Random Fourier series as rough paths and applications to a class of SPDEs

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Workshop *Stochastic Methods in Finance and Physics*,
Heraklion, Crete,
July 19, 2013
Outline

What are (Gaussian) rough paths?

Random Fourier series

Applications to SPDEs
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Applications to SPDEs
Motivation for rough paths: robustness

Consider map

$$\omega \mapsto I_V(\omega, \xi)$$

where $\omega$ Brownian trajectories and $I_V(\cdot, \xi)$ solution to

$$dY_t = V_0(Y_t) \, dt + \sum_{i=1}^{d} V_i(Y_t) \circ dB_t^i; \quad Y_0 = \xi \in \mathbb{R}^m.$$
What are (Gaussian) rough paths?

\[(\Xi, \varrho) \xrightarrow{I_V(\cdot, \xi)} (\tilde{\Xi}, \varrho)\] (continuous)

\[S_{\mu^d} \uparrow \text{measurable} \]

\[(C_0([0, T]; \mathbb{R}^d), \mathcal{B}, \mu^d) \xrightarrow{I_V(\cdot, \xi)} C_0([0, T]; \mathbb{R}^m)\]

(not continuous)

\[\mu^d = \mu \otimes \cdots \otimes \mu \text{ d-dimensional Wiener measure}\]

- (\Xi, \varrho) metric space, polish, rough paths space
- \(S_{\mu^d}\) injective, lift map
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What are (Gaussian) rough paths?

\[(\Xi, \varrho) \xrightarrow{I_V(\cdot, \xi)} (\tilde{\Xi}, \varrho)\]

- \(S_{\mu^d}\) measurable
- \((C_0([0, T]; \mathbb{R}^d), \mathcal{B}, \mu^d) \xrightarrow{I_V(\cdot, \xi)} C_0([0, T]; \mathbb{R}^m)\)

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\(d\)-dimensional Wiener measure

- \((\Xi, \varrho)\) metric space, polish, rough paths space
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What are (Gaussian) rough paths?

- generalizes to centered Gaussian measures
  \[ \gamma = \gamma_1 \otimes \cdots \otimes \gamma_d \]

Theorem (Friz, Victoir; AIHP 2010)

If covariances

\[ (s, t) \mapsto R_{\gamma_i}(s, t) = \int \omega(s)\omega(t) \gamma_i(d\omega), \quad i = 1, \ldots, d \]

have finite 2D \( \rho \)-variation, \( \rho < 2 \), then \( \exists \) lift map \( S_{\gamma} \) to some rough paths space “in a natural way”.

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SDEs driven by Gaussian rough paths

Can solve

\[ dY_t = V_0(Y_t) \, dt + \sum_{i=1}^{d} V_i(Y_t) \, dX_t^i \equiv V(Y_t) \, dX_t; \quad Y_0 = \xi \]

with \( X_t(\omega) = S_\gamma(t, \omega(t)) \),

\[
\begin{align*}
(\Xi, \varrho) & \quad \xrightarrow{I_V(\cdot, \xi)} \quad (\tilde{\Xi}, \varrho) \\
S_\gamma \uparrow \text{measurable} & \quad \downarrow \pi \\
(C_0([0, T]; \mathbb{R}^d), B, \gamma) & \quad \xrightarrow{\exists l_V(\cdot, \xi)} \quad C_0([0, T]; \mathbb{R}^m)
\end{align*}
\]
Some more results about Gaussian rough paths:

- stochastic integration theory
- sharp tail estimates for stochastic integrals (and related objects)
- numerical approximation theory for SDEs
- support theorems
- Freidlin-Wentzell large deviations
- Hörmander Theorem
- ...
What is 2D $\rho$-variation?

- Have to make sense of integrals of the form

$$\int_{[0,T]^2} R(s, t) \, dR(s, t) \quad = \quad \int_{[0,T]^2} R(s, t) \, \partial_s \partial_t R(s, t) \, ds \, dt$$

- If $\partial_s \partial_t R(s, t)$ is finite measure, integral should exist
- This is the case if $R$ has finite 1-variation. Example: Brownian motion

$$\mathbb{E} \left| \int_0^T \tilde{B}_s \, dB_s \right|^2 \; = \; \int_{[0,T]^2} (s \wedge t) \, \delta_{s,t} \, ds \, dt \; = \; \int_0^T s \, ds$$

- Typically, $R$ “smooth” outside the diagonal $D$
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- Typically, $R$ “smooth” outside the diagonal $D$
• Notation:

\[ R([s, t] \times [u, v]) := \text{Cov}(X_t - X_s, X_v - X_u) = \mathbb{E}(X_t - X_s)(X_v - X_u), \]

\[ \sigma^2(s, t) := R([s, t]^2) = \text{Var}(X_t - X_s) \]

• If \( \varphi \) positive test function,

\[
\langle \partial_s \partial_t R(s, t), \varphi \rangle = \lim_{|\Pi| \to 0} \sum_{t_i \in \Pi} \varphi(t_i, t_i) \sigma^2(t_i, t_{i+1}) + \sum_{t_i, t_j \in \Pi} \varphi(t_i, t_j) R([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \geq 0
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Observation:

- If $X$ has **positively correlated increments** (equivalently: $\partial_s \partial_t R(s, t)$ positive measure away from diagonal $D$),
  $\Rightarrow \partial_s \partial_t R(s, t)$ positive Radon measure on $[0, T]^2$ (**Theorem of Schwartz**)

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Theorem (Friz, Gess, Gulisashvili, R.)

Assume: \( \mu = \partial_s \partial_t R(s, t) \) Radon measure on \((0, T)^2 \setminus D\) with decomposition \( \mu = \mu_+ - \mu_- \).

1. If \( \mu_- \) has finite mass, then \( R \) has finite (2D) 1-variation.
2. If
   
   (i) \( \mu_+ \) has finite mass,
   (ii) \( R([u, v] \times [s, t]) \geq 0 \) for all \([u, v] \subseteq [s, t]\),
   (iii) \( \sigma^2 \) has finite (1D) \( \rho \)-variation, \( \rho \in [1, \infty) \),

   then \( R \) has finite (2D) \( \rho \)-variation.

In particular, for \( \rho \in [1, 2) \), \( X \) can be lifted to process with trajectories in rough paths space (in the sense of Friz–Victoir).

- Condition (iii) first appeared in [Jain, Monrad; AOP ’83] in the context of sample path regularity.
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Remarks:

- Stable under smooth perturbations $X \mapsto X + X^{\text{smooth}}$
- Same assumptions imply finite (1D) $q$-variation ($q = \frac{2}{\rho^{-1} + 1} < 2$) for paths in Cameron–Martin space (cf. [Friz, Gess, R; Preprint ’13])
- If $\sigma^2(s, t) = \sigma^2(t - s)$ (stationary increments) and $\sigma^2$ concave,

$$\partial^2_{s,t} R(s, t) = -\partial^2_{s,t} \sigma^2(t - s) = (\sigma^2)^{''}(t - s) \leq 0$$

$\Rightarrow \mu_+ = 0 \Rightarrow (i)$,

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Examples:

- fractional Brownian motion \((H > 1/4)\)
- bifractional Brownian motion (no stationary increments, no Volterra kernel)
- (fractional) bridges
- (fractional) Ornstein–Uhlenbeck processes
- ...
- random Fourier series!
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Random Fourier series

\[ \psi(t) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{\infty} \alpha_k Y^k \sin(k t) + \alpha_{-k} Y^{-k} \cos(k t) \in \mathbb{R}, \]

where \( \mathbb{E} Y^k Y^{-l} = \delta_{k,l} \) for all \( k, l \in \mathbb{Z} \).

Set \( \Delta a_k := a_{k+1} - a_k \), \( \Delta^2 := \Delta \circ \Delta \).

Theorem (Friz, Gess, Gulisashvili, R.)

If

1. \( a_k := \alpha_k^2 = O(|k|^{-(1+1/\rho)}) \) and decreasing for \( |k| \to \infty \), some \( \rho \geq 1 \)
2. \( \Delta^2(k^2 a_k) \leq 0 \) for all \( k \in \mathbb{Z} \)
3. \( k^3 |\Delta^2 a_k| + k^2 |\Delta a_k| = o(1) \) for \( |k| \to \infty \)

Then \( R_\psi \) has finite \( \rho \)-variation (actually, Hölder controlled finite \( (1, \rho) \)-variation).
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Then \( R_\psi \) has finite \( \rho \)-variation (actually, Hölder controlled finite \((1, \rho)\)-variation).
Proof

- By robustness results, suffices to show claim for 
  $R(s, t) = K(t - s)$ where

\[K(x) = \sum_{k=0}^{\infty} a_k \cos(kx)\]

- From $\sigma^2(s, t) = 2(K(0) - K(t - s))$, suffices to find criteria for $a_k$ such that $K$ is convex and $1/\rho$ Hölder.
Corollary

Assume

- $\Psi = (\Psi^1, \ldots, \Psi^d)$ Gaussian, independent copies
- $a_k^i = |k|^{-(1+1/\rho_i)}$ and $\rho_1 \lor \ldots \lor \rho_d =: \rho < 2$

Then $\Psi$ has a lift (in the sense of Friz–Victoir) to a rough paths space.

Result robust: if $b_k$ “behaves similar to” $a_k$, same result applies for corresponding Fourier series.
Corollary

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Result robust: if \( b_k \) “behaves similar to” \( a_k \), same result applies for corresponding Fourier series.
Approximation: truncated Fourier series

\[ \psi^N(t) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{N} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt) \in \mathbb{R} \]

Proposition

Under the same assumptions, on a set of full measure,

\[ \varrho(S_{\psi^N}(\omega), S_{\psi}(\omega)) = \mathcal{O}(N^{-\eta}) \]

for any \( \eta < \frac{1}{\rho} - \frac{1}{2} \).

Other approximations of \( \psi \) possible.
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Applications to SPDEs
• $\Psi = (\Psi^1, \ldots, \Psi^d)$, each $\Psi^i$ solves fractional stochastic heat equation

$$d\Psi^i_t = (-(-\Delta)^\alpha)\Psi^i_t \, dt + \sigma dW^i_t; \quad \Psi^i_0 = 0$$

where
- $dW^i$ space-time white noise, uncorrelated
- spatial variable $x \in [0, 2\pi]$, time variable $t \in [0, T]$
- $(-(-\Delta)^\alpha)$ fractional Laplacian with periodic (P), Dirichlet (D) or Neumann (N) $\partial$-conditions ($\alpha = 1 \sim$ usual Laplacian $\Delta$)

• can also consider stationary solution $\Psi$ (with shifted spectrum of $(-\Delta)^\alpha$ if necessary)

• $\Psi^i$ can be written as random Fourier series:
  - $\alpha_k^2 = -\alpha_{-k}^2$ in case of (P)
  - sine series in case of (D)
  - cosine series in case of (N)
What are (Gaussian) rough paths?

- $\psi = (\psi^1, \ldots, \psi^d)$, each $\psi^i$ solves \textit{fractional stochastic heat equation}

\[
d\psi^i_t = (-(-\Delta)^\alpha)\psi^i_t \, dt + \sigma dW_t^i; \quad \psi^i_0 = 0
\]

where

- $dW_t^i$ space-time white noise, uncorrelated
- spatial variable $x \in [0, 2\pi]$, time variable $t \in [0, T]$ 
- $(-(-\Delta)^\alpha)$ fractional Laplacian with periodic (P), Dirichlet (D) or Neumann (N) $\partial$-conditions ($\alpha = 1 \sim$ usual Laplacian $\Delta$)

- can also consider stationary solution $\Psi$ (with shifted spectrum of $(-\Delta)^\alpha$ if necessary)

- $\psi^i$ can be written as random Fourier series:
  - $\alpha_k^2 = \alpha_{-k}^2$ in case of (P)
  - sine series in case of (D)
  - cosine series in case of (N)
\( \psi = (\psi^1, \ldots, \psi^d) \), each \( \psi^i \) solves fractional stochastic heat equation

\[
d\psi^i_t = \left(-(-\Delta)^\alpha\right)\psi^i_t \, dt + \sigma \, dW^i_t; \quad \psi^i_0 = 0
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- \( dW^i \) space-time white noise, uncorrelated
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- \( (-(-\Delta)^\alpha) \) fractional Laplacian with periodic (P), Dirichlet (D) or Neumann (N) \( \partial \)-conditions (\( \alpha = 1 \) \( \sim \) usual Laplacian \( \Delta \))

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- \( \psi^i \) can be written as random Fourier series:
  - \( \alpha_k^2 = \alpha_{-k}^2 \) in case of (P)
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Corollary

Consider stationary solution $\Psi$. Then for every

- $t \in [0, T]$
- $\alpha \in (3/4, 1]$

there exist lift of $\Psi_t$ to a process with rough sample paths (in space) in the sense of Friz–Victoir. Moreover,

$$t \mapsto S_{\Psi_t}(\omega)$$

is continuous in rough-paths topology.
• **Application:** solution $\Psi$ can be used as “building block” for solving non-linear (Burger’s-like) SPDE with additive noise:

$$
\begin{align*}
\frac{du}{dt}_t &= (-(-\Delta)^{\alpha})u_t \, dt + f^i(u) \, dt + \sum_{j=1}^{n} g^j(u) \partial_x u^j \, dt + \sigma dW_t^i;
\end{align*}
$$

$u^i_0$ smooth; $i = 1, \ldots, n$

where $(-(-\Delta)^{\alpha})$ fractional Laplacian with $(P), (D)$ or $(N)$ $\partial$-conditions

• extends results from [Hairer; CPAM] (there: $\alpha = 1$, $(P)$)

• SPDE has relevance in path sampling for diffusions
**Application:** solution $\Psi$ can be used as “building block” for solving non-linear (Burger’s-like) SPDE with additive noise:

$$du^i_t = (-(-\Delta)^\alpha) u^i_t \, dt + f^i(u) \, dt + \sum_{j=1}^{n} g^i_j(u) \partial_x u^j \, dt + \sigma \, dW^i_t;$$

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Thank you for your attention.