New formulas for the Maslov canonical operator near focal points and caustics and applications

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Common work
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The Maslov canonical operator $\psi = K_{A_n}^h A$:

ASYMPTOTIC SOLUTIONS FOR EQUATIONS:

$$i\hbar \frac{\partial \psi}{\partial t} = L(x, -i\hbar \frac{\partial}{\partial x}, \hbar)\psi \quad \text{or} \quad L(x, -i\hbar \frac{\partial}{\partial x}, \hbar)\psi = E\psi, \quad x \in \mathbb{R}^n, \quad 0 < \hbar << 1$$

Cauchy problem

Scattering and spectral problems, the Green function

$L(x, p, h) \in C^\infty(\mathbb{R}^{2n}_{px} \times (0, 1])$ is the symbol,

the principle symbol $H(x, p) = L(x, p, 0)$ is the Hamiltonian in $2n - D$- phase space $\mathbb{R}^{2n}_{px}$

The basic objects: n-D Lagrangian manifold ("wave front" in $\mathbb{R}^{2n}_{px}$)

$\Lambda^n = \{p = P(\alpha), x = X(\alpha), \alpha = (\alpha_1, \ldots, \alpha_n)\}$ in $\mathbb{R}^{2n}_{px}$,

(locally $\exists S = \int P \, dX$) and

a function $A(\alpha) \in C^\infty(\Lambda^n)$

Asymptotic solutions: $\psi = K_{A_n}^h A$
The Maslov canonical operator = Algorithm

Regular, singular (focal) points, Lagrangian singularities (caustics)

Jacobians: \( J(\alpha) = J^{(1,2)}(\alpha) = \det \frac{\partial X}{\partial \alpha} \)

\( \alpha = (P(\alpha), X(\alpha)) \) is regular if \( J(\alpha) \neq 0 \) and singular (focal) if \( J(\alpha) = 0 \).

The set of focal points gives Lagrangian singularities, their projections to \( \mathbb{R}^n_x \) give caustics.

The other Jacobians (restrict ourselves to the case \( n = 2 \))

\( J^{(-1,2)}(\alpha) = \det \frac{\partial (P_1, X_2)}{\partial \alpha}, J^{(1,-2)}(\alpha) = \det \frac{\partial (X_1, P_2)}{\partial \alpha}, J^{(-1,-2)}(\alpha) = \det \frac{\partial (P_1, P_2)}{\partial \alpha} \)

**Lemma** (Kucherenko, Arnold) \( |J^{(1,2)}| + |J^{(-1,2)}| + |J^{(1,-2)}| + |J^{(-1,-2)}| \neq 0 \)

There exist charts \( \Omega^I_j, I = \{(1, 2), (-1, 2), (1, -2), (-1, -2)\} \) and corresponding portion of unity \( e_j(\alpha) \) such that \( J^I \neq 0 \) in \( \Omega^I_j \), \( \bigcup_j \Omega^I_j = \Lambda^n \)
The influence of charts into the canonical operator:

Regular charts $I = (1, 2)$ (WKB):

$$K_j(Ae_j) = e^{-i\pi m_j/2} \frac{e^{\frac{i}{\hbar} \int_{\alpha_0}^{\alpha(x)} P dX}}{\sqrt{|J^{(1,2)}(\alpha(x))|}} (Ae_j)(\alpha(x)), \quad \alpha(x) : \quad X(\alpha) = x$$

Singular charts $I = (-1, 2)$ (the partial Fourier transform)

$$K_j(Ae_j) = e^{-i\pi m_j/2} \sqrt{\frac{i}{2\pi\hbar}} \int_{\mathbb{R}} \frac{e^{\frac{i}{\hbar} \left( \int_{\alpha_0}^{\alpha(p_1,x_2)} P dX + p_1(x_1-X_1(\alpha(p_1,x_2))) \right)}}{\sqrt{|J^{(-1,2)}(\alpha(p_1,x_2))|}} (Ae_j)(\alpha(p_1, x_2)) dp_1,$$

$$\alpha(p_1, x_2) : \quad P_1(\alpha) = p_1, X_2(\alpha) = x_2$$

Here $m_j = m_j(\Omega^I_j)$ is the Maslov index.

The Maslov canonical operator: $K^h_{\Lambda_n}A = \sum_j K_j(Ae_j)$.

Practical defects:
1) there is no advice what $I$ is better (moreover it possible to rotate the coordinates $x$); 2) sometimes it is necessary to use too many charts.
The catastrophe theory and “general position”

The caustic in general position:

Nongeneral position:

Example 1 \((n = 1)\) : \(\Lambda^1 = \{p = \alpha, x = \xi = \text{const}\}\)
Example 2 ($n = 2$), $\alpha = (\tau, \psi)$:

$\Lambda^2 = \{ P = n(\phi), X = \tau n(\phi), n(\phi) = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \phi \in S^1, \tau \in \mathbb{R} \}$
Example 2 ($n = 2$), $\alpha = (\tau, \psi)$:

\[ \Lambda^2 = \{ P = n(\phi), X = \tau n(\phi), n(\phi) = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \phi \in S^1, \tau \in \mathbb{R} \} \]

$J^{(1,2)} = \tau$, $J^{(-1,2)} = \sin^2 \phi$, $J^{(1,-2)} = \cos^2 \phi$, $J^{(-1,-2)} = 0$, $S = \tau$

The singular charts $I = (-1, 2)$

\[ K_j(Ae_j) = e^{-i\pi m_j/2} \sqrt{\frac{i}{2\pi h}} \int_{\mathbb{R}} \frac{e^{i h \left( \tau(p_1, x_2) + \cos \phi(x_1 - \tau(p_1, x_2) \cos \phi(p_1, x_2)) \right)}}{|\sin \phi(p_1, x_2)|} (Ae_j)(\alpha(p_1, x_2)) \, dp_1, \]

$(\tau, \phi)(p_1, x_2): \cos \phi = p_1, \tau \sin \phi = x_2$

The change of variables: $p_1 = \cos \phi$

\[ K_j(Ae_j) = \sqrt{\frac{i}{2\pi h}} \int_{\mathbb{R}} e^{\frac{i h}{h} (x_1 \cos \phi + x_2 \sin \phi)} (Ae_j)(\tau(x_2, \phi), \phi) \, d\phi \]

\[ \sum_j K_j(Ae_j) = \sqrt{\frac{i}{2\pi h}} \int_{0}^{2\pi} e^{\frac{i h}{h} (x_1 \cos \phi + x_2 \sin \phi)} A(\tau(x, \phi), \phi) \, d\phi \]

Let $A = 1 \implies K_{\Lambda^2}^h 1 = \sqrt{\frac{2\pi i}{h}} \mathcal{J}_0\left(\frac{|x|}{h}\right) !!!!!
Generalization

The eikonal-coordinates: let $P \, dX$ be nondegenerated on $\Lambda^2$  \implies  
\exists  \ (local) coordinates \quad \tau = \int P \, dX \text{ and } \phi

Lemma. \quad |J(\tau, \phi)| + |\tilde{J}(\tau, \phi)| \neq 0, \quad \tilde{J} = \det(P_\phi, P) \text{ and}

$$K_j(Ae_j) = e^{-i\pi m_j/2} \sqrt{\frac{i}{2\pi \hbar}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} \tau(x, \phi)} \sqrt{|\det(P_\phi, P)(\tau(x, \phi), \phi)|} (Ae_j) \, d\phi,$$

$$\tau(x, \phi) : \quad \langle P(\tau, \phi), x - X(\tau, \phi) \rangle = 0$$

Relationship with Hörmander type Fourier integral operators (locally):

$$u(x, k) = \int e^{\frac{i}{\hbar} \Phi(x, \theta)} A(x, \theta) \, d\theta \implies$$

Many beautiful properties proved by Hörmander could be used
Representation near standard caustics

**Lemma** Suppose that $A$ is supported in a sufficiently small neighborhood of the focal point $\alpha^* = \alpha(\tau^*, \phi^*)$.

1. If $X_{\phi\phi}^* \neq 0$ (case (a)), then the following asymptotic formula holds for $x$ in the $O(k^{-5/6})$-neighborhood of the point $x^*$:

$$[K_A A_j](x, h) = \frac{2^{5/6} \sqrt{\pi}}{\sqrt[6]{\hbar} e^{i\pi/4}} A(\tau^*, \phi^*) \sqrt{\frac{\mu(\tau^*, \phi^*) |P^*||P^*_\phi|}{3 |\langle P^*, X^*_{\phi\phi} \rangle|}} \times \exp \left( \frac{i}{\hbar} (\tau^* + \langle P^*, x - X^* \rangle) \right) \text{Ai} \left( \frac{2^{1/3} \langle P^*_\phi, x - X^* \rangle}{\hbar^{2/3} \sqrt[3]{\langle P^*, X^*_{\phi\phi} \rangle}} \right) + O(\sqrt[6]{\hbar}).$$
2. If \( X^{*}_{\phi\phi} = 0 \) but \( X^{*}_{\phi\phi\phi} \neq 0 \) (case (b)), then the following asymptotic formula holds for \( x \) in the \( O(k^{-5/6}) \)-neighborhood of the point \( x^{*} \):

\[
[K_{\lambda} A_{j}](x, h) = \frac{\sqrt{6}}{\sqrt{\hbar e^{i\pi/4}}} A(\tau^{*}, \phi^{*}) \frac{\sqrt{\mu(\tau^{*}, \phi^{*})} |P^{*}| |P^{\phi}|}{\sqrt{\langle P^{*}, X^{*} \rangle}} \exp \left( \frac{i}{\hbar} (\tau^{*} + \langle P^{*}, x - X^{*} \rangle) \right)
\]

\[
\times P^{\pm} \left( \sqrt{\frac{24}{\hbar^{3} |\langle P^{*}, X^{*} \rangle|}} \langle P^{*}, x - X^{*} \rangle, \sqrt{\frac{6}{\hbar |\langle P^{*}, X^{*} \rangle|}} \langle P^{\phi}, x - X^{*} \rangle \right) + O(1),
\]

where \( P(v, y) \) is the Pearcy function,

\[
P^{+}(v, y) = P(v, y), \quad P^{-}(v, y) = \overline{P}(-v, -y).
\]
Eikonal-coordinates, classical and quantum Maupertuis–Jacobi correspondence, Finsler metric, etc

\[ \langle P, X_\tau \rangle = 1, \langle P, X_\phi \rangle = 1, \quad |J| = \frac{|X_\phi|}{|P|} \quad \text{(the “Green” law)} \]

Regular charts

\[ K_j(Ae_j) = e^{-\pi m_j/2}e^{i\tau(x)}\sqrt{|P(\tau(x), \phi(x))|} \left( \frac{\sqrt{|P(\tau(x), \phi(x))|}}{\sqrt{|X_\phi(\tau(x), \phi(x))|}} \right)(Ae_j)(\tau(x), \phi(x)) \]

\[ \tau(x), \phi(x) : \quad X(\tau, \phi) = x \]

Let the Hamiltonian be \( H(x, p) = |p|C(x) \) \( \implies \) then \( \tau \) is a “propper” time of the Hamilton system

\[ \dot{p} = -H_x, \quad \dot{x} = H_p \]
Example: stationary Schrödinger equation

\[-\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi = E\psi, \quad E > V, \quad x \in \mathbb{R}^2\]

We put \( C^2 = \frac{1}{2m(E - V(x))}, \quad \hat{p} = -i\hbar \frac{\partial}{\partial x} \) and get

\[(-\hbar^2 C^2(x) \Delta - 1)\psi(x, h) = (\hat{L} + 1)(\hat{L} - 1)\psi = 0, \quad \implies \]

\[(\hat{L} - 1)\psi = 0, \quad \hat{L} = \sqrt{-\hbar^2 C^2(x) \Delta} = \sqrt{\frac{2}{C^2(x)}} \hat{p}^2 = L(\frac{2}{1}, \frac{1}{x}, \frac{1}{\hat{p}}, h)\]

The symbol \( L(x, p, h) = H(x, p) + hL_1(x, p) + h^2 L_2(x, p) + \ldots, \)

\[H(x, p) = C(x)|p|, \quad L_1(x, p) = \frac{i}{2H} \frac{\partial H}{\partial p} \frac{\partial H}{\partial x} \equiv \frac{i}{2} \frac{\partial^2 C(x, p)}{|p|} \equiv \frac{i}{2} \text{tr} H_{px}\]
Example: the Dirac-type system for graphene

\[
(-i\hbar \frac{\partial}{\partial x_1} + \hbar \frac{\partial}{\partial x_2})v + U(x)u = \varepsilon u,
\]

\[
(-i\hbar \frac{\partial}{\partial x_1} - \hbar \frac{\partial}{\partial x_2})u + U(x)v = \varepsilon v
\]

The reduction to scalar (electronic) equation:

\[
w(x, h) = \hat{\chi}\psi \equiv \chi(x, -i\mu \frac{\partial}{\partial x}, h)\psi(x, \mu), \quad \hat{L}\psi \equiv L(x, -i\mu \frac{\partial}{\partial x}, h)\psi = \psi,
\]

\[
L = H(x, p) + \hbar L_1(x, p) + \ldots, \quad H = |p|C(x), \quad L^1 = \frac{U_{x_1}p_2 - U_{x_2}p_1}{2(\varepsilon - U(x))p^2}
\]

\[
\chi^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} p_1 + ip_2 \\ |p| \end{pmatrix}, \quad C = \frac{1}{\varepsilon - U(x)}
\]
**Example:** Bessel beams in paraxial approximation.

The Cauchy problem for the Schrödinger type equation (paraxial approximation in optics, \(c, k\) are physical constants)

\[
    i\hbar \frac{\partial v}{\partial t} = -i\hbar c \frac{\partial v}{\partial x_3} - \frac{\hbar^2}{2} \left( \frac{c}{2k} \frac{\partial^2 v}{\partial x_1^2} + \frac{c}{2k} \frac{\partial^2 v}{\partial x_2^2} \right), \quad v|_{t=0} = v_0.
\]

We put \(v(x_1, x_2, x_3, t) = u(x_1, x_2, z - ct)\) \(\implies\)

2-D Schrödinger equation

\[
    i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right), \quad m = \frac{2k}{c}, \quad u|_{t=0} = v_0(x_1, x_2, z) = K_h^{A_0(z)} A.
\]
The Lagrangian manifolds

\[ \Lambda^2_t(z) = \{ p = P(\tau, \phi) \equiv \lambda(z)n(\phi), x = (\tau + t\frac{\lambda(z)}{m})n, \tau \in \mathbb{R}, \phi \in S \} \]

The asymptotic solution:

\[ u = \pi \left( \frac{i}{2\pi \hbar} \right)^{1/2} e^{- \frac{i t \lambda^2(z)}{2m \hbar}} \lambda(z) f(z) \left( (g\left( \frac{|x|}{\lambda(z)} - t\frac{\lambda(z)}{2m} \right) + g\left( -\frac{|x|}{\lambda(z)} - t\frac{\lambda(z)}{2m} \right)) J_0\left( \frac{\lambda(z)|x|}{\hbar} \right) + \right. \]

\[ \left. i \left( g\left( \frac{|x|}{\lambda(z)} - t\frac{\lambda(z)}{2m} \right) - g\left( -\frac{|x|}{\lambda(z)} - t\frac{\lambda(z)}{2m} \right) \right) J_1\left( \frac{\lambda(z)|x|}{\hbar} \right) \right) \]

\[ f(z), g(\tau) \text{ are smooth and finite} \]
The other application: Correction to asymptotic of generalized function of 2-D Schrödinger equation and influence of the tic into measurement in electronic microscope


S. Yu. Dobrokhotov, G. Makrakis, V. E. Nazaikinskii, New formulas for Maslov’s canonical operator in the multidimensional case (submitted to V.A.Marchenko volume)
THANK YOU FOR YOUR ATTENTION!