Boundary value problems and integrability: a Riemann-Hilbert approach

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The motivation for this work:

- "Integrable" PDEs → PDE that admit a Lax pair formulation.
- Large class of nonlinear PDEs of mathematical physics
- Find a method to solve Boundary Value Problems, as good as the scattering transform, i.e.,
  - applicable to this class
  - can produce exact solutions
  - yields good qualitative information, effective approximation schemes
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*Find a method to solve Boundary Value Problems, as good as the scattering transform, i.e.,*

▶ applicable to this class
▶ can produce exact solutions
▶ yields good qualitative information, effective approximation schemes

▶ How does the integrability (exact solvability) of a PDE depend on domain/boundary conditions?
Unifying approach: a Riemann-Hilbert formulation

What is a RH problem? The reconstruction an analytic function from the prescribed jump across a curve.

Given

(a) a contour $\Gamma \subset \mathbb{C}$ that divides $\mathbb{C}$ into the simply connected subdomains $\mathbb{C}^+$ and $\mathbb{C}^-$

(b) a scalar/matrix valued $G(\lambda)$, $\lambda \in \Gamma$

find $H(z)$ analytic off $\Gamma$ (with prescribed normalization, e.g. $H \sim I$ at infinity) that admits limits $H^\pm$ on $\Gamma$ such that

$$H^+(\lambda) = H^-(\lambda)G(\lambda) \quad (\lambda \in \Gamma)$$
A **PDE problem is integrable** if its solution can be reduced to a RH problem, i.e. to the reconstruction of an analytic function from the known structure of its singularities.

Why is it good to have a RH formulation?

- explicit solution in the **scalar case**, that can be written additively

\[
\log H^+ = \log H^- + \log G \iff h^+ - h^- = g
\]

\[
\Rightarrow h(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta)}{\zeta - \lambda} d\zeta \quad (\text{Plemelj formula}), \quad H = \exp(h).
\]

- else can be reduced to the analysis of a **linear integral equation**
- **complex analytic tools**, asymptotic computations (Deift-Zhou)
- numerical computations
A method to study BVPs

This method (Fokas, from 1994), based on the heuristic identification of integrability with a RH formulation, can be exploited for answering a broader set of questions.

I will only consider the case of two independent variables, and will discuss two examples and related results:

**linear:** \( q_t = q_{xxx}, \quad t > 0, \ 0 < x < L \)

"can the solution be expressed as an infinite series?"

**nonlinear:** \( q_{xx} + q_{yy} = \sin q, \quad x > 0, \ 0 < y < L \)

"can we find an explicit solution?"

**Note:** Integrable PDE are ubiquitous as models of physical phenomena - this question is very important for many applications.
A key example - the relation of the two parts:

Fourier Transform on $\mathbb{R}$

$q : \mathbb{R} \to \mathbb{R}$,  \textit{smooth, decaying as} $|x| \to \infty$

Direct:

$$\hat{q}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} q(x) dx, \quad \lambda \in \mathbb{R}$$

Inverse:

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{q}(\lambda) d\lambda, \quad x \in \mathbb{R}$$

The essential tool e.g. in the solution of initial value problems for linear evolution PDEs
Linear initial value problem - the FT solution

\[ q_t(x, t) + q_{xxx}(x, t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad q(x, 0) = q_0(x) \]

(\( + \) some smoothness and decay)

Solution scheme (via writing the PDE as \( \left[ e^{-i\lambda^3 t} \hat{q}(\lambda, t) \right]_t = 0 \)):

\[
q_0(x) \xrightarrow{\text{Direct map}} \hat{q}_0(\lambda) \xrightarrow{\text{Inverse map}} q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t} \hat{q}_0(\lambda) d\lambda
\]

Generally applicable, yields an exact solution representation, and qualitative information (explicit \( x \) and \( t \) dependence)
RH proof of the Fourier inversion theorem

Fourier Transform on $\mathbb{R}$ via the spectral analysis of

$$\mu_x - i\lambda \mu = q(x), \quad \mu = \mu(x, \lambda), \quad q(x) \text{ smooth and decaying, arbitrary}$$

We seek a solution bounded $\forall \lambda \in \mathbb{C}$

$$\mu_+(x, \lambda) = \int_{-\infty}^{x} e^{i\lambda(x-y)} q(y) \, dy, \quad \text{bounded for } \text{Im}(\lambda) \geq 0$$

$$\mu_-(x, \lambda) = -\int_{x}^{\infty} e^{i\lambda(x-y)} q(y) \, dy \quad \text{bounded for } \text{Im}(\lambda) \leq 0$$

$\triangleright \mu_\pm$ is analytic in $\mathbb{C}^\pm$
$\triangleright \mu_\pm = O \left( \frac{1}{\lambda} \right) \text{ as } \lambda \to \infty,$
$\triangleright$ For $\lambda \in \mathbb{R},$

$$(\mu_+ - \mu_-)(x, \lambda) = e^{i\lambda x} \int_{-\infty}^{\infty} e^{-i\lambda y} q(y) \, dy = e^{i\lambda x} \hat{q}(\lambda)$$
\[ \mu_x - i\lambda \mu = q(x) \]

\[ (\lambda \text{ plane}) \quad \mu_+ \quad \mathbb{C}_+ \]

\[ \mu_+ - \mu_- \quad \text{bounded for Im}(\lambda) \geq 0 \]

\[ \mathbb{C}_- \quad \mu_- \quad \text{bounded for Im}(\lambda) \leq 0 \]

\[ \mu_+(x, \lambda) = \int_{-\infty}^{x} e^{i\lambda(x-y)} q(y) dy, \quad \text{bounded for Im}(\lambda) \geq 0 \]

\[ \mu_-(x, \lambda) = - \int_{x}^{\infty} e^{i\lambda(x-y)} q(y) dy \quad \text{bounded for Im}(\lambda) \leq 0 \]

\[ \mu^\pm \sim \frac{iq(x)}{\lambda}, \quad |\lambda| \to \infty. \]
Solution (Plemelj formula):

\[ \mu(\lambda, x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\mu_+ - \mu_-)(\zeta)}{\zeta - \lambda} d\zeta = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\zeta x} \hat{q}(\zeta)}{\zeta - \lambda} d\zeta \]

\[ \Rightarrow q(x) = \mu_x - i\lambda \mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\zeta x} \hat{q}(\zeta) d\zeta, \quad x \in \mathbb{R}. \]

This procedure is rigorously justified as soon as \( q(x) \) is Hölder continuous - simple example of a Riemann-Hilbert problem

**MORAL**: The ODE \( \mu_x - i\lambda \mu = q(x) \) encodes the Fourier transform
Can the classical FT approach be applied more widely?

Example 1: nonlinear initial value problems:

\[ iq_t + q_{xx} + q|q|^2 = 0 \]

\[ \Rightarrow \left[ e^{i\lambda^2 t} \hat{q} \right]_t = i e^{i\lambda^2 t} \hat{q}|q|^2 \]

Example 2: boundary value problem

\[ q_t(x, t) + q_{xxx}(x, t) = 0, \ t > 0, \ x > 0, \ q(x, 0) = q_0(x) \]

Solution:

\[ q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t} \hat{q}_0(\lambda) d\lambda + \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x + i\lambda^3 t} \left[ \int_{0}^{t} e^{-i\lambda^3 s} (q_{xx}(0, s) + i\lambda q_x(0, s) - \lambda^2 q(0, s)) \, ds \right] d\lambda. \]
The direct application of the classical Fourier transform methods fails in more general cases. I want to describe how this failure can be rescued: the RH derivation of the transform indicates the way to generalize the approach to more general situations

(a): ivp for nonlinear PDEs that can be written in terms of a so-called Lax pair - aka integrable PDEs
This amounts to the famous inverse scattering transform

(b): linear boundary value problems, defined in fixed or time-dependent domains

(a)+(b): nonlinear integrable boundary value problems
Recall: The ODE $\mu_x - i\lambda \mu = q(x)$ encodes the Fourier transform.

Same idea, but use a matrix-valued ODE $\rightarrow$ matrix-valued (noncommutative) RH problem:

$$M_x + i\lambda [\sigma_3, M] = Q M, \quad M(x, \lambda) \text{ a } 2 \times 2 \text{ matrix}$$

$$Q = \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix}, \quad \sigma_3 = \text{diag}(1, -1), \quad [\sigma_3, M] = \sigma_3 M - M \sigma_3$$

Find $M$ by solving the associated RH problem $\Rightarrow$

$$q = 2i \lim_{|\lambda| \to \infty} (\lambda M)_{12},$$

(linear case: $\mu \sim iq/\$ as $\lambda \to \infty$)

**a nonlinear Fourier transform**
Nonlinear IVP: the nonlinear Schrödinger equation on $\mathbb{R}$

$$iq_t + q_{xx} + q|q|^2 = 0 \quad x \in \mathbb{R}, \ t > 0$$

$PDE \iff M_{xt} = M_{tx}$ \quad Lax pair

with

$$\begin{cases} M_x + i\lambda [\sigma_3, M] = QM, \\ M_t + 2i\lambda^2 [\sigma_3, M] = (2\lambda Q - iQ_x\sigma_3 - i|q|^2\sigma_3)M \end{cases}$$

RH from 1st ODE + time evolution of $M$ is linear $\rightarrow$ solve for $M$

Solution of the initial value problem:

$$q(x, t) = -\frac{1}{\pi} \int_{\mathbb{R}} \left[ e^{-i(4\lambda^2t + 2\lambda x)} \gamma(\lambda) M_{11}^+ + |\gamma(\lambda)|^2 M_{12}^+ \right] (\lambda) d\lambda$$

$$\gamma(\lambda) \longleftrightarrow q(x, 0)$$

Note: this FAILS for boundary value problems
Nonlinear BVP: the elliptic sine-Gordon equation

\[ q_{xx} + q_{yy} = \sin q, \quad q = q(x, y), \quad \text{also integrable:} \]

**Lax pair:**

\[
\psi_x + \frac{1}{4i} \left( \lambda - \frac{1}{\lambda} \right) [\sigma_3, \psi] = Q(x, y, \lambda)\psi,
\]

\[
\psi_y + \frac{1}{4i} \left( \lambda + \frac{1}{\lambda} \right) [\sigma_3, \psi] = iQ(x, y, -\lambda)\psi,
\]

\[
Q(x, y, \lambda) = \frac{i}{4} \begin{pmatrix}
\frac{1}{\lambda} (1 - \cos q) & q_x - iq_y + \frac{i \sin q}{\lambda} \\
q_x - iq_y - \frac{i \sin q}{\lambda} & -\frac{1}{\lambda} (1 - \cos q)
\end{pmatrix}
\]

no "time-like" variable - which one of the two ODE in the Lax pair to use?

What is the "evolution of the spectral data"?
** Two ideas:

(1) integral transform for solving IVP ↔ RH problem coded by an ODE
(2) integrable PDE ↔ pair of ODEs (Lax pair)

** Combine these two ideas:
set up the RH problem from both ODEs in the Lax pair and obtain a transform that involves both $x$ and $t$ as parameters

If the solution involves boundary values, the second approach is more natural ⇒ spectral data in terms of (initial and) boundary conditions (need to be only the known bc though)

This approach has proved valuable also for solving BVPs for linear PDEs, that admit trivially a Lax pair
Starting point: the PDE admits a Lax pair

A PDE problem is integrable if its solution can be reduced to a RH problem: the reconstruction of an analytic function from the known structure of its singularities (the prescribed jump across a given curve)

**Part (A):** Lax pair $\Rightarrow$ RH problem $\Rightarrow$ formal integral representation

if a PDE admits a Lax pair $\rightarrow$ can always obtain an integral representation of the solution of the PDE in a given domain, in terms of all its boundary values

**Part (B):** Lax pair $\Rightarrow$ global relation. The analysis of the global relation yields the determination of the spectral data ("singularity structure") in terms of the given boundary values
Lax representation always exists for linear PDEs:

\[ q_t = q_{xxx} \iff \mu_{xt} = \mu_{tx} \]

with \( \mu(x, t, \lambda) \) solving the Lax pair

\[
\begin{cases}
\mu_x - i\lambda \mu = q \\
\mu_t - i\lambda^3 \mu = q_{xx} + i\lambda q_x - \lambda^2 q
\end{cases}
\]

Express the solution formally via a RH Problem formulated for \( \mu(x, t, \lambda) \)
Given the domain where the PDE is posed

\[ q(0, t) = f_0(t) \]
\[ q_x(0, t) = f_1(t) \]

\[ q_t = q_{xxx} \quad q(L, t) = g_0(t) \]

\[ q(x, 0) = q_0(x) \quad x = L \]

\[ (T, L \text{ finite or infinite}) \]

\[ q(x, t) = \frac{1}{2\pi} \left[ \int_{\mathbb{R}} e^{i\lambda x + i\lambda^3 t} \hat{q}_0(\lambda) d\lambda - \int_{\partial D_+} e^{i\lambda x + i\lambda^3 t} (\tilde{f}_2 + i\lambda \tilde{f}_1 - \lambda^2 \tilde{f}_0) d\lambda \right. \]
\[ \left. - \int_{\partial D_-} e^{i\lambda (x-1) + i\lambda^3 t} (\tilde{g}_2 + i\lambda \tilde{g}_1 - \lambda^2 \tilde{g}_0) d\lambda \right] \]
The RH problem

Figure: The contours $\partial D^+$, $\partial D^-$ for the equation $q_t = q_{xxx}$ for $0 < x < 1$
global relation: $\lambda \in \mathbb{C}$ (where $f_0 = f_1 = g_0 = 0$)

$$\tilde{f}_2 - e^{-i\lambda}(\tilde{g}_2 + i\lambda \tilde{g}_1) = \hat{q}_0(\lambda) - e^{-i\lambda^3 T} \hat{q}(\lambda, T) \quad (*)$$

crucial properties (complex plane needed!):

(i) each $\tilde{\cdot}$ function is a function of $\lambda^3$, hence invariant under

$$\lambda \rightarrow \omega \lambda, \quad \omega = e^{2i\pi/3}$$

(ii) the term $e^{-i\lambda^3 T} \hat{q}(\lambda, T)$ is a ghost whose trace disappears from the integral (analyticity properties)

$\Rightarrow$ evaluate $(*)$ at $\lambda, \omega \lambda, \omega^2 \lambda$ to get in $D$ a system of 3 equations for the 3 unknown (red) functions
Explicit solution representation

\[ q_t = q_{xxx}, \quad q(x, 0) = q_0(x), \quad q(0, t) = q(1, t) = q_x(0, t) = 0. \]

\[
2\pi q(x, t) = \int_{\mathbb{R}} e^{-i\lambda^3 t + i\lambda x} \hat{q}_0(\lambda) d\lambda + \int_{\partial D^+} e^{-i\lambda^3 t + i\lambda x} \tilde{F}(\lambda) d\lambda \\
+ \int_{\partial D^-} e^{-i\lambda^3 t + i\lambda(x-1)} \tilde{G}(\lambda) d\lambda
\]

with

\[
\tilde{F} = \frac{\hat{q}_0(\lambda)e^{i\lambda} + \omega \hat{q}_0(\omega \lambda)e^{i\omega \lambda} + \omega^2 \hat{q}_0(\omega^2 \lambda)e^{i\omega^2 \lambda}}{\Delta(\lambda)}
\]

\[ \tilde{G} \] has a similar expression, and

\[
\Delta(\lambda) = (e^{i\lambda} + \omega e^{i\omega \lambda} + \omega^2 e^{i\omega^2 \lambda}), \quad \omega = e^{2\pi i/3}.
\]

This integral is not equivalent to a series representation.
The zeros of $\Delta(\lambda)$ are infinitely many by Hadamard, accumulating only at infinity, and their location (asymptotically) is given e.g. by Levin’s results, and are not on the integration contour. Zeros are outside $D$ and unreachable $\rightarrow$ underlying differential operator does not admit a basis of eigenfunctions (D. Smith).

No series representation of the solution exists.
Part (A): Lax pair $\Rightarrow$ RH problem $\Rightarrow$ formal integral representation

goes through for any PDE that admits a Lax pair $\Rightarrow$ can always obtain an integral representation

Part (B): analysis of the global relation $\Rightarrow$ determination of unknown boundary values

spectral functions in general are characterised implicitly by a nonlinear system. However, there exist special boundary conditions such that spectral data have invariance properties in $\mathbb{C}$ that allow their explicit computation (linearisable bc)
Nonlinear example: the elliptic sine-Gordon equation

PART (A):

\[ q_{xx} + q_{yy} = \sin q, \quad 0 < x < \infty, \quad 0 < y < L, \]

admits the Lax pair:

\[
\begin{align*}
\psi_x + \frac{1}{4i} \left( \lambda - \frac{1}{\lambda} \right) [\sigma_3, \psi] &= Q(x, y, \lambda) \psi, \\
\psi_y + \frac{1}{4} \left( \lambda + \frac{1}{\lambda} \right) [\sigma_3, \psi] &= iQ(x, y, -\lambda) \psi,
\end{align*}
\]

\[
Q(x, y, \lambda) = \frac{i}{4} \begin{pmatrix} \frac{1}{\lambda} (1 - \cos q) & q_x - iq_y + \frac{i\sin q}{\lambda} \\ q_x - iq_y - \frac{i\sin q}{\lambda} & -\frac{1}{\lambda} (1 - \cos q) \end{pmatrix}
\]
The BVP problem in a semistrip: $0 < x < \infty$, $0 < y < L$

Solution:

$$q_x(x, y) - iq_y(x, y) = 2 \lim_{\lambda \to \infty} (\lambda \psi)_{21},$$

$$\cos q(x, y) = 1 + 4i(\lim_{\lambda \to \infty} (\lambda \psi_x)_{11}) - 2(\lim_{\lambda \to \infty} (\lambda \psi)_{21})^2,$$

where the $2 \times 2$ matrix $\psi(x, y, \lambda)$ is the solution of the RH problem $\psi_- = \psi_+ J$.
The jump matrix

\[
J^0 = \begin{pmatrix}
a_2(\lambda) & b_3(-\lambda) e^{-\theta(x, y, \lambda)} \\
\frac{a_2(\lambda)}{a_1(-\lambda) a_3(\lambda)} & \frac{b_3(-\lambda)}{a_3(\lambda)} e^{-\theta(x, y, \lambda)} \\
-\frac{e^{-\omega(\lambda) L} b_1(\lambda)}{a_1(-\lambda)} e^{\theta(x, y, \lambda)} & 1
\end{pmatrix}
\]

\[
J^{\pi/2} = \begin{pmatrix}
1 & \frac{b_2(-\lambda)}{a_1(\lambda) a_3(\lambda)} e^{-\theta(x, y, \lambda)} \\
0 & 1
\end{pmatrix}
\]

\[
J^{3\pi/2} = \begin{pmatrix}
1 & 0 \\
\frac{b_2(\lambda)}{a_1(-\lambda) a_3(-\lambda)} e^{\theta(x, y, \lambda)} & 1
\end{pmatrix}
\]

\[a_1, b_1 \leftrightarrow q(x, L), q_y(x, L); \quad a_2, b_2 \leftrightarrow q(0, y), q_x(0, y);\]

\[a_3, b_3 \leftrightarrow q(x, 0), q_y(x, 0); \quad \theta(x, y, \lambda) = \frac{1}{2i} \left( \lambda - \frac{1}{\lambda} \right) x + \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right) y.\]
The global relation

PART (B)

The global relation, valid for $\lambda \in \mathbb{C}^+$, is:

$$a_1(\lambda) = a_2(-\lambda)a_3(\lambda) - b_2(-\lambda)b_3(\lambda),$$

$$b_1(\lambda)e^{-\omega(\lambda)L} = a_2(\lambda)b_3(\lambda) - a_3(\lambda)b_2(\lambda),$$

where

$$\omega(\lambda) = \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right).$$
In this case, it is possible to compute the spectral functions in closed form.

(1): additional invariance properties:

\[
\begin{align*}
    a_1(-\frac{1}{\lambda}) &= a_1(\lambda); \quad a_3(-\frac{1}{\lambda}) = a_3(\lambda); \\
    b_1(-\frac{1}{\lambda}) &= b_1(\lambda); \quad b_3(-\frac{1}{\lambda}) = b_3(\lambda); \\
    a_2\left(\frac{1}{\lambda}\right) &= \tilde{F}_\lambda[a_2(\lambda) - F_\lambda b_2(\lambda) + F_\lambda e^{-\omega(\lambda)L}b_2(-\lambda) - F_\lambda^2 e^{-\omega(\lambda)L}a_2(-\lambda)]; \\
    b_2\left(\frac{1}{\lambda}\right) &= \tilde{F}_\lambda[b_2(\lambda) - F_\lambda a_2(\lambda) + F_\lambda e^{-\omega(\lambda)L}a_2(-\lambda) - F_\lambda^2 e^{-\omega(\lambda)L}b_2(-\lambda)],
\end{align*}
\]

where \( F_\lambda = i\frac{1-\lambda^2}{1+\lambda^2} \tan \frac{d}{2} \), and \( \tilde{F}_\lambda = \frac{1}{1-F_\lambda^2} \).

(2): these equations yield a scalar RH problem for the spectral functions that can be solved explicitly in terms of the constant \( L \) and \( d \).
The explicit form of the jump matrices:

\[ \tilde{J}_0 = \begin{pmatrix}
1 - \frac{G^2(\lambda)}{h(\lambda)h(-\lambda)}e^{-\omega(\lambda)L} & -\frac{G(\lambda)}{h(\lambda)}e^{-\theta(x,y,\lambda)} \\
e^{-\omega(\lambda)L} \frac{G(\lambda)}{h(-\lambda)}e^{\theta(x,y,\lambda)} & 1
\end{pmatrix} \]

\[ \tilde{J}_{\pi/2} = \begin{pmatrix}
1 & -\frac{G(\lambda)}{h(\lambda)} \left(1 + e^{\omega(\lambda)L}\right) e^{-\theta(x,y,\lambda)} \\
0 & 1
\end{pmatrix} \]

\[ \tilde{J}_{3\pi/2} = \begin{pmatrix}
1 & 0 \\
\frac{G(\lambda)}{h(-\lambda)} \left(1 + e^{-\omega(\lambda)L}\right) e^{\theta(x,y,\lambda)} & 1
\end{pmatrix} \]

\[ G(\lambda) = \frac{i(1 - \lambda^2)}{1 + \lambda^2} \frac{e^{\omega(\lambda)L} + e^{-\omega(\lambda)L} - 2}{e^{\omega(\lambda)L} - e^{-\omega(\lambda)L}} \tan \frac{d}{2}, \]

\[ h(\lambda) = e^{H(\lambda)}, \quad H(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \ln[1 - G^2(\lambda')] \frac{d\lambda'}{\lambda' - \lambda} \]
Some bibliography