let \( R_{2}^{\mu_0^N} \) denote the marginal
of \( R_{2}^{\mu_0^N} \) on \( L^\infty([0,T];\mathcal{M}_+) \). Then \( R_{2}^{\mu_0^N} \) is concentrated on paths \( \sigma \) with the property that there exist \( L^2(I \times \mathbb{T}^d) \) functions denoted
by \( \partial_j \sigma \), \( j = 1, \ldots, d \), such that

\[
\int_0^T \int_{\mathbb{T}^d} \partial_j H_t(x) \sigma(t,x) dx dt = - \int_0^T \int_{\mathbb{T}^d} H_t(x) \partial_j \sigma(t,x) dx dt
\]

and

\[
\int_0^T \int_{\mathbb{T}^d} \left\| \nabla \sigma(t,x) \right\|^2 \sigma(t,x) dx dt < +\infty.
\]

In particular \( R_{2}^{\mu_0^N} \{ L^\infty([0,T];H^1(\mathbb{T}^d)) \} = 1 \).
The one block estimate

In order to close the equation $\partial_t \pi = \nabla \sigma$ one must manage to replace the trajectories $\sigma$, on which the second marginal $R_{\mu_0}^{\mu N}$ is concentrated, as functions of the mass distribution $\pi$. 
The one block estimate

In order to close the equation \( \partial_t \pi = \nabla \sigma \) one must manage to replace the trajectories \( \sigma \), on which the second marginal \( R_{2,0}^{\mu N} \) is concentrated, as functions of the mass distribution \( \pi \).

With this in mind we have extended the one-block estimate to the case of finite critical density \( \rho_c < \infty \) for bounded jump rate functions \( g \),
The one block estimate

In order to close the equation $\partial_t \pi = \nabla \sigma$ one must manage to replace the trajectories $\sigma$, on which the second marginal $R_{20}^{\mu_0 N}$ is concentrated, as functions of the mass distribution $\pi$.

With this in mind we have extended the one-block estimate to the case of finite critical density $\rho_c < \infty$ for bounded jump rate functions $g$, which allows the replacement of the process $\sigma^N$ by the process

$$\sigma^{\Phi, N, \ell}(t, \eta) := \frac{1}{Nd} \sum_{x \in \mathbb{T}^d_N} \Phi(\eta^\ell_t(x)) \delta_{x/N}$$

as $N \to \infty$ and then $\ell \to \infty$, 

The one block estimate

In order to close the equation $\partial_t \pi = \nabla \sigma$ one must manage to replace the trajectories $\sigma$, on which the second marginal $R_{t0}^{\mu_N}$ is concentrated, as functions of the mass distribution $\pi$.

With this in mind we have extended the one-block estimate to the case of finite critical density $\rho_c < \infty$ for bounded jump rate functions $g$, which allows the replacement of the process $\sigma^N$ by the process

$$\sigma^{\Phi,N,\ell}(t, \eta) := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Phi(\eta^\ell(x)) \delta_{\frac{x}{N}}$$

as $N \to \infty$ and then $\ell \to \infty$, where

$$\eta^\ell(x) = \frac{1}{(2\ell + 1)^d} \sum_{y : |y-x| \leq \ell} \eta(y).$$
The One Block Estimate

**Proposition**

Let $g$ be a bounded jump rate function and let $\{\mu_0^N \in \mathbb{P}_1 \mathcal{M}_N^d\}$ be a sequence of initial distributions associated to a macroscopic profile $\mu_0 \in \mathcal{M}_+(\mathbb{T}^d)$, satisfying the $O(N^d)$-entropy assumption. Then

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} \mathbb{P}_{\mu_0^N} \left\{ \left| \int_0^T \langle f_s, \sigma^N - \sigma_s^{\Phi,N,\ell} \rangle ds \right| > \epsilon \right\} = 0$$

for all $G \in L^1([0, T]; C(\mathbb{T}^d))$ and all $\epsilon > 0$. 
The One-Block estimate is one of the main ingredients in the Relative Entropy method of Yau. The other main ingredient is existence and appropriate regularity results for the hydrodynamic equation that is to be proved.
The One-Block estimate is one of the main ingredients in the Relative Entropy method of Yau. The other main ingredient is existence and appropriate regularity results for the hydrodynamic equation that is to be proved. An appropriate interpretation of the non-linear heat equation $\partial_t \mu = \Phi(\mu)$ that allows measure-valued solutions $\mu$ has been given rigorously in dimension $d = 1$ by Fornaro, Lisini, Savare and Toscani in 2010, that (almost) satisfies our needs.
The One-Block estimate is one of the main ingredients in the Relative Entropy method of Yau. The other main ingredient is existence and appropriate regularity results for the hydrodynamic equation that is to be proved. An appropriate interpretation of the non-linear heat equation \( \partial_t \mu = \Phi(\mu) \) that allows measure-valued solutions \( \mu \) has been given rigorously in dimension \( d = 1 \) by Fornaro, Lisini, Savare and Toscani in 2010, that (almost) satisfies our needs.

**Definition**

Let \( \mathcal{M}_{+,\rho_c}(M) \), \( M = \mathbb{R}^d \) or \( \mathbb{T}^d \) and denote the set of all finite measures of finite quadratic moment whose absolutely continuous part has a continuous representative \( \rho \in C(M; [0, \rho_c]) \) such that

\[
\mu^\perp(\{\rho < \rho_c\}) = 0, \quad L^1(M \setminus \{\rho < \rho_c\}) = 0 \quad \text{and} \quad \mu^{ac} = \rho dx.
\]
Let \( \Phi \in C^1(\mathbb{R}_+) \) be an increasing function with \( \Phi(0) \) and \( \lim_{\rho \to \infty} \Phi(\rho) =: \Phi(\infty) < \infty \) and let \( V : \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) driving potential such that

\[
\inf_{x \in \mathbb{R}} V''(x) > 0 \quad \text{and} \quad \lim \inf_{|x| \to \infty} \frac{V(x)}{|x|^2} \geq 0.
\]
Current Work

Let \( \Phi \in C^1(\mathbb{R}_+) \) be an increasing function with \( \Phi(0) \) and \( \lim_{\rho \to \infty} \Phi(\rho) =: \Phi(\infty) < \infty \) and let \( V : \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) driving potential such that

\[
\inf_{x \in \mathbb{R}} V''(x) > 0 \quad \text{and} \quad \lim_{|x| \to \infty} \frac{V(x)}{|x|^2} \geq 0.
\]

Definition

For \( \mu = \rho + \mu^\perp \in \mathcal{M}_{+, \rho_c} \) we set

\[
J(\mu) = \int \left| \frac{\nabla (\Phi(\rho))}{\rho} + V' \right|^2 \rho dx + \int |V'|^2 d\mu^\perp
\]

if \( \Phi(\rho) \in W_{\text{loc}}^{1,1} \) and \( J(\mu) = +\infty \) otherwise.
Current Work

Definition

A curve $\mu \in C([0, \infty); \mathcal{M}_{+,2}(\mathbb{R}))$ is a Wasserstein solution to the drift-diffusion equation

$$\partial_t \mu - \text{div}(\nabla \Phi(\rho) + V' \mu) = 0$$

if
A curve $\mu \in C([0, \infty); \mathcal{M}_{+2}(\mathbb{R}))$ is a Wasserstein solution to the drift-diffusion equation

$$\partial_t \mu - \text{div}(\nabla \Phi(\rho) + V' \mu) = 0$$

if

(a) $\mu_t = \rho_t + \mu_t^\perp \in \mathcal{M}_{+\rho_c}(\mathbb{R})$ a.s. $\forall \ t \geq 0$,
Definition

A curve $\mu \in C([0, \infty); \mathcal{M}_{+,2}(\mathbb{R}))$ is a Wasserstein solution to the drift-diffusion equation

$$\partial_t \mu - \text{div}(\nabla \Phi(\rho) + V' \mu) = 0$$

if

(a) $\mu_t = \rho_t + \mu_t^\perp \in \mathcal{M}_{+,\rho_c}(\mathbb{R})$ a.s. $\forall \ t \geq 0$,

(b) $\int_0^T J(\mu_t) dt < +\infty \ \forall \ T > 0$, 

A Hydrodynamic Limit for the Zero-Range Process in the presence of Condensation
A curve $\mu \in C([0, \infty); \mathcal{M}_{+,2}(\mathbb{R}))$ is a \textit{Wasserrstein solution to the drift-diffusion equation}

$$\partial_t \mu - \text{div}(\nabla \Phi(\rho) + V'\mu) = 0$$

if

(a) $\mu_t = \rho_t + \mu_t^\perp \in \mathcal{M}_{+,\rho_c}(\mathbb{R}) \text{ a.s. } \forall \ t \geq 0$,

(b) $\int_0^T J(\mu_t) dt < +\infty \ \forall \ T > 0$,

and for all $G \in C_c^\infty((0, \infty) \times \mathbb{R})$

(c) $\int_0^\infty \int_{\mathbb{R}} \left[ \partial_t G - (\partial_x G)V' \right] d\mu_t dt + \int_0^\infty \int_{\mathbb{R}} (\partial_{xx}^2 G) \Phi(\rho_t) dx dt = 0$. 
Fornaro, Lisini, Savare and Toscani prove existence and uniqueness of the solutions of the diffusion-drift equation in the class $\mathcal{M}_{+,\rho_c}(\mathbb{R})$ ($\rho_c = \infty$) by studying the Wasserstein gradient flow of the energy functional

$$
E(\mu) = \int_{\mathbb{R}} E_\Phi(\rho)dx + \int V(x)d\mu(x), \quad \mu = \rho dx + \mu^\perp,
$$

where

$$
E_\Phi(\rho) = -\Phi(\rho) - \rho \int_\rho^\infty \frac{\Phi'(r)}{r}dr.
$$
Fornaro, Lisini, Savare and Toscani prove existence and uniqueness of the solutions of the diffusion-drift equation in the class $\mathcal{M}_{+\rho_c}(\mathbb{R})$ ($\rho_c = \infty$) by studying the Wasserstein gradient flow of the energy functional

$$
\mathcal{E}(\mu) = \int_{\mathbb{R}} E_{\Phi}(\rho) dx + \int V(x) d\mu(x), \quad \mu = \rho dx + \mu^\perp,
$$

where

$$
E_{\Phi}(\rho) = -\Phi(\rho) - \rho \int_\rho^\infty \Phi'(r) \frac{dr}{r}.
$$

Similar uniqueness and existence results for the torus $\mathbb{T}$ in place of $\mathbb{R}$, and without a driving potential $V$ if proven, can be helpful in proving the the hydrodynamic limit of the ZRP.
As far as the Guo-Papanikolaou-Varadhan approach is concerned, if one manages to show that the solutions \( \mu_t = \rho_t + \mu^\perp \) of the continuity equation (14) are contained in the class \( \mathcal{M}_{+,\rho_c} \) and then achieves in replacing \( \sigma_t \) by \( \Phi(\rho_t) \) in the continuity equation, then due to the uniqueness of solutions, he has proved the hydrodynamic description of the ZRP.
Current Work

As far as the Guo-Papanikolaou-Varadhan approach is concerned, if one manages to show that the solutions $\mu_t = \rho_t + \mu_\perp$ of the continuity equation (14) are contained in the class $\mathcal{M}_{+,,\rho_c}$ and then achieves in replacing $\sigma_t$ by $\Phi(\rho_t)$ in the continuity equation, then due to the uniqueness of solutions, he has proved the hydrodynamic description of the ZRP.

In the diffusive time scale, the condensate is not expected to move. Assuming that the solution $\mu_t = \rho_t + \mu_\perp$ of the equation $\partial_t \mu_t = \nabla \Phi(\mu)$ with initial condition $\mu_0 = \rho_0 + a(0)\delta_0$ is of the form $\mu_t = \rho_t + a(t)\delta_0$ with $\rho_t \in C^{2+\epsilon}$ for some $\epsilon > 0$ we try to apply the relative entropy method of Yau to conclude the conservation of local equilibrium in $\mathbb{T} \setminus \{0\}$, according to the following.
Proposition

Let \( \{\mu_0^N\} \) be a sequence of probability measures on \( \mathbb{P} \mathbb{M}_N^d \) and let \( \mu_t^N, t \in [0, T] \) denote the law at time \( t \) of the diffusively rescaled ZRP starting from \( \mu_0^N \). We suppose that for all \( \epsilon > 0 \)

\[
H(\mu_t^N, \epsilon | \nu_t^N, \epsilon) = o(N^d)
\]

where \( \mu_t^N, \epsilon \) denotes the projection of \( \mu_t^N \) on \( \mathbb{Z}_+^{\{ |\cdot| > [N\epsilon]\}} \). Then \( \mu_t^N \) is a weak local equilibrium of profile \( \rho_t \) in \( \mathbb{T}^d \setminus \{0\} \), i.e. for all \( H \in C_c(\mathbb{T}^d \setminus \{0\}) \) and all bounded cylinder functions \( \Psi \),

\[
\lim_{N \to \infty} \int \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H\left( \frac{x}{N} \right) \tau_x \Psi - \int H(u) \tilde{\Psi}(\rho_t(u)) \, du \right| \, d\mu_t^N = 0, \]

where \( \tilde{\Psi}(a) = \int \Psi \, d\nu_{a \wedge \rho_c} \) for all \( a \geq 0 \).
Thank You for your Attention!