Well-Balanced Central Schemes for the SWEs

R. Touma

Lebanese American University

January 14, 2013

SciComp 2013
Central schemes for hyperbolic systems

Hyperbolic systems of conservation laws
Central Nessyahu and Tadmor Schemes
Unstaggered central scheme
Central schemes in two space dimensions

Well-Balanced Central Schemes for the SWE systems

Well-Balanced Central schemes for the 1D SWE systems
Well-Balanced Central schemes for the 2D SWE systems
Numerical Experiments

On going work
Hyperbolic conservation laws

Let $\vec{U} : \mathbb{R}^d \times [0, \infty[ \rightarrow \Omega \subseteq \mathbb{R}^p$ and $\vec{F}(\vec{U}) = (f_1, ..., f_d)$ be smooth functions from $\Omega$ to $\mathbb{R}^p$. Denote by $A_j(\vec{U})$ the Jacobian matrix of $f_j$. The system:

$$\vec{U}_t + \nabla \cdot \vec{F}(\vec{U}) = 0, \quad t > 0$$

is said to be hyperbolic if for all $\vec{U} \in \Omega$ and all $\omega = (\omega_1, ..., \omega_d) \in \mathbb{R}^d, |\omega| = 1$, the matrix

$$A(\vec{U}, \omega) = \sum_{j=1}^{d} \omega_j A_j$$

has $p$ real eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_p$ and $p$ linearly independent eigenvectors $r_1, ..., r_p$. 
Godunov-type methods for Hyperbolic problems

Consider the one-dimensional initial value problem

\[
\begin{cases}
  u_t + f(u)_x = 0, & t > 0; \\
  u(x, t = 0) = u_0(x).
\end{cases}
\]

- A piecewise constant numerical solution $u^n_i$ is evolved on the control cells $C_i = [x_{i-1/2}, x_{i+1/2}]
- The numerical solution is computed using the exact solution of the local Riemann problems arising at the cell interfaces (i.e., at points $x_{i\pm1/2}$), a time consuming process!!
- First-order accuracy both in time and space.
- The stability condition is: $\frac{\Delta t}{\Delta x} \max(\|\lambda_i\|)_{i=1,\ldots,p} \leq 1$ where the $\lambda_i$ are the eigenvalues of the jacobian matrix $A = \frac{\partial f}{\partial u}$.
The numerical solution has the following form:

\[ u_{i+1}^{n+1} = u_{i+1}^n - \frac{\Delta t}{\Delta x} \left[ f(\mathcal{R}(u_{i+1}^n, u_{i+2}^n, t^n)) - f(\mathcal{R}(u_i^n, u_{i+1}^n, t^n)) \right] \]
The staggered Lax-Friedrichs scheme avoids the resolution of the Riemann problems arising at the cell interfaces by evolving a piecewise constant numerical solution on an original grid and a staggered grid at successive time steps:

- $u^n_i$ is computed at time $t^n$ on $C_i = [x_{i-1/2}, x_{i+1/2}]$
- $u^{n+1}_{i+1/2}$ is computed at time $t^{n+1}$ on the staggered dual cells $D_{i+1/2} = [x_i, x_{i+1}]$
- No Riemann problem to be solved
- The stability condition is: $\frac{\Delta t}{\Delta x} \max(\lambda_i)_{i=1,..,p} \leq 0.5$
- First-order accuracy both in time and space.
The numerical solution is calculated as follows:

\[
 u_{i+1/2}^{n+1} = \frac{1}{2} (u_i^n + u_{i+1}^n) - \frac{\Delta t}{\Delta x} [f(u_{i+1}^n) - f(u_i^n)]
\]
To improve the accuracy of the Lax-Friedrichs scheme, Nessyahu and Tadmor proposed a method that evolves a piecewise linear numerical solution on two staggered grids:

- Using van Leer’s MUSCL limiter (to avoid spurious oscillations) the piecewise linear solution is defined by

\[ L_i(x, t^n) = u_i^n + (x - x_i) \frac{u_i^{lim}}{\Delta x} \]

where \( u_i^{lim}/\Delta x \) denotes a limited numerical derivative.

- The resolution of the Riemann problems at the cell interfaces is by-passed thanks to the use of the staggered cells.

- Second-order accurate in space (piecewise linear interpolants) and in time (midpoint rule for the flux integration in time).

- Same stability condition as in the Lax-Friedrichs scheme.
The numerical solution is computed at time $t^{n+1}$ as follows:

$$u_{i+1/2}^{n+1} = \frac{1}{2}(u_i^n + u_{i+1}^n) + \frac{1}{8}(u_{i,lim}^n - u_{i+1,lim}^n) - \frac{\Delta t}{\Delta x}(f_{i+1/2}^{n+1/2} - f_{i}^{n+1/2})$$

while the solution at time $t^{n+2}$ is calculated as follows:

$$u_i^{n+2} = \frac{1}{2}(u_{i-1/2}^{n+1} + u_{i+1/2}^{n+1}) + \frac{1}{8}(u_{i-1/2}^{n+1,lim} - u_{i+1/2}^{n+1,lim}) - \frac{\Delta t}{\Delta x}(f_{i+1/2}^{n+3/2} - f_{i-1/2}^{n+3/2})$$

Figure: Geometry of the 1D Nessyahu-Tadmor scheme
In many cases, the numerical solutions obtained using a numerical base scheme requires additional treatment in order to satisfy certain physical conditions (Ex. SWE with a variable bottom topography, MHD, SMHD, and others).

When we apply the ”NT” scheme, the numerical solution alternates between 2 grids. The treatment of $u^n_i$ requires the solution at previous time $u^{n-1}_{i+1/2}$ and thus a synchronization problem appears.
The unstaggered central scheme (UCS) is an adaptation of the Nessyahu and Tadmor scheme that:

1. Evolves a piecewise linear numerical solution on ”one grid”
2. Avoids the resolution of the Riemann problems arising at the cell interfaces by implicitly using a ”ghost” staggered grid

**Figure:** Geometry of the 1D unstaggered central scheme
Given the solution $u_i^n$, we apply the first step of the "NT" scheme and obtain the solution $u_{i+1/2}^G$ on the staggered ghost grid at time $t^{n+1}$.

We use a piecewise linear reconstruction of the obtained solution values at the nodes $x_{i+1/2}$ to define the solution $u_i^{n+1}$ at the centers $x_i$ of the cells $C_i$ of the computational domain as follows:

$$u_i^{n+1} = \frac{1}{2}(u_{i-1/2}^G + u_{i+1/2}^G) + \frac{1}{8}(u_{i-1/2}^{G,\text{lim}} - u_{i+1/2}^{G,\text{lim}}).$$

The resulting adaptation of the "NT" scheme is an unstaggered central scheme (UCS) that is second-order accurate, non-oscillatory, with a stability condition CFL number equals to 0.5.
Central schemes in two space dimensions

Central schemes are used to solve two-dimensional hyperbolic systems of the form

\[ u_t + f(u)_x + g(u)_y = 0 \]

- Two staggered grids are required in order to avoid the resolution of the Riemann problems at the cell interfaces.
- Piecewise linear numerical solution defined at the cell centers is evolved.
- Gradient limiting process is needed to avoid spurious oscillations in the neighborhood of jumps and discontinuities.
- Second-order accurate (both in both in space and time) provided an appropriate CFL condition is maintained.
Figure: Geometry of the 2D staggered central scheme
Knowing the solution \( u_{ij}^n \) at time \( t = t^n \) at the center of the cell \( C_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}] \), the solution at the next time step is obtained using a careful discretization of the following integrals

\[
\begin{align*}
  u_{i+1/2,j+1/2}^{n+1} &= \frac{1}{\Delta x \Delta y} \left[ \int_{D_{i+1/2,j+1/2}} u(x, y, t^n) \, dx \, dy 
  - \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+1/2,j+1/2}} (f(u), g(u)) \cdot \vec{n} \, dAdt \right]
\end{align*}
\]

- \( D_{i+1/2,j+1/2} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) is a staggered dual cell.
- \( \vec{n} \) is the unit outward normal vector to the boundary \( \partial D_{i+1/2,j+1/2} \).
- \( u_{ij}^{n+2} \) will be computed at the centers of the \( C_{ij} \) cells using a similar formula at time \( t = t^{n+2} \).
Unstaggered central schemes, 2D

In order to avoid staggering, one can project the numerical solution \( u_{i+1/2,j+1/2}^{n+1} \) obtained at time \( t = t^{n+1} \) back onto the original grid and obtain \( u_{ij}^{n+1} \).

First we define the linear interpolant \( u_{i+1/2+\alpha,j+1/2+\beta}^L \) at the nodes \((x_{i+1/2}, y_{j+1/2})\) using a first-order Taylor expansion as follows

\[
\begin{align*}
    u_{i+1/2+\alpha,j+1/2+\beta}^L &= u_{i+1/2,j+1/2}^{n+1} + \alpha u_{i+1/2,j+1/2; x}^{n+1, lim} + \beta u_{i+1/2,j+1/2; y}^{n+1, lim} \\
\end{align*}
\]

where \( 0 \leq \alpha, \beta \leq \frac{1}{2} \).

Next we calculate \( u_{ij}^{n+1} \)

\[
\begin{align*}
    u_{i,j}^{n+1} &= \frac{1}{4} \left( u_{i-1/2+\alpha,j-1/2+\beta}^L + u_{i+1/2-\alpha,j-1/2+\beta}^L \\
    &\quad + u_{i+1/2-\alpha,j+1/2-\beta}^L + u_{i-1/2+\alpha,j+1/2-\beta}^L \right)
\end{align*}
\]
In this work we set $\alpha = \beta = 1/4$.

The unstaggered central scheme is second-order accurate, and shares the same stability conditions as the two-dimensional staggered central scheme.

Figure: Geometry of the 2D unstaggered central scheme
Central schemes on unstructured schemes

Figure: Geometry of central schemes on unstructured triangular grids
The shallow water equations represent a good mathematical model for the hydrodynamics of coastal oceans, simulation of flows in channels and rivers, and are used to model propagating tsunamis and inundation waves. Written in their conservation form, the two-dimensional shallow water equations (SWE) are:

$$\frac{\partial}{\partial t} \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} hu^2 + \frac{1}{2}gh^2 \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} hv \\ huv \\ huv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\partial z}{\partial x} \\ -gh\frac{\partial z}{\partial y} \end{pmatrix}$$

$h(x, y, t)$ is the water height, $(u(x, y, t), v(x, y, t))$ is the velocity field, and $z = b(x, y)$ is the bottom topography function.
Some properties of the SWE system:

- In the case of a flat bottom topography the source term vanishes and the resulting SWE system becomes a homogeneous hyperbolic system with real eigenvalues.
- The SWE system admits steady state solutions like the lake at rest case or a case with steady flow.
- C-property: In the case of a quiescent flow \((u, v) = (0, 0)\) and \(h + z = \text{constant}\) should be exactly maintained. (Lake at rest)

**Goal:** Adapt the unstaggered central scheme to the case of SWE systems and ensure the steady state and/or the C-property requirements.
The one-dimensional SWE system is

\[ \frac{\partial}{\partial t} \begin{pmatrix} h \\ hv \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \end{pmatrix} \].

At an equilibrium state (lake at rest) we have \( v \equiv 0 \) and \( H = h + z \equiv \text{constant} \); the SWE system reduces to the hydrostatic balance \( (gh^2/2)_x + ghz_x = 0 \) which can be rewritten as \( g(h + z)_x = gH_x = 0 \). Usually numerical schemes do not preserve the hydrostatic balance at the discrete level and generate spurious oscillations and numerical storms.
The one-dimensional SWE system has the form

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= S(u, x), \quad t > 0, x \in \Omega \subset \mathbb{R} \\
u(x, 0) &= u_0(x)
\end{align*}
\] (1)

- The computational domain \( \Omega \subset \mathbb{R} \) is discretized using control cells \( C_i = [x_{i-1/2}, x_{i+1/2}] \) centered at the nodes \( x_i \).
- The dual cells \( D_{i+1/2} = [x_i, x_{i+1}] \) are centered at \( x_{i+1/2} \).
- Assuming that the numerical solution \( u_i^n \) at time \( t^n \) is known, we construct the piecewise linear interpolant \( L_i(x, t^n) \) on the cells \( C_i \); we have:

\[
u(x, t^n) \approx L_i(x, t^n) \text{ for } x \in C_i \text{ and } u_i^n = \int_{x_{i-1/2}}^{x_{i+1/2}} L_i(x, t^n) \, dx.
\]

We integrate the balance law over the domain \( D_{i+1/2} \times [t^n, t^{n+1}] \),

=R. Touma

Well-Balanced Central Schemes for the SWEs
we apply Green's theorem, and then we solve for $u_{i+1/2}^{n+1}$, we obtain:

$$
u_{i+1/2}^{n+1} = u_{i+1/2}^n - \frac{\Delta t}{\Delta x} \left[ f(u_{i+1/2}^{n+1}) - f(u_{i+1/2}^n) \right]$$

$$+ \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(u(x, t), x) \, dx \, dt$$

$u_{i+1/2}^n$ is the projected solution at time $t^n$ on $D_{i+1/2}$ and is obtained using a spatial Taylor expansion:

$$u_{i+1/2}^n = \frac{1}{2} \left( u_i^n + u_{i+1}^n \right) + \frac{\Delta x}{8} \left( (u_i^n)' - (u_{i+1}^n)' \right)$$

the required predicted values at time $t^{n+1/2}$ are obtained using a first-order Taylor expansion in time and the balance law as follows

$$u_{i+1/2}^{n+1} = u_i^n + \frac{\Delta t}{2\Delta x} \left( -F_i' + S_i^n \cdot \Delta x \right)$$

with $S_i^n$ discretizes the source term.
The integral of the source term is approximated using centered differences and the midpoint quadrature rule as follows:

\[
\int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(u(x, t), x)) \, dx \, dt \approx \Delta t \, \Delta x \, S(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}}) = \\
\Delta t \, \Delta x \begin{pmatrix} 0, g \frac{h_i^{n+\frac{1}{2}} + h_{i+1}^{n+\frac{1}{2}}}{2} (-\frac{z_{i+1} - z_i}{\Delta x}) \end{pmatrix}^T
\]

Finally \( u_i^{n+1} \) is projected back onto the original grid using linear interpolants as follows:

\[
u_i^{n+1} = \frac{1}{2} \left( u_i^{-\frac{1}{2}} + u_i^{n+1} \right) + \frac{\Delta x}{8} \left( (u_i^{-\frac{1}{2}})' - (u_i^{n+1})' \right)
\]
In order to ensure well-balancing $S^n_i$ should be discretized according to the discretization of the flux integral $S^n_i = -gh^n_i \left( S^n_{i,L} + S^n_{i,R} + S^n_{i,C} \right)$, with

$$
S^n_{i,L} = \frac{s_i^2(1 - s_i)(2 - s_i)}{6} \begin{pmatrix} 0 \\ \theta \frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix},
$$
$$
S^n_{i,R} = \frac{s_i^2(1 + s_i)(2 - s_i)}{2} \begin{pmatrix} 0 \\ -\theta \frac{z_{i+1} - z_i}{\Delta x} \end{pmatrix},
$$
$$
S^n_{i,C} = \frac{s_i(s_i^2 - 1)}{6} \begin{pmatrix} 0 \\ \theta \frac{z_{i+1} - z_{i-1}}{2\Delta x} \end{pmatrix}.
$$
The parameter $s_i \in \{-1, 0, 1, 2\}$ forces the discretization of $dz/dx$ in $S(u, x)$ according to the discretization of $\partial h/\partial x$ prescribed by the MC-$\theta$ limiter as follows:

1. $s_i = -1$ if $h'_i = \theta(h^n_i - h^n_{i-1})/\Delta x$
2. $s_i = 0$ if $h'_i = 0$,
3. $s_i = 1$ if $h'_i = \theta(h^n_{i+1} - h^n_i)/\Delta x$
4. $s_i = 2$ if $h'_i = (h^n_{i+1} - h^n_{i-1})/2\Delta x$

If the quiescent flow conditions ($h(x, 0) + z(x) = \text{constant}$ and $v(x, 0) = 0$ for all $x \in \Omega$) are satisfied at the discrete level at time $t^n$ by $u^n_i$, then one can show that the numerical solution $u^{n+1}_{i+1/2}$ satisfies the equations $u^{n+1/2}_i = u^n_i$, and $u^{n+1}_{i+1/2} = u^n_{i+1/2}$.

Finally the back-projection step $u^{n+1}_{i+1/2} \rightarrow u^{n+1}_i$ should be properly performed in order to ensure that the quiescent flow constraint is satisfied $u^{n+1}_i$, i.e., $h^{n+1}_i + z_i = h^n_i + z_i = \text{const}$. 
Central schemes fail to satisfy the C-property because of the linearity of the water bed and water level function on the $C_i$'s.

The surface gradient method is a solution: Calculate the numerical derivative of the water height component in terms of the water level function $H(x, t) = z(x) + h(x, t)$

We limit the numerical derivatives of $H_i = h_i + z_i$ and then calculate $h'_i$ as follows:

$$h'_i = H'_i - \frac{1}{\Delta x} (z_{i+1/2} - z_{i-1/2}).$$

This reformulation of $h'_i$ is necessary to calculate the first component of the projected solution on the staggered grid.

A similar reformulation of $h'_{i+1/2}$ is necessary for the back projection step on the cells $C_i$ using is obtained using the formula

$$h'_{i+1/2} = \tilde{H}'_{i+1/2} - \frac{1}{\Delta x} (z_{i+1} - z_i).$$
Well-Balanced Central Schemes for the SWE systems

Figure: Water height of the lake at rest problem.
Outline
Central schemes for hyperbolic systems

Well-Balanced Central Schemes for the SWE systems

On going work

Well-Balanced Central schemes for the 1D SWE systems

Well-Balanced Central schemes for the 2D SWE systems

Numerical Experiments

Figure: Dam break problem.

Movie: Well-balanced scheme
Well-balanced central schemes in two space dimensions

Here we consider the 2D SWE system of balance laws

$$u_t + f(u)_x + g(u)_y = S(u, x, y)$$

Assuming that $u^n_{i,j}$ is known, we integrate the balance law on $D_{i+1/2,j+1/2} \times [t^n, t^{n+1}]$, we obtain:

$$u^{n+1}_{i+1/2,j+1/2} = u^n_{i+1/2,j+1/2}$$

$$- \frac{1}{(\Delta x)^2} \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+1/2,j+1/2}} (f(u), g(u)) \cdot \vec{n} \ dA dt$$

$$+ \frac{1}{(\Delta x)^2} \int_{t^n}^{t^{n+1}} \int_{D_{i+1/2,j+1/2}} S(u, x, y) \ dx dy dt$$

(2)

The above integrals will be evaluated with second-order accuracy.
Transformations from the original to the staggered cells are obtained using linear interpolants and gradients limiting as follows

\[
u_{i+\frac{1}{2},j+\frac{1}{2}}^n = \frac{1}{4} \left( u_{ij}^n + u_{i+1j}^n + u_{ij+1}^n + u_{i+1j+1}^n \right) + \frac{1}{16} \left( \delta_{ij}^n + \delta_{ij+1}^n - \delta_{i+1j}^n - \delta_{i+1j+1}^n \right) + \frac{1}{16} \left( \sigma_{ij}^n - \sigma_{ij+1}^n + \sigma_{i+1j}^n - \sigma_{i+1j+1}^n \right)
\]

with \((\nabla u)_ij^n \approx (\delta_{ij}^n, \sigma_{ij}^n)\) is a limited numerical gradient of \(u_{ij}^n\).

The flux integrals are calculated using the midpoint rule at intermediate time steps using \(u_{ij}^{n+1/2}\)

\[
u_{ij}^{n+1/2} = u_{ij}^n + \frac{\Delta t}{2} \left( -\frac{f_{ij}'}{\Delta x} - \frac{g_{ij}'}{\Delta y} + S_{ij}^n \right)
\]

\(S_{ij}^n\) approximates the source term.
The solution at time $t^{n+1}$ will be calculated as follows:

$$
\begin{align*}
    u_{i+\frac{1}{2}j+\frac{1}{2}}^{n+1} &= u_{i+\frac{1}{2}j+\frac{1}{2}}^n - \frac{\Delta t}{2\Delta x} \left[ f_{i+1,j}^{n+\frac{1}{2}} + f_{i+1,j+1}^{n+\frac{1}{2}} - f_{i,j}^{n+\frac{1}{2}} - f_{i,j+1}^{n+\frac{1}{2}} \right] \\
    &\quad - \frac{\Delta t}{2\Delta y} \left[ g_{i,j+1}^{n+\frac{1}{2}} + g_{i+1,j+1}^{n+\frac{1}{2}} - g_{i,j}^{n+\frac{1}{2}} - g_{i+1,j}^{n+\frac{1}{2}} \right] \\
    &\quad + \Delta t \cdot S(u_{ij}^{n+\frac{1}{2}}, u_{i+1,j}^{n+\frac{1}{2}}, u_{i,j+1}^{n+\frac{1}{2}}, u_{i+1,j+1}^{n+\frac{1}{2}})
\end{align*}
$$

$\triangleright$ $S(u_{ij}^{n+\frac{1}{2}}, u_{i+1,j}^{n+\frac{1}{2}}, u_{i,j+1}^{n+\frac{1}{2}}, u_{i+1,j+1}^{n+\frac{1}{2}})$ approximates the integral of the source term with second-order of accuracy.

$\triangleright$ $u_{ij}^{n+1}$ is obtained by projecting $u_{i+\frac{1}{2}j+\frac{1}{2}}^{n+1}$ back onto the original grid.
In the case of a lake at rest \((h + z = \text{const.}, (u, v) = (0, 0))\), the SWE system reduces to

\[
\frac{\partial}{\partial t} \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \\ 0 \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\partial z}{\partial x} \\ -gh\frac{\partial z}{\partial y} \end{pmatrix}
\]

The balancing between the flux divergence and the source term should be exactly maintained at the discrete level.

One should carefully define

- \(S^n_{ij}\) in the predictor step
- \(S(u^n_{ij}, u_{i+1,j}^n, u_{i,j+1}^n, u_{i+1,j+1}^n)\) in the corrector step
In the context of the MC-θ limiter, we define the sensor functions $s_1 = 0$, $s_2$, and $s_3$ to direct the discretization of $\frac{\partial z}{\partial x}|_{ij}$ and $\frac{\partial z}{\partial y}|_{ij}$ according to the discretizations of $h_x|_n^i,j$ and $h_y|_n^i,j$, respectively.

\[ s_2 = \begin{cases} 
-1 & \text{if } \partial_x h|_{ij} \approx \theta \left( h_{i,j}^n - h_{i-1,j}^n \right)/\Delta x, \\
0 & \text{if } \partial_x h|_{ij} \approx 0, \\
1 & \text{if } \partial_x h|_{ij} \approx \theta \left( h_{i+1,j}^n - h_{i,j}^n \right)/\Delta x, \\
2 & \text{if } \partial_x h|_{ij} \approx \left( h_{i+1,j}^n - h_{i-1,j}^n \right)/(2\Delta x).
\end{cases} \]

and \[ s_3 = \begin{cases} 
-1 & \text{if } \partial_y h|_{ij} \approx \theta \left( h_{i,j}^n - h_{i,j-1}^n \right)/\Delta y, \\
0 & \text{if } \partial_y h|_{ij} \approx 0, \\
1 & \text{if } \partial_y h|_{ij} \approx \theta \left( h_{i,j+1}^n - h_{i,j}^n \right)/\Delta y, \\
2 & \text{if } \partial_y h|_{ij} \approx \left( h_{i,j+1}^n - h_{i,j-1}^n \right)/(2\Delta y).
\end{cases} \]
Now the $S^n_{ij} = (S_1 = 0, S_2, S_3)$ term in the predictor step is defined componentwise as follows:

$$S_2 = \begin{cases} 
-g h^n_{i,j} \frac{z_{i,j} - z_{i-1,j}}{\Delta x} & \text{if } s_2 = -1, \\
0 & \text{if } s_2 = 0, \\
-g h^n_{i,j} \frac{z_{i+1,j} - z_{i,j}}{\Delta x} & \text{if } s_2 = 1, \\
-g h^n_{i,j} \frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x} & \text{if } s_2 = 2.
\end{cases}$$

and

$$S_3 = \begin{cases} 
-g h^n_{i,j} \frac{z_{i,j} - z_{i,j-1}}{\Delta y} & \text{if } s_3 = -1, \\
0 & \text{if } s_3 = 0, \\
g h^n_{i,j} \frac{z_{i,j+1} - z_{i,j}}{\Delta y} & \text{if } s_3 = 1, \\
-g h^n_{i,j} \frac{z_{i,j+1} - z_{i,j-1}}{2\Delta y} & \text{if } s_3 = 2.
\end{cases}$$
As regarding the integral of the source term (in the SWE case), we propose the second-order accurate approximation:

\[ \int_{t}^{t+1} \int_{D_{i+1/2,j+1/2}} (0, -ghz_x, -ghz_y) \, dt \, dA \approx \]

\[ \Delta t \, S(u_{ij}^{n+1/2}, u_{i+1,j}^{n+1/2}, u_{i,j+1}^{n+1/2}, u_{i+1,j+1}^{n+1/2}) = \]

\[ \frac{\Delta t}{2} \left( \begin{array}{cccc}
   -g & h_{i+1,j}^{n+1/2} - h_{i,j}^{n+1/2} & 0 & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} \\
   -g & h_{i,j+1}^{n+1/2} - h_{i,j}^{n+1/2} & 0 & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} \\
   -g & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} & 0 & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} \\
   -g & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} & 0 & h_{i+1,j+1}^{n+1/2} - h_{i,j+1}^{n+1/2} \\
\end{array} \right) \]
With the proposed formulas for \( S(u_{ij}^{n+\frac{1}{2}}, u_{i+1,j}^{n+\frac{1}{2}}, u_{i,j+1}^{n+\frac{1}{2}}, u_{i+1,j+1}^{n+\frac{1}{2}}) \) and \( S^n_{ij} \), tedious calculations show that (in a quiescent flow context)

\[ u_{ij}^{n+1/2} = u_{ij}^n \]
\[ u_{i+1/2,j+1/2}^{n+1} = u_{i+1/2,j+1/2}^n \]

We conclude that if the initial discretization satisfies the quiescent flow condition, this condition will be preserved if the procedures of passing from the original grid to the staggered one and vice versa are defined in an appropriate way.
The projection formulas are based on piecewise linear interpolants:

\[
    u^n_{i+\frac{1}{2}j+\frac{1}{2}} = \frac{1}{4} \left( u^n_{ij} + u^n_{i+1j} + u^n_{ij+1} + u^n_{i+1j+1} \right) \\
    + \frac{1}{16} \left[ (\delta^n_{ij} + \delta^n_{ij+1} - \delta^n_{i+1j} - \delta^n_{i+1j+1}) + (\sigma^n_{ij} - \sigma^n_{ij+1} + \sigma^n_{i+1j} - \sigma^n_{i+1j+1}) \right]
\]

and

\[
    u^{n+1}_{ij} = \frac{1}{4} \left( u^{n+1}_{i-\frac{1}{2}j-\frac{1}{2}} + u^{n+1}_{i+\frac{1}{2}j-\frac{1}{2}} + u^{n+1}_{i-\frac{1}{2}j+\frac{1}{2}} + u^{n+1}_{i+\frac{1}{2}j+\frac{1}{2}} \right) \\
    + \frac{1}{16} \left( \delta^{n+1}_{i-\frac{1}{2}j-\frac{1}{2}} + \delta^{n+1}_{i-\frac{1}{2}j+\frac{1}{2}} - \delta^{n+1}_{i+\frac{1}{2}j-\frac{1}{2}} - \delta^{n+1}_{i+\frac{1}{2}j+\frac{1}{2}} \right) \\
    + \frac{1}{16} \left( \sigma^{n+1}_{i-\frac{1}{2}j-\frac{1}{2}} + \sigma^{n+1}_{i-\frac{1}{2}j+\frac{1}{2}} - \sigma^{n+1}_{i+\frac{1}{2}j-\frac{1}{2}} - \sigma^{n+1}_{i+\frac{1}{2}j+\frac{1}{2}} \right)
\]

fail to fulfill the quiescent flow condition, and the numerical central scheme needs some modifications.
Since in the case of a quiescent flow $hu = hv = 0$, the modifications of the numerical scheme will be done just for the first component $h$ of the vector solution $u$.

\[
\frac{\partial}{\partial t} \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} 0 \\ \frac{1}{2} gh^2 \\ 0 \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh \frac{\partial z}{\partial x} \\ -gh \frac{\partial z}{\partial y} \end{pmatrix}
\]

The linearization of the water height $h$ will be made indirectly by first linearizing the water level $H = h + z$ and then by using the relation $h(x, y) = H(x, y) - z(x, y)$. 
Surface gradient method

- Values of the bottom topography are required at the corners of the cells $C_{ij}$.

- Follow the balanced central scheme previously derived

$$u_{i+rac{1}{2},j+rac{1}{2}}^{n+1} = u_{i+rac{1}{2},j+rac{1}{2}}^{n} - \frac{\Delta t}{2\Delta x} \left[ f_{i+1,j}^{n+\frac{1}{2}} + f_{i+1,j+1}^{n+\frac{1}{2}} - f_{i,j}^{n+\frac{1}{2}} - f_{i,j+1}^{n+\frac{1}{2}} \right]$$

$$- \frac{\Delta t}{2\Delta y} \left[ g_{i,j+1}^{n+\frac{1}{2}} + g_{i+1,j+1}^{n+\frac{1}{2}} - g_{i,j}^{n+\frac{1}{2}} - g_{i+1,j}^{n+\frac{1}{2}} \right]$$

$$+ \Delta t \cdot S \left( u_{i,j}^{n+\frac{1}{2}}, u_{i+1,j}^{n+\frac{1}{2}}, u_{i,j+1}^{n+\frac{1}{2}}, u_{i+1,j+1}^{n+\frac{1}{2}} \right)$$

The first component $h_{i+rac{1}{2},j+rac{1}{2}}^{n}$ of $u_{i+rac{1}{2},j+rac{1}{2}}^{n}$ is calculated as follows:
\[ h^n_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{4} \left( h^n_{i,j} + h^n_{i+1,j} + h^n_{i,j+1} + h^n_{i+1,j+1} \right) \]
\[ + \frac{1}{16} \left( h_x|_{i,j} + h_x|_{i,j+1} - h_x|_{i+1,j} - h_x|_{i+1,j+1} \right) \]
\[ + \frac{1}{16} \left( h_y|_{i,j} - h_y|_{i,j+1} + h_y|_{i+1,j} - h_y|_{i+1,j+1} \right) \]

\( h_x|_{ij} \) and \( h_y|_{ij} \) are indirectly calculated as follows:

\[ h_x|_{ij} = H_x|_{ij} - z_x|_{ij} \]

\[ h_y|_{ij} = H_y|_{ij} - z_y|_{ij} \]

A similar treatment is applied for the back projection step when calculating \( h^{n+1}_{ij} \).

We can show that with these modifications we have \( u^{n+1}_{ij} = u^n_{ij} \), for all \((x_i, y_j)\) and the steady state requirement is exactly maintained at the discrete level.
Dam Break over a rectangular bump

In this example, we consider a rapidly varying flow over a discontinuous bottom. The riverbed is described by

\[ z(x, y) = \begin{cases} 
8 & \text{if } |x - \frac{1500}{2}| < \frac{1500}{4}, \\
0 & \text{otherwise}.
\end{cases} \]

The initial water level is discontinuous:

\[ H(x, y, 0) = h(x, y) + z(x, y) = \begin{cases} 
20 & \text{if } x < \frac{1500}{2}, \\
15 & \text{otherwise}.
\end{cases} \]

The final time is \( t = 15 \text{s} \), the computational domain is discretized using 600x10 grid points.
Figure: Dam Break over a Rectangular Bump: Water height profile at time $t = 15$ s obtained using the numerical base scheme.

Movie: Original staggered central scheme
**Figure:** Dam Break over a Rectangular Bump: Cross section in the $x$—direction of the water height at the final time.
Figure: Dam Break over a Rectangular Bump: Water height at the final time $t = 15$ s obtained using the well balanced scheme.
Figure: Dam Break over a Rectangular Bump: Cross section in the $x-$direction of the water height at the final time $t = 15s$ obtained using the well balanced scheme and the interface type reformulation

Movie: Well-balanced scheme
Figure: Dam Break problem: Well balanced + Interface-Type: 1D vs 2D
Lake at rest state problem

In this example, we consider a lake at rest over an irregular bottom topography.

- The computational domain $[0; 1500] \times [0; 1500]$ is discretized using 200x200 grid points.
- The computations are performed at $t = 0.6s$.
- The initial water level is $H(x, y) = 90$ for all $(x, y)$.
- The initial velocity field is $(u, v) = (0, 0)$. 
Figure: Lake at rest state problem: Initial water height and the profile of the waterbed
Figure: Lake at rest problem over irregular topography: The numerical base scheme doesn’t satisfy the steady state condition
Figure: Lake at rest problem over irregular topography: Cross section of the water height obtained using the well balanced scheme and the interface-type reformulation at time $t = 0.6s$ using the MC-$\theta = 1.5$ limiter.
In this example, we consider a dam break problem over a non variable bottom topography.

- Computational domain: $[0;40] \times [0;40]$ discretized using 100x100 grid points.
- The initial water height is defined as follows:

$$h(x, y) = \begin{cases} 
2.5 & \text{, if } (x - 20)^2 + (y - 20)^2 < 2.5, \\
0.5 & \text{, otherwise}.
\end{cases}$$

and $(u, v) = (0, 0)$.
- Final time is $t = 4.7s$ and the MC-($\theta = 1.5$) limiter is used.
Figure: Toro’s problem at $t=0.69s$, 200x200 gridpoints, $MC-\theta = 1.5$
Figure: Toro’s problem at $t=1.7s$, 200x200 gridpoints, $MC-\theta = 1.5$
**Figure:** Toro’s problem at $t=4.7s$, 100x100 gridpoints, $MC-\theta = 1.5$

**Movie**
2D Toro’s Dam Break problem: Solution at t=4,7, section along y=10

**Figure:** Toro’s problem at t=4.7s: Central base scheme vs well balanced scheme
On going work
Well-balanced central schemes on unstructured grids

Figure: Geometry of the 2D unstaggered central scheme on unstructured grid.
Central schemes for the SWE system with wet/dry states

Figure: Fully and partially flooded cells
Central schemes on nonuniform grids

Figure: Geometry of the 1D unstaggered central scheme on nonuniform grids
Thank you for your attention