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1. Setup

We consider the problem of imaging extended reflectors in waveguides using an array of sensors. Our array is composed of \( N \) elements that can play the role of sources and receivers. Our data for solving the inverse problem is the array response matrix, \( \tilde{\mathbf{V}}(\vec{t},\vec{x}) \), corresponding to the "scattered" acoustic pressure field recorded as a function of time at receiver location \( \vec{z} \), and due to a pulse emitted from a source located at \( \vec{t} \). We assume that the waveguide has a simple geometry (horizontal boundaries) and the medium is homogeneous, i.e. the wave speed is constant, \( c(\vec{r}) = c_0 \).

The reflector that we wish to image will be modeled as an acoustically hard scatterer occupying the domain \( \Omega \). We have \( \Delta t = 0 \) in \( \Omega \).

\[ \Delta t = 0, \quad \vec{z}_n \in \Omega \]

2. Array response matrix

We compute the array response matrix by solving the wave equation numerically. We have,

\[ \tilde{\mathbf{V}}(\vec{t},\vec{x}) = p_{\text{inc}}(\vec{t},\vec{x}) \mathbf{e}_t + p_{\text{scat}}(\vec{t},\vec{x}) - p_{\text{tot}}(\vec{t},\vec{x}) \]

Here \( p_{\text{inc}} \), \( p_{\text{scat}} \) and \( p_{\text{tot}} \) correspond to the scattered, total and incident fields, respectively. The total field is given by solving (1) while \( \mathbf{e}_t \) is the solution of (2).

\[ \begin{align*}
  \frac{\partial^2 p_{\text{tot}}(\vec{t},\vec{x})}{\partial t^2} - \beta_0^2 \nabla^2 p_{\text{tot}}(\vec{t},\vec{x}) &= \delta(t-t_0) \mathbf{e}_t \\
  \frac{\partial^2 p_{\text{inc}}(\vec{t},\vec{x})}{\partial t^2} - \beta_0^2 \nabla^2 p_{\text{inc}}(\vec{t},\vec{x}) &= \delta(t-t_0) \\
  \frac{\partial^2 p_{\text{scat}}(\vec{t},\vec{x})}{\partial t^2} - \beta_0^2 \nabla^2 p_{\text{scat}}(\vec{t},\vec{x}) &= 0
\end{align*} \]

where \( \beta_0 \) is the central frequency of the source and \( t_0 \) the time at which the source attains its maximum.

\[ p_{\text{tot}}(\vec{t},\vec{x}) = p_{\text{inc}}(\vec{t},\vec{x}) + p_{\text{scat}}(\vec{t},\vec{x}) \]

3. Imaging

To construct an image we use Kirchhoff migration,

\[ \hat{\mathbf{K}}(\vec{g},\omega) = \int_{-\infty}^{\infty} \hat{\mathbf{G}}(\vec{t'},\vec{r'},\omega) \hat{\mathbf{V}}(\vec{t'},\vec{r'},\omega) \hat{\mathbf{G}}(\vec{g},\vec{r'},\omega) d\vec{t'} \]

The points \( \vec{g} = (g_x,g_y) \) span the search domain \( \Gamma \) and \( \hat{\mathbf{G}}(\vec{t'},\vec{r'},\omega) \) is the Green's function in the waveguide evaluated at \( \vec{r'} \), for a point source at \( \vec{r'} \) and a given frequency \( \omega \).

\[ \hat{\mathbf{G}}(\vec{t'},\vec{r'},\omega) = \frac{1}{2 \pi} \frac{1}{\sqrt{\left( \omega / c_0 \right)^2 - \beta_0^2}} \exp \left( \frac{i \left( \omega t' - \beta_0 \sqrt{\left( \omega / c_0 \right)^2 - \beta_0^2} \right)}{2} \right) X(x,t) \hat{X}(y,t), \]

where \( \beta_0 = (\omega / c_0)^2 \), \( c_0 = \sqrt{\mu / \epsilon} \), and \( X(x,t) = \sqrt{\mu / \epsilon} \) and \( \hat{X}(y,t) = \sqrt{\mu / \epsilon} \). \n
When the array spans the whole depth of the waveguide, and is far from the reflector, we may, alternatively, introduce a matrix \( \hat{\mathbf{M}}(\omega) \), such that:

\[ \hat{\mathbf{M}}(\omega) = \frac{\mathbf{e}_t}{i \omega} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{G}}(\vec{t},\vec{r},\omega) \hat{\mathbf{V}}(\vec{t},\vec{r},\omega) d\vec{t} d\vec{r} \]

\[ \hat{\mathbf{M}} = \int_{-\infty}^{\infty} \hat{\mathbf{M}}(\omega) d\omega \]

\[ \hat{\mathbf{M}} = \mathbf{A} \mathbf{Q} \quad \text{with} \quad \mathbf{Q}^2 = \mathbf{I} \]

where \( \mathbf{A} \) is the diagonal matrix \( \mathbf{A} = \delta_{ij} e^{i \beta_0 L} \).

Now, studying the SVD of \( \hat{\mathbf{M}}(\omega) \), reduces to studying the SVD of \( \mathbf{A} \). An asymptotic analysis indicates that \( \mathbf{A} \) has \( M \) significant singular values, all equal to one, and the corresponding singular vectors are \( \mathbf{V}^2 = \{ \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_M \} \), with \( \mathbf{V}_j^2 = \frac{\beta_0 L}{\lambda_j} \).

To create a filtered version of \( \hat{\mathbf{M}}(\omega) \), we use:

\[ D \hat{\mathbf{M}}(\omega) = \sum_{j=1}^{M} \mathbf{V}_j \hat{\mathbf{M}}(\omega) \mathbf{V}_j^* \]

where \( \mathbf{V}_j \) is the orthonormal basis of \( L^2 \).

In that case the Kirchhoff migration functional at the correct range reduces to

\[ \hat{\mathbf{K}}(\vec{g},\omega) = (\sum_{j=1}^{M} \mathbf{V}_j \hat{\mathbf{V}} \mathbf{V}_j^*)^{1/2} \]

4. The D.O.R.T. Method

The main idea behind D.O.R.T. is to:

- Use the SVD of the response matrix \( \hat{\mathbf{M}}(\omega) \)
- Use a filtered version of the response matrix to create the image.

The SVD of \( \hat{\mathbf{M}}(\omega) \) is:

\[
\hat{\mathbf{M}}(\omega) = \mathbf{U} \mathbf{D} \mathbf{V}^* = \sum_{j=1}^{M} \mathbf{U}_j \mathbf{D}_j \mathbf{V}_j^*
\]

where \( \mathbf{D}_j \) is the diagonal matrix \( \mathbf{D}_j = \delta_{ij} e^{i \beta_0 L} \).

5. Numerical Results

- Central frequency: \( f_1 = 75 \text{Hz} \)
- Bandwidth: \( B = 18 \text{Hz} \)
- \( L = 450 \text{cm} \), \( h = 200 \text{cm} \), \( c_0 = 1500 \text{m/s} \)
- \( h_0 = 20 \text{cm} \), \( b = 40 \text{cm} \)
- 28 propagating modes.

Here, \( \omega \) denotes the image produced after projecting on the \( j \)-th singular vector.

\[
\hat{\mathbf{K}}(\vec{g},\omega) = (\sum_{j=1}^{M} \mathbf{V}_j \hat{\mathbf{V}} \mathbf{V}_j^*)^{1/2}
\]

6. Theoretical Approach

From a theoretical point of view and using the single scattering Born approximation on the reflector, we can show that for a crack located at \( \vec{r} \), and \( \omega \) \( \omega \), we have:

\[
\hat{\mathbf{M}}(\omega) = \sum_{M=1}^{M} \mathbf{U}_j \mathbf{D}_j \mathbf{V}_j^* \]

\[
\hat{\mathbf{M}}(\omega) = \sum_{j=1}^{M} \mathbf{U}_j \mathbf{D}_j \mathbf{V}_j^* \]

7. Comparison

On the left column, we see the images produced with data acquired by our numerical simulations, while on the right one, we see those created by our theoretical approach.

References